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Indices

There are five basic rules of indices

(a) $a^p \times a^q = a^{p+q}$

(b) $\frac{a^p}{a^q} = a^{p-q}$

(c) $(a^p)^q = a^{pq}$

(d) $a^{\frac{1}{q}} = \sqrt[q]{a}$

(e) $a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p$

Example 1

Evaluate the following

(a) $2^2 \times 2^3$

(b) $\frac{4^3}{4^2}$

(c) $(3^2)^3$

(d) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$

(e) $\sqrt[3]{27}$

(f) $125^{\frac{2}{3}}$

Solution

(a) $2^2 \times 2^3 = 2^{2+3} = 2^5 = 32$

(b) $\frac{4^3}{4^2} = 4^{3-2} = 4^1 = 4$

(c) $(3^2)^3 = 3^{2 \times 3} = 3^6 = 729$

(d) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$

(e) $\sqrt[3]{27} = (3^3)^{\frac{1}{3}} = 3^{3 \times \frac{1}{3}} = 3^1 = 3$

(f) $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$

Example 2

Evaluate the following

(a) $\left(\frac{125}{27}\right)^{\frac{4}{3}}$

(b) $81^{\frac{3}{4}}$

Solution

(a) $\left(\frac{125}{27}\right)^{\frac{4}{3}} = \left(\frac{125^{\frac{4}{3}}}{27^{\frac{4}{3}}}\right) = \left(\frac{\left(\sqrt[3]{125}\right)^4}{\left(\sqrt[3]{27}\right)^4}\right) = \frac{625}{81}$

(b) $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3 = 27$

The zero index

From $\frac{a^p}{a^p} = a^{p-p} = a^0 = 1$

∴ Any number raised to power zero =1

i.e. $100^0 = 529^0 = 83^0 = 1$

Negative indices

It can be shown that

$\frac{1}{a} = \frac{a^0}{a^1} = a^{0-1} = a^{-1}$

Also

$\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$

Hence a negative index is the inverse of a given number

Example 3

Evaluate the following

(a) $16^{\frac{-3}{2}}$

(b) $\left(\frac{64}{27}\right)^{-\frac{2}{3}}$

Solution

(a) $16^{\frac{-3}{2}} = \left(\frac{1}{16}\right)^{\frac{3}{2}} = \left(\frac{1}{\sqrt{16}}\right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$

(b) $\left(\frac{64}{27}\right)^{-\frac{2}{3}} = \left(\frac{27}{62}\right)^{\frac{2}{3}} = \left(\frac{\sqrt[3]{27}}{\sqrt[3]{64}}\right)^2 = \frac{9}{16}$

Solving equations with unknown indices

It involves making appropriate substation after expressing terms containing powers in simplified form

Example 4

Solve the equation

$2^{2x+1} - 7(2^x) + 6 = 0$

Solution

$2^{2x+1} - 7(2^x) + 6 = 0$

$2^1.2^{2x} - 7(2^x) + 6 = 0$

$2(2^x)^2 - 7(2^x) + 6 = 0$

Let p = 2^x

⇨ $2p^2 - 7p + 6 = 0$
$(2p - 3)(p - 2) = 0$

Either 2p -3 = 0

p = $\frac{3}{2}$

or

p – 2 =0

p = 2

when p= $\frac{3}{2}$ => $2^x = \frac{3}{2}$

$\log 2^x = \log \frac{3}{2}$

$x \log 2 = \log \frac{3}{2}$

$x = \frac{\log \frac{3}{2}}{\log s} = 0.585$

When p = 2

$2^x = 2 = 2^1$

x = 1

Hence x = 1 and x = 0.585 (3d.p)

Example 5

Show that

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x} = 4$

Solution

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x} = 4$

Handling terms on the LHS

$\frac{3(2^{x+1})-4(2^{x-1})}{2^{x+1}-2^x}$

$= \frac{3(2^x x 2^1)-4(2^x.2^1)}{2^x.2^1-2^x}$

$= \frac{2^x(3\,.\,2^1-4\,.\,2^1)}{2^x(2^1-1)} = \frac{6-2}{1} = 4$

Example 5

Solve $x^{\frac{4}{3}} = 81$

$x^{\frac{4}{3}x\frac{3}{4}} = 81^{\frac{3}{4}}$

x = $\left(\sqrt[4]{81}\right)^2 = 3^3 = 27$

Solving equations with squares

Example 6

$\sqrt{2x+5} = x + 1$

Square both sides

$\left(\sqrt{2x+5}\right)^2 = (x+1)^2$

$2x + 5 = x^2 + 2x + 1$

$x^2 = 4$

$x = \pm 2$

Testing/checking using -2

$\sqrt{2x+5} = x + 1$

$\sqrt{2\,x - 2 + 5} = -2 + 1$

$1 \neq -1$

Hence -2 is **not** a solution to the equation

Testing/checking using 2

$\sqrt{2x+5} = x + 1$

$\sqrt{2\,x2 + 5} = 2 + 1$

$3 = 3$

Hence 2 is the solution to the equation

Example 7

Solve for x: $\sqrt{x+2} = 4$

Square both sides

$\left(\sqrt{x+2}\right)^2 = 4^2$

x + 2 = 16

x = 14

Finding square roots of terms containing rational and irrational quantities

When finding roots of terms expressed in the form a + $\sqrt{b}$, where a is a rational and b is an irrational quantity, we let the root to be in the form of $\pm(\sqrt{x_1} + \sqrt{x_2})$ where x_1 and x_2 are integers.

Example 8

Find the square root of 6 +2$\sqrt{5}$

Let $\pm(\sqrt{x_1} + \sqrt{x_2})$ be square root of 6 +2$\sqrt{5}$

⇨ $\pm(\sqrt{x_1} + \sqrt{x_2}) = \sqrt{6 + 2\sqrt{5}}$

Squaring both sides

$(\sqrt{x_1} + \sqrt{x_2})^2 = \left(\sqrt{6+\sqrt{5}}\right)^2$

$x_1 + x_2 + 2\sqrt{x_1.x_2}$ =6 +2$\sqrt{5}$

Comparing terms on the two sides

$x_1 + x_2 = 6$

$x_1 = 6 - x_2$ .....(i)

$x_1.x_2 = 5$.....(ii)

Substituting eqn. (i) into eqn. (ii)

$(6 - x_2)x_2$ = 5

${x_1}^2 - 6x_2 + 5 = 0$

${x_1}^2 - x_2 - 5x_2 + 5 = 0$

$x_2(x_2 - 1) - 5(x_2 - 1) = 0$

$(x_2 - 1)((x_2 - 5) = 0$

Either: $x_2 - 1$ = 0 => $x_2 = 1$

Or $x_2 - 5$ = 0 => $x_2 = 5$

When$x_2 = 1$: $x_1 = 6 - 1$ = 5

When$x_2 = 5$: $x_1 = 6 - 5$ = 1

Hence the square root of 6 +2$\sqrt{5}$ is $\pm(1 + \sqrt{5})$

Example 9

Find the square root of 8 - 2$\sqrt{15}$

et $\pm(\sqrt{x_1} - \sqrt{x_2})$ be square root of 8 -2$\sqrt{15}$

$\pm(\sqrt{x_1} + \sqrt{x_2}) = \sqrt{8 - 2\sqrt{15}}$

Squaring both sides

$(\sqrt{x_1} - \sqrt{x_2})^2 = \left(\sqrt{8 - 2\sqrt{15}}\right)^2$

$x_1 + x_2 - 2\sqrt{x_1.x_2} = 8 - 2\sqrt{15}$

Comparing terms on the two sides

$x_1 + x_2 = 8$

$x_1 = 8 - x_2$ .....(i)

$x_1.x_2 = 15$.....(ii)

Substituting eqn. (i) into eqn. (ii)

$(8 - x_2)x_2$ = 15

${x_1}^2 - 8x_2 + 15 = 0$

${x_1}^2 - 3x_2 - 5x_2 + 15 = 0$

$x_2(x_2 - 3) - 5(x_2 - 3) = 0$

$(x_2 - 3)((x_2 - 5) = 0$

Either: $x_2 - 5$ = 0 => $x_2 = 5$

Or $x_2 - 3$ = 0 => $x_2 = 3$

When$x_2 = 5$: $x_1 = 8 - 5$ = 3

When$x_2 = 3$: $x_1 = 8 - 3$ = 5

Hence the square root of 8 -2$\sqrt{15}$ is $\pm(\sqrt{5} - \sqrt{3})$

Revision exercise

1. Simplify
   (i) $9a^2 \div 27a^{-4}\left[\frac{2}{3}a^6\right]$
   (ii) $(6a^{-3}) \div (9a^{-4})^2\left[\frac{2}{27}a^5\right]$
   (iii) $\frac{2a^{-3}b^2}{7c^{-4}d^2}$ $\left[\frac{2b^2c^4}{7a^3d^2}\right]$

(iv) $(x^4yz^{-3})^2 \times \sqrt{x^{-5}y^2z} \div (xz)^{\frac{1}{2}}$ $[x^5yz^{-6}]$

(v) $\sqrt[4]{y^3}x\sqrt{y^{\frac{1}{2}}}$ $\left[y^{\frac{5}{4}}\right]$

2. Evaluate
   (a) $(64)^{-\frac{3}{2}}$ [16]
   (b) $\left(\frac{8}{27}\right)^{-\frac{1}{3}}\left[\frac{3}{2}\right]$
   (c) $\left(\frac{1}{25}\right)^{\frac{1}{2}}\left[\frac{1}{5}\right]$
   (d) $\left(\frac{8}{27}\right)^{\frac{2}{3}}\left[\frac{4}{9}\right]$
   (e) $\left(\frac{243}{512}\right)^{-\frac{2}{3}}$ $[1.6445]$
3. Solve the following equations
   (a) $98x^2 = 2$ [x= 0.1429]
   (b) $x^{-3} = 8\left[x = \frac{1}{2}\right]$
   (c) $\frac{1}{32}x^3 = 8x^{-1}$[x = 4]
   (d) $\frac{9}{25}x = \frac{5}{3}x^{-2}\left[x = \frac{5}{3}\right]$
   (e) $\frac{2}{14}x^{-2} + 14x = 0$ [x = -0.2169]
4. Solve for x
(a) $3^{2x+1} + 3 = 10(3^x)$ [x = 1 or x = -1]
(b) $2^{2x-1} + \frac{3}{2} = 2^{x+1}$[x =0, x =1.585]
(c) $7^x = 3^{1-x}$ [x=0.3608]
(d) $7x^{\frac{1}{2}}+2 = 0\left[x = \frac{4}{49}\right]$
(e) $5x^{\frac{2}{3}} = x^{-\frac{1}{3}}\left[x = \frac{1}{5}\right]$
(f) $4x^{-\frac{1}{3}} = 5x^{\frac{1}{6}}\left[x = \frac{16}{25}\right]$
(g) $6x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{2}} = 0$ [x = 0.077]
(h) $8x^{-2} = 343x^{\frac{1}{2}}$ [x=0.003562]

5. Show that
(a) $\frac{(2^{2x}-3.2^{2x-2})(3^x-2.3^{x-2})}{3^{x-4}(4^{x+3}-2^{2x})} = \frac{1}{4}$
(b) $\frac{(1+a)^{\frac{1}{2}}-\frac{1}{3}a(1+a)^{-\frac{2}{3}}}{(1+a)^{\frac{2}{3}}} = \frac{3+2a}{3(1+a)^{\frac{4}{3}}}$
(c) $(a - a^{-1})\left(a^{\frac{4}{3}} - a^{\frac{2}{3}}\right) = \frac{a^2-a^{-2}}{a^{-\frac{1}{3}}}$
(d) $\frac{a^{\frac{1}{2}}+ab}{ab-b^2} - \frac{\sqrt{a}}{\sqrt{a-b}} = \sqrt{\frac{a}{b}}$
6. Solve
   (a) $x^{\frac{1}{3}} - 3 = 28x^{-\frac{1}{3}}$ [x =-64, x = 343]
   (b) $2x^{\frac{1}{4}} = 9 - 4x^{-\frac{1}{4}}$[x =$\frac{1}{16}$, x = 256]
   (c) $x^3 + 8 = 9x^{\frac{3}{2}}$ [x = 1, x= 4]
   (d) $2x^{\frac{1}{3}} = \frac{81}{8}x^{-1}$ [x = 8.6967]
   (e) $49x^{-\frac{5}{6}} - \frac{8}{81}x^{\frac{7}{6}} = 0$ [x=22.2739]
   (f) $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 2 = 0$[x=-1]
   (g) $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} + 6 = 0$[x=-1]
   (h) $6x^{\frac{1}{3}} + 5 + x^{-\frac{1}{3}} = 0\left[x = \frac{1}{2}, x = \frac{1}{3}\right]$
7. Solve for x
   (a) $\sqrt{x+2} - x = 0$[x=2]
   (b) $\sqrt{1+x} = 1 + \sqrt{1-x}\left[x = \frac{\sqrt{3}}{2}\right]$
   (c) $(3-x)^{\frac{1}{2}} = (1+x)^{\frac{1}{2}} + (2-x)^{\frac{1}{2}}$ [x = -0.92665]
   (d) $\sqrt{x+6} = \sqrt{1-3x} - \sqrt{4-x}$ [-5]
8. Without using mathematical tables or calculators, find the value of
   $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]
9. Find the square root of the following
   (a) $6 + 2\sqrt{5}\left[\pm\left(1+\sqrt{5}\right)\right]$
   (b) $18 - 2\sqrt{12}\left[\pm\left(\sqrt{0.695} - \sqrt{17.303}\right)\right]$
   (c) $18 - 2\sqrt{2}\left[\pm\left(\sqrt{0.1118} - \sqrt{17.8882}\right)\right]$

Thank you

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Complex numbers

Equations $ax^2 + bx + c = 0$ where $a \neq 0$ and $b^2 < 4ac$ can only be solved by introducing $\sqrt{(-1)}$ = i or j; and this is the basis of complex numbers.

Solving equations involving complex numbers

Given $ax^2 + bx + c = 0$

$$x^2 + \left(\frac{b}{a}\right)x = \left(-\frac{c}{a}\right)$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = \frac{b^2-4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{4a^2}$$

If $b^2 > 4ac$; the equation has two roots which are both real.

If $b^2 = 4ac$; the equation has got two roots which are repeated and real.

If $b^2 < 4ac$; the equation has got two roots which are not real.

In complex numbers equations whose roots are not real are considered

Hence

$$x = -\frac{b}{2a} \pm i\frac{\sqrt{b^2-4ac}}{4a^2}$$

We let x = p ± iq

Where p = $-\frac{b}{2a}$ and q = $\frac{\sqrt{b^2-4ac}}{4a^2}$

Example 1

(a) $x^2 + 4 = 0$

Solution

$x^2 = -4$

$x = \sqrt{-4} = \pm 2i$

(b) $x^2 + 2x + 5 = 0$

Solution

From $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x = \frac{-2\pm\sqrt{4-4\,(1)(5)}}{2\,(1)} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm4i}{2}$

x = -1 ± 2i

(c) $2x^2 - 3x + 4 = 0$

$x = \frac{3\pm\sqrt{9-4\,(2)(4)}}{2\,(2)} = \frac{3\pm\sqrt{-25}}{4} = \frac{3\pm5i}{4}$

$x = \frac{3}{4} \pm \frac{5}{4}i$

The above simple examples reveal the following

(a) A complex number Z may be written as Z = x + iy, where x and y are real numbers. The real part of Z is x and the imaginary part is y. i.e. *x = Re(Z)* and *y = im(Z).*

(b) Solution of such equations whose roots are complex occur in pairs, called conjugates. For instance, compare

Z = $\pm 2i$, -1 ± 2i and $\frac{3}{4} \pm \frac{5}{4}i$

(i) If Z = -1 + 2i is one root of an equation, then its conjugate $\bar{Z}$ = -1 – 2i is another root.

(ii) An equation cannot have odd number of complex roots

(c) If x + iy = 0, then x = 0 and y = 0

This implies that two complex numbers are equal if and only if their corresponding real and imaginary parts are equal
i.e. $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$, then $Z_1 = Z_2$, only and only if $x_1 = x_2$ and $y_1 = y_2$.
Proof: If Z1 = Z2
$x_1 + iy_1 = x_2 + iy_2$

i.e. $(x_1 – x_2) + (y_1 – y_2) = 0$

=> $x_1 – x_2 = 0$ and $y_1 – y_2 = 0$

$x_1 = x_2$ and $y_1 = y_2$

Algebra of complex numbers

Like in the real plane, complex numbers can be subjected to four basic operations of addition, subtraction, multiplications and division.

Using $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$

Addition of complex number

When adding two or more complex numbers, real parts are added separately and imaginary parts are added separately

$Z_1 + Z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 – 2i$

$Z_1 + Z_2 = (2 + 1) + i(3 + -1)$

$= 3 + 2i$

Subtraction of complex numbers

When subtracting two or more complex numbers, real parts are subtracted from real parts and imaginary parts subtracted from imaginary parts.

$Z_1 – Z_2 = (x_1 – x_2) + i(y_1 – y_2)$

Illustration

If $Z_1 = 2 + 3i$ and $Z_2 = 1 – 2i$

$Z_1 + Z_2 = (2 – 1) + i(3 – -1)$

$= 1 + 4i$

Geometrically, the operations of addition and subtraction of complex numbers are diagonals of a parallelogram. (see the Argand diagrams)

Multiplication of complex numbers

When multiplying complex numbers the following properties should be observed

$i^2 = \sqrt{-1} \times \sqrt{-1} = -1$;

$i^3 = -1 \times i = -1$

$i^4 = i^2 \times i^2 = -1 \times -1 = 1$

$i^5 = i^3 \times i^2 = -1 \times -1 = 1$

$i^6 = i^3 \times i^3 = -1 \times -1 = i^2 = -1$

$i^8 = i^4 \times i^4 = 1 \times 1 = 1$

Now

$Z_1Z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$= (x_1x_2 – y_1y_2) + i(x_1y_2 + x_2y_1)$

Division of complex numbers

When diving a complex number with another, the result is also a complex number. It can be done in two ways

Either:

Let $\frac{Z_1}{Z_s} = p + iq$ for some real numbers p and q.

⇨ $Z_1 = (p + iq)Z_2$

$x_1 + iy_1 = (p + iq)(x_2 + iy_2)$

$= px_2 + ipy_2 + iqx_2 + i2qy_2$

$= px_2 – qy_2 + i(py_2 + qx_2)$

Equating the corresponding real and imaginary parts of the resulting complex number

$x_1 = px_2 – qy_2$ and $x_2 = py_2 + qx_2$

Solving these equation simultaneously for p and q

p = $\frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2}$ and q = $\frac{x_1x_2-y_1y_2}{x_2{}^2+y_2{}^2}$

Or:

'Realizing' the denominator by multiplying through by it conjugate

$$\frac{Z_1}{Z_s} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{x_1x_2-ix_1y_2-ix_2y_1\ y_1y_2}{x_2{}^2+y_2{}^2}$$

$$= \frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2} + i\frac{(x_1x_2-y_1y_2)}{x_2{}^2+y_2{}^2}$$

Example 2

Given that $Z_1 = 3 - 2i$ and $Z_2 = 2 + 3i$, by expressing your answer in the form a + bi, find

(a) $Z_1 + Z_2$
Solution
$Z_1 + 3Z_2 = (3 - 2i) + 3(2 + 3i) = 9 + 7i$

(b) $2Z_1 - Z2$
Solution
$2(3 - 2i) - (2 + 3i) = 4 - 7i$

(c) If Z1 = 2 – 3i and Z2 = 4 + i, find the values of

(i) $Z_1{}^3Z_2$
Solution

$Z_1{}^3Z_2 = (2 - 3i)^3(4 + i)$

Using Pascal's triangle

$= [2^3 + 3(2)^2(-3i) + 3(2)(-3i)^2 + (-3i)^3](4+ i)$

$=(8 - 36i - 54 + 27i)(4 + i)$

$=(-46 - 9i)(4 + i)$

$= -184 - 46i - 36i + 9$

$= -175 - 82i$

(ii) $\frac{Z_2{}^1}{Z_1}$

Solutions

$$\frac{Z_2{}^1}{Z_1} = \frac{(4+i)^2}{(2-3i)} = \frac{16+8i-1}{(2-3i)} = \frac{15+8i}{(2-3i)}$$

$$= \frac{(15+8i)(2+3i)}{(2-3i)(2+3i)} = \frac{30+45i+16i-24}{4+9}$$

$$= \frac{6}{13} + i\frac{61}{13}$$

Forming quadratic equations given roots

When given roots to a complex number, the quadratic equation say Z can be obtained by finding the sum and the products of the roots

i.e. the equation is given by
Z^2 – (sun of roots)Z + product of roots

Not that if Z is a root of the equation, then its conjugate $\bar{Z}$ is also a root

Example 3

Form a quadratic equation in Z given the following roots

(a) 1 + 5i
Solution
Let Z = 1 + 5i, the Z* = 1 – 5i
Sum of roots (1 + 5i) + (1 - 5i) = 2
Product of roots (1 + 5i)(1 – 5i)= 26
Equation: $Z^2 - 2Z + 26 = 0$
Or
Let Z = 1 + 5i => Z – 1 – 5i = 0
Z = 1 – 5i => Z – 1 + 5i = 0
⇨ (Z – 1 – 5i)(Z – 1 + 5i) = 0
$Z^2 - 2Z + 26 = 0$

(b) 4 – 3i
Solution
Let Z = 4 – 3i, the Z* = 4 + 3i
Sum of roots (4 - 3i) + (4 + 3i) = 8
Product of roots (4 - 3i)(4 - 3i)= 41
Equation: $Z^2 - 8Z + 41 = 0$
Or
Let Z = 4 - 3i => Z – 4 + 3i = 0
Z = 4 + 3i => Z – 4 - 3i = 0
⇨ (Z – 4 + 3i)(Z – 4 - 3i) = 0
$Z^2 - 8Z + 41 = 0$

Solving cubic equations

It is quite difficult to solve cubic equations, however, we use the approach of inspection, i.e. we substitute the assumed factors into the given equation. If the unction is f(x) = 0 and x – a = 0 is a root, then f(a) = 0.

We then get another equation (quadratic) using long division which can easily be solved to obtain other factors.

Example 4

Solve the following equations

(a) $x^3 - 4x^2 + x + 26 = 0$

Solution

Let $f(x) = x^3 - 4x^2 + x + 26 = 0$

$f(1) = 2 - 4 + 1 + 26 \neq 0$; => $x - 1$ is not a factor

$f(2) = 8 - 8 + 2 + 26 \neq 0$; => $x - 2$ is not a factor

$f(-2) = -8 - 16 - 2 + 26 = 0$; $x + 2$ is a factor

Using long division

$$
\begin{array}{r}
x^2 - 6x + 13 \\
(x+2)\overline{)\,x^3 - 4x^2 + x + 26} \\
-\ \underline{x^3 + 2x^2} \\
-6x^2 + x + 26 \\
-\ \underline{-6x^2 - 12x} \\
13x + 26 \\
-\ \underline{13x + 26} \\
0 + 0
\end{array}
$$

$\therefore x^3 - 4x^2 + x + 26 = (x+2)(x^2 - 6x + 13) = 0$

Either $x + 2 = 0$; $x = -2$

Or $x^2 - 6x + 13 = 0$

Solving $x^2 - 6x + 13 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{16\,x - 1}}{2} = 3 \pm 2i$

The roots of the equation are $x = -2, 3 \pm 2i$

Alternatively

Given the equation $x^2 - 6x + 13 = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 6$

$2x = 6$; $x = 3$

Product of roots $= (x + iy)(x - iy) = 13$

$x^2 + y^2 = 13$

Substituting for x;

$9 + y^2 = 13$; $y = \pm 2$

The roots of the equation are $x = -2, 3 \pm 2i$

(b) $2x^3 - 12x^2 + 25x - 21 = 0$

Solution

Let $f(x) = 2x^3 - 12x^2 + 25x - 21 = 0$

$f(1) = 2 - 12 + 25 - 12 = -6 \neq 0$; $x - 1$ is not a factor

$f(2) = 16 - 48 + 50 - 21 = -3 \neq 0$; $x - 2$ is not a factor

$f(3) = 54 - 108 + 75 - 21 = 0$; $x - 3$ is a factor

Using long division

$$
\begin{array}{r}
2x^2 - 6x + 7 \\
(x-3)\overline{)\,2x^3 - 12x^2 + 25x - 21} \\
-\ \underline{2x^2 - 6x^2} \\
-6x^2 + 25x - 21 \\
-\ \underline{-6x^2 + 18x} \\
7x - 21 \\
-\ \underline{7x - 21} \\
0 + 0
\end{array}
$$

$\therefore \quad 2x^3 - 12x^2 + 25x - 21$

$= (x-3)(2x^2 - 6x + 7) = 0$

Either $x - 3 = 0$; $x = 3$

Or $2x^2 - 6x + 7 = 0$

$x = \frac{6 \pm \sqrt{36 - 4(2)(7)}}{2(2)} = \frac{6 \pm \sqrt{4\,x\,5\,x - 1}}{4} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Alternatively

Given the equation $2x^2 - 6x + 7 = 0$

Or $x^2 - 3 + \frac{7}{2} = 0$

Let the roots be $x \pm 2i$

Sum of roots $x + iy + x - iy = 3$

$$2x = 3;\ x = \frac{3}{2}$$

Product of roots = $(x + iy)(x - iy) = \frac{7}{2}$

$$x^2 + y^2 = \frac{7}{2}$$

Substituting for x;

$\frac{9}{4} + y^2 = \frac{7}{2}$; $y = \frac{\pm\sqrt{5}}{2}$

The roots of the equation are $x = 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

Finding the square root of a complex number

Suppose that the square root of Z is a + bi

Then $a + bi = Z^{\frac{1}{2}}$ => $(a + bi)^2 = Z$

Example 5

Find the square roots of

(a) -5 + 12i

Solution

Approach 1

Let $\sqrt{(-5 + 12i)} = x + iy$

By squaring both sides

$-5 + 12i = x^2 - y^2 + i2xy$

Equating corresponding parts

$x^2 - y^2 = -5$ ……. (i)

$2xy = 12$

$x = \frac{6}{y}$…… (ii)

Substituting eqn.(ii) into eqn. (i)

$$\left(\frac{6}{y}\right)^2 - y^2 = -5$$

$36 - y^4 = -5y^2$

Let k = y2

$k^2 - 5k - 36 = 0$

(k – 9)(k + 4) = 0 i.e. k = 9 or k = -4

When k = -4, $y^2 = -4$ which is inadmissible since y must be real.

When k = 9; $y^2 = 9$ => y = ±3

⇨ $x = \frac{6}{\pm 3} = \pm 2$

∴ $\sqrt{(-5 + 12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2 + 3i)$

Approach 2

Let $\sqrt{(-5 + 12i)} = x + iy$

$x^2 - y^2 + i2xy = -5 + 12i$

Equating corresponding parts

$x^2 - y^2 = -5$ ……. (i)

$2xy = 12$…… (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$x^4 - 2x^2y^2 + y^4 = 25$

$+ \quad 4x^2y^2 = 144$

$x^4 + 2x^2y^2 + y^4 = 169$

$$(x^2 + y^2)^2 = 169$$

$x^2 + y^2 = 13$ ……(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8$; x = ±2

Substituting for x in eqn. (iii)

$4 + y^2 = 13$; y = ±3

∴ $\sqrt{(-5 + 12i)} = 2 + 3i$ or $-2 - 3i \equiv \pm(2 + 3i)$

(b) 3 + 4i

Let $\sqrt{(3 + 4i)} = x + iy$

$x^2 - y^2 + i2xy = 3 + 4i$

Equating corresponding parts

$x^2 - y^2 = 3$ ……. (i) and $2xy = 4$…… (ii)

Eqn. $(i)^2$ + eqn. $(ii)^2$

$x^4 - 2x^2y^2 + y^4 = 9$

$+ \quad 4x^2y^2 = 16$

$x^4 + 2x^2y^2 + y^4 = 25$

$(x^2 + y^2)^2 = 25$

$x^2 + y^2 = 5$ ……(iii)

Eqn.(i) + eqn. (iii)

$2x^2 = 8$; x = ±2

Substituting for x in eqn. (iii)

$4 + y^2 = 5$; y = ±1

∴ $\sqrt{(3+4i)}$ = 2 + i or -2 – i ≡ $\pm(2+i)$

Other equations

Example 6

Solve the simultaneous equations

$Z_1 + Z_2 = 8$

$4Z_1 – 3iZ_2 = 26 + 8i$

Solution

$Z_1 + Z_2 = 8$

$Z_1 = 8 – Z_2$ …… (i)

$4Z_1 – 3iZ_2 = 26 + 8i$…… (ii)

Substituting eqn. (i) into eqn. (ii)

$4(8 – Z_2) + 3iZ_2 = 26 + 8i$

$32 – 4Z_2 – 3iZ_2 = 26 + 8i$

$(4 + 3i)Z_2 = 6 – 8i$

$z_2 = \frac{(6-8i)(4-3i)}{(4+3i)(4-3i)} = \frac{-50i}{25} = -2i$

Substituting for Z_2 into (i)

$Z_1 = 8 + 2i$

∴ $Z_1 = 8 + 2i$ and $Z_2 = -2i$

Example 7

Solve the equation 6x2 – 2(1+ 2i)x – 1 = 0

Solution

Using $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x = \frac{2(1+2i) \pm \sqrt{4(1+2i)^2 - 4(6)(-1)}}{2(6)}$$

= $\frac{1+2i\pm\sqrt{(3+4i)}}{6}$

But $\sqrt{(3+4i)} = \pm(2+i)$

⇨ $x = \frac{1+2i\pm(2+i)}{6}$

Either $x = \frac{1}{2} + \frac{1}{2}i$ or $x = -\frac{1}{6} + \frac{1}{6}i$

Example 8

Solve the equation $\left(\frac{Z-1}{Z+1}\right)^2 = i$, If Z is a complex number in form a + bi

Solution

$(Z – 1)^2 = i(Z + 1)^2$

$Z^2 – 2Z + 1 = iZ^2 + 2iZ + i$

$(i – 1)Z^2 + 2(i + 1)Z + i = 0$

⇨ $Z^2 – 2iZ + 1 = 0$

$Z = \frac{2i\pm\sqrt{-4-4}}{2} = \frac{2i\pm2i\sqrt{2}}{2} = i \pm \sqrt{2}$

∴ $Z = (1 \pm \sqrt{2})i$

Revision exercise 1

1. Simplify each of the following and express your answer in the form a + bi
   (a) (3 + 4i) + (2 + 3i) [5 + 7i]
   (b) (3 + 4i) – (2 – 3i) [1+ 7i]
   (c) $(2i)^2$ [-4]
   (d) (2 + 3i)(2 – 3i) [13]
   (e) (3 + 4i)(1 – 2i) [11 – 2i]
2. If Z is a complex number in form (a + bi) solve
   (a) $\left(\frac{Z-1}{Z+1}\right)^2 = i$ [$Z = (1 \pm \sqrt{2})i$]
   (b) $\frac{Z}{Z+1} = 1 + 2i$ $\left[-1 + \frac{1}{2}i\right]$

(c) Express each of the following complex numbers in the form a + bi
   (a) $Z_1 = (1 – i)(1 + 2i)$ [3 + i]
   (b) $Z_2 = \frac{2+6i}{3-i}$ [2i]
   (c) $Z_3 = \frac{-4i}{1-i}$ [2 – 2i]

(d) Find the square root of
   (a) 12i – 5 [2 +3i or -2 – 3i]
   (b) 5 + 2i [±(3 +2i)]
   (c) 15 + 8i [±(4 + i)]

(d) 7 – 24i [±(4 – 3i)]

(e) Given that Z = 3 + 4i, find the value of the expression $Z + \frac{25}{Z}$ [6]

(f) Given that the complex number Z* and its complex conjugate satisfy the equation ZZ* + 2iZ = 12 + 6i, find the possible values of Z [3 + 3i; 3 – i]

(g) Solve the simultaneous equation

$Z_1 + Z_2 = 8$

$4Z_1 – 3iZ_2 = 26 + 8i$

[Z_1 = 8 + 2i and Z_2 = -2i]

(h) Express each of the following in the form a + bi

(a) $\frac{2+3i}{1-i}$ $\left[-\frac{1}{2}+\frac{5}{2}i\right]$

(b) $\frac{3-i}{1+2i}$ $\left[\frac{1}{5}-\frac{7}{5}i\right]$

(c) $\frac{4}{1+i}$ [-1 + i]

(d) $\frac{3+4i}{3-4i}-\frac{3-4i}{4+4i}$ $\left[\frac{48i}{25}\right]$

(i) Solve the following equations

(a) $2x^2 + 32 = 0$ [±4i]

(b) $4x^2 + 9 = 0$ $\left[\pm\frac{3}{2}i\right]$

(c) $x^2 + 2x + 5 = 0$ [-1 ± 2i]

(d) $x^2 – 4x + 5 = 0$ [2 ± i]

(e) $2x^2 + x + 1 = 0$ $\left[\frac{-1\pm\sqrt{71}}{4}\right]$

(j) Form quadratic equations having roots

(a) 3i, -3i [$x^2 + 9 = 0$]

(b) 1 + 2i, 1- 2i [$x^2 – 2x + 5 = 0$]

(c) 2 + I, 2 – i[$x^2 – 4x + 5 = 0$]

(d) 3+4i, 3 – 4i [$x^2 – 6x + 25 = 0$]

(k) Solve the following equations if each has at least one real root

(a) $x^3 – 7x^2 + 19x – 13 = 0$ [1, 3 ± 2i]

(b) $2x^3 – 2x^2 – 3x – 2 = 0$ $\left[2, -\frac{1}{2}\pm\frac{1}{2}i\right]$

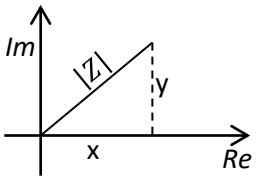
(c) $x^3 + 3x^2 + 5x + 3 = 0$ [-1, $\pm\sqrt{2}i$]

The Argand diagram

Complex numbers can be represented graphically just as coordinates are. However, instead of the x – axis and the y – axes, thereal (*Re*) and the imaginary (*Im*) exes are used respectively instead. This representation was first suggested by a mathematician named R. Argand, hence the Argand diagram. Each complex number is represented by a line of certain length in a particular direction. Thus each complex number is shown as a vector on Argand diagram.

The sum and the difference of the two complex numbers can be shown on an Argand diagram in the same way as we show vectors which are normally added or subtracted.

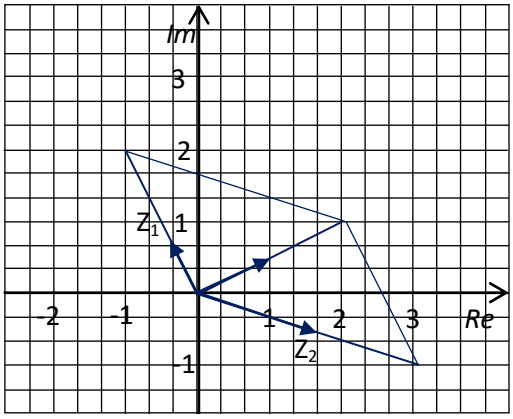
The complex number x = x + iy is represented as shown below



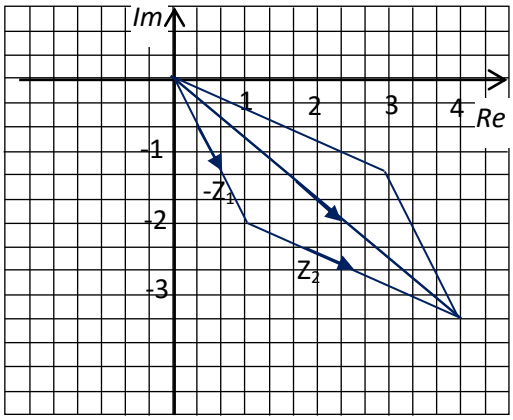
Example 8

Given Z_1 = -1 + 2i and Z_2 = 3 – i show $Z_1 + Z_2$ and $Z_2 – Z_1$ using Argand diagram

(a) $Z_1 + Z_2$ = -1 + 2i + 3 – i = 2 + i



(b) $Z_2 - Z_1$ = (3 – i) – (-1 + 2i) = 4 – 3i



The modulus and argument of a complex number

From the above diagram

$r^2 = |Z|^2 = x^2 + y^2$

i.e. r = $|Z| = \sqrt{x^2 + y^2}$ which is modulus Z = mod(Z) and tanθ = $\frac{x}{y}$, i.e. $\theta = \tan^{-1}\frac{y}{x}$, which is the argument, arg(Z) or amplitude, amp(Z).

Note: for principal values, $-\pi \leq Arg(Z) \leq \pi$

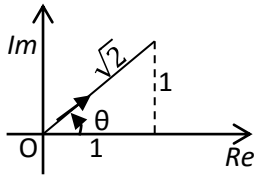
Example 9

Find the modulus and argument of the following

(a) $Z_1 = 1 + i$

Solution

$|Z_1| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

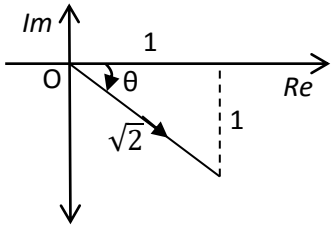


Arg (Z_1) =$\tan^{-1}\left(\frac{1}{1}\right) = 45^0 \; or \; \frac{\pi}{4}$

(b) $Z_2 = 1 - i$

Solution

$|Z_1| = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$

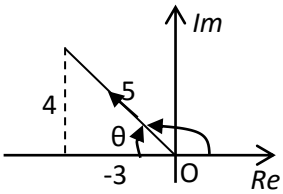


Arg(Z_2) = $\tan^{-1}(-1) = 45^0 \; or \; -\frac{\pi}{4}$

(c) $Z_3 = -3 + 4i$

Solution

$|Z_2| = \sqrt{((-3)^2 + 4^2)} = 5$

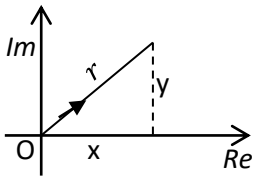


Arg(Z_3) = $\tan^{-1}\left(\frac{4}{3}\right)$ = 180^0 - 53.1^0 = 126.9^0

Note: Due care must be taken in obtaining the argument of complex numbers as above example shows. Although the moduli for Z_1 and Z_2 are the same, the arguments differ because the fall in different quadrants. A sketch may be thus may be a necessary pre-requisite.

The polar (modulus – argument) form of a complex number.

Given Z = x + iy (the Cartesian form)



x = rcosθ and y = rsinθ

∴Z= x + iy = rcosθ + rsinθ

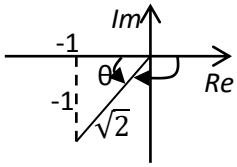
i.e. Z = rcosθ + rsinθ or simply Z = rcis. This form is referred to as polar form of Z.

Example 10

Express $Z_1 = -1 - i$ and $Z_2 = -\sqrt{3} + 3i$

Solution

$(a)|Z_1| = \sqrt{((-1)^2 + (-1)^2)} = \sqrt{2}$

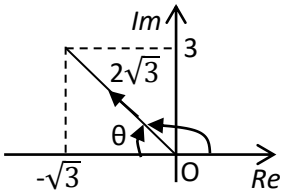


Arg(Z_1) = $\tan^{-1}\left(\frac{-1}{-1}\right) = 225^0 \; = -135^0 or -\frac{\pi}{4}$

$Z_1 = \sqrt{2}(\cos(-135^0) + isin135^0)$

$= \sqrt{2}\left(cos\frac{3\pi}{4} - isin\frac{3\pi}{4}\right)$

$(b)|Z_2| = \sqrt{\left((-\sqrt{3})^2 + 3^2\right)} = 2\sqrt{3}$



Arg(Z_2) = $\tan^{-1}\left(\frac{3}{-\sqrt{3}}\right)$= $120^0 = \frac{2\pi}{3}$

$Z_2 = 2\sqrt{3}(\cos 120^0 + i\sin 120^0)$

$Z_2 = 2\sqrt{3}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

Multiplication and division of polar form

The two operations offer valuable results in complex plane

Let $|Z_1| = r_1$ and arg(Z_1) = θ_1 i.e.

$|Z_1| = r_1(cos\theta_1 + isin\theta_1)$

Let $|Z_2| = r_2$ and arg(Z_2) = θ_2 i.e.

$|Z_2| = r_2(cos\theta_2 + isin\theta_2)$

(a) Multiplication of polar form

$Z_1Z_2 = r_1r_2(cos\theta_1 + isin\theta_1)(cos\theta_2 + isin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2)$

$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\}$

$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deductions:

(i) $|Z_1Z_2| = r_1r_2 = |Z_1||Z_2|$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= \arg(Z_1) + \arg(Z_2)$
(iii) $|Z^2| = |Z||Z| = |Z|^2$
(iv) $arg(Z^2) = \arg(Z) + \arg(Z) = 2\arg(Z)$

In general

(i) $|Z_1Z_2Z_3 \ldots Z_n| = |Z_1||Z_2||Z_2| \ldots\ldots |Z_n|$
(ii) $|Z^n| = |Z||Z||Z| \ldots |Z| = |Z|^n$
(iii) $arg(Z_1Z_2Z_3 \ldots Z_n)$

$= \arg(Z_1) + \arg(Z_2) + \arg(Z_3) \ldots \arg(Z_n)$

(iv) $arg(Z^n) = n\arg(Z)$

Example 11

Given that $Z_1 = 3 + 4i$ and $Z_2 = 1 - i$, find

(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

arg(Z_1) =$\tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

arg(Z_2) =$\tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 5\sqrt{2}$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1$
$= 53.1^0 + (-45^0) = 8.1^0$

Example 12

Given that $Z_1 = 3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
And $Z_2 = 5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ find
(i) $|Z_1Z_2|$
(ii) $Arg(Z_1Z_2)$
Solution
For

$Z_1 = 3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$; $|Z_1| = 3\ and\ \arg(Z_1) = \frac{\pi}{3}$

For

$Z_2 = 5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$; $|Z_2| = 5\ and\ \arg(Z_2) = \frac{\pi}{4}$

(i) $|Z_1Z_2| = |Z_1||Z_2| = 3x5 = 15$
(ii) $arg(Z_1Z_2) = \theta_1 + \theta_1 = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$

(b) Division of polar form

$\frac{Z_1}{Z_2}$

$= \frac{r_1}{r_2}\left\{\frac{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta_2}{(cos^2\theta_2 + sin^2\theta_2)}\right\}$

$= \frac{r_1}{r_2}\{cos\theta_1cos\theta_2 - icos\theta_1sin\theta_2 + isincos\theta_2 - sin\theta_1sin\theta_2\}$

$= \frac{r_1}{r_2}\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Deduction

(a) $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$
(b) $arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$

$= \arg(Z_1) - \arg(Z_2)$

Example 13

Given that $Z_1 = 3 + 4i$ and $Z_2 = 1 - i$, find

(i) $\left|\frac{Z_1}{Z_2}\right|$

(ii) $Arg\left(\frac{Z_1}{Z_2}\right)$

Solution

$|Z_1| = \sqrt{3^2 + 4^2} = 5$

$\arg(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{1}{1}\right) = -45^0$

(i) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

(ii) $arg\left(\frac{Z_1}{z_2}\right) = \theta_1 - \theta_1$

$= 53.1^0 - (-45^0) = 98.1^0$

Example 14

Three complex numbers are given as

$Z_1 = -1 - i$; $Z_2 = 3 - i\sqrt{3}$; $Z_3 = -1 + i\sqrt{3}$

Find

(a) $\left|\frac{Z_1}{Z_2}\right|$

(b) $arg\left(\frac{Z_2}{z_1 Z_3}\right)$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right|$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

Solution

$|Z_1| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\arg(Z_1) = \tan^{-1}\left(\frac{-1}{-1}\right) = -135^0 = -\frac{3\pi}{4}$

$|Z_2| = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$

$\arg(Z_2) = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -30^0 = -\frac{\pi}{6}$

$|Z_3| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\arg(Z_1) = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = 120^0 = -\frac{2\pi}{3}$

(a) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{\sqrt{2}}{2}$

(b) $arg\left(\frac{Z_2}{z_1 Z_3}\right) = arg Z_2 - arg(z_1 Z_3)$

$= arg Z_2 - arg Z_1 - arg Z_3$

$= -30^0 + 135^0 - 120^0 = -15 = -\frac{\pi}{12}$

(c) $\left|\frac{{Z_1}^2}{{Z_2}^3 Z_3}\right| = \frac{|Z_1|^2}{|Z_2|^3 |Z_3|} = \frac{2}{24\sqrt{3}(2)} = \frac{\sqrt{3}}{72}$

(d) $arg\left(\frac{{Z_3}^3 {Z_2}^2}{Z_1}\right)$

$= arg\left({Z_3}^2\right) + \arg\left({Z_2}^2\right) - \arg(Z_1)$

$= 3\arg(Z_3) + 2\arg(Z_2) - \arg(Z_1)$

$= 3(120^0) - 2(30^0) + 135^0 = 75^0 = \frac{5\pi}{12}$

Note: the reader should correlate the computations of arguments Z_1Z_2 and $\frac{Z_1}{Z_2}$ with the laws of logarithms

Example 15

(a) Given that $Z_1 = 3 + 4i$, $Z_2 = 12 + 5i$ find

(i) $|Z_1 Z_2|$

(ii) $\arg(Z_1 Z_2)$

(iii) $\left|\frac{Z_1}{Z_2}\right|$

(iv) $\arg(Z_1 Z_2)$

Solution

$|Z_1| = \sqrt{(3)^2 + (4)^2} = 5$

$\text{Arg}(Z_1) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^0$

$|Z_3| = \sqrt{(12)^2 + (3)^2} = 13$

$\text{Arg}(Z_1) = \tan^{-1}\left(\frac{5}{12}\right) = 22.6^0$

(i) $|Z_1 Z_2| = |Z_1||Z_2| = 5\ x\ 13 = .65$

(ii) $\arg(Z_1 Z_2) = \arg(Z_1) + \arg(Z_2)$

$= 53.1^0 + 22.6^0 = 75.7^0$

(iii) $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{5}{13}$

(iv) $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2)$

$= 53.1^0 - 22.6^0 = 30.5^0$

(b) Express Z = -1 -i$\sqrt{3}$ in modulus-urgument form. Hence find $\frac{1}{Z}$ in form a + bi where a and b are real numbers

Solution

$$|Z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$$

Arg(Z) =$\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = -120^0 = -\frac{2}{3}\pi$

$\therefore Z = 2\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)$

Hence, $\frac{1}{Z} = Z^{-1} = 2^{-1}\left(cos\frac{2\pi}{3} - isin\frac{2\pi}{3}\right)^{-1}$

$$.= \frac{1}{2}\left(cos\left(-\frac{2\pi}{3}\right) - isin\left(-\frac{2\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(cos\left(\frac{2\pi}{3}\right) + isin\left(\frac{2\pi}{3}\right)\right)$$

$$= -\frac{1}{4} + i\frac{\sqrt{3}}{4}$$

Solving equations of higher order

When one or more roots of a complex number is(are) given, their conjugates are also the roots of the same equation

To prove that a certain root is a factor of a complex function, we either use long division to show that the quotient of division is zero or we substitute the factor given into the equation to note the remainder or we get the product of the already existing roots and then carry on long division to observe the remainder or we use the synthetic approach

Example 16

Given that (1+ 3i) and (2 – i) are roots of the equation

$aZ^4 + bZ^3 + cZ^2 + dZ + e = 0$, find

(i) The other two roots
(ii) The sum of the four roots
(iii) The product of the four roots

Solution

(i) Since 1 + 3i and 2 – I are roots of the equation, then their conjugates are also roots of the same equations. Hence the other two roots are 1 – 3i and 2 + i.
(ii) Sum of roots = (1 + 3i) + (1 – 3i) + (2 – i) + (2 + i) = 6
(iii) Products of roots
= (1 + 3i)(1 – 3i)(2 – i (2 + i) = 6
= $(1 - 9i^2)(4 - i^2)$
=(1 + 9)(4 + 1)= 50

Example 17

Show that Z_1 = 2 + 3i is a root of the equation $Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Hence find the other remaining roots

Solution

Approach 1

Given 2 + 3i is a root, its conjugate 2 - 3i is also a root of the equation

The equation of these two roots

Sum of roots 2 + 3i + 2 – 3i = 4

Product of roots = (2 + 3i)(2 – 3i) = 13

The equation is $Z^2 - 4Z + 13 = 0$

Using long division

Let $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13$

$$\begin{array}{r} Z^2 - Z + 1 \\ Z^2 - 4Z + 13 \overline{)\ Z^4 - 5Z^3 + 18Z^2 - 17Z + 13} \\ \underline{Z^4 - 4Z^3 + 13Z^2} \\ -Z^3 + 5Z^2 - 17Z + 13 \\ \underline{-Z^3 + 4Z^2 - 13Z} \\ Z^2 - 4Z + 13 \\ \underline{Z^2 - 4Z + 13} \\ 0 + 0 + 0 \end{array}$$

Since the remainder is zero, $Z^2 - Z + 1$is also a factor of f(Z)

⇨ $f(Z) = (Z^2 - 4Z + 13)(Z^2 - Z + 1)$

Either $Z^2 - 4Z + 13 = 0$ or $Z^2 - Z + 1 = 0$

Factors of $Z^2 - Z + 1$

$Z = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$

The roots are 2 ± 3i and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 2

Taking Z = 2 + 3i and substituting in the given equation

LHS = $(2 + 3i)^4 - 5(2 + 3i)^3 + 18(2 + 3i)^2 - 17(2 + 3i) + 13 = 0$ (RHS)

Z1 = 2 + 3i is a root

This means $Z_2 = 2 - 3i$ is also a root of the equation as it is a conjugate of the given root.

Given $f(Z) = Z^4 - 5Z^3 + 18Z^2 - 17Z + 13 = 0$

Sum of roots = 5 and

Product of roots = 13

Let the other roots be α and β

Sum of the roots = 2 + 3i + 2 – 3i + α + β = 5

4 + α + β = 5

α + β = 1

α = (1 – β) …….. (i)

Product of roots = (2 + 3i)(2 – 3i)αβ = 13

13αβ = 13

αβ = 1 …….. (ii)

Substituting (i) into (ii)

(1 – β)β = 1 i.e. β2 – β + 1 = 0

$\beta = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm i\sqrt{3}}{2}$

∴ The roots are 2 ± 3i and $\frac{1 \pm i\sqrt{3}}{2}$

Approach 3: using synthetic method

2 + 3i	1	-5	18	-17	13
	0	2+3i	-15-3i	15+3i	-13
	1	-3 + 3i	3 -3i	-2+3i	0

Since the last value in the table is zero, therefore 2 + 3i is a root of the equation. Other roots can be obtained as shown above.

Illustration of the synthetic method

Procedure

(a) Write down the root and coefficients of the expression in the first row (as shown above)
(b) Write zero immediately below the first coefficient (this is the only entry in the second row that is simply written, the other are to be obtained by multiplication)
(c) At the first coefficient to zero to get the first entry in the third row
(d) Obtain the second entry in the second row by multiplying 1 by 2 + 3i, then add to -5 to get the second entry in the third row i.e. -3 + 3i
(e) Repeat (d) by multiplying -3 + 3i to get the third entry in the second row, this entry is added to 18 to get to get the third row i.e. 3 – 3i. continue with this trend up to the last entry. If the last entry in the table is zero, then the value being tested is a root.

Example 18

Show that Z1 = 1 + I is a root of the equation $Z^4 - 6Z^3 + 25Z^2 - 34Z + 26 = 0$

Solution

Using synthetic division

1 + i	1	-6	25	-34	26
	0	2+i	-6-4i	21+13i	-26
	1	-5 + i	17 - 4i	-3+ 13i	0

Since the last entry in the table is zero, 1 + I is a root.

The conjugate 1 – i is also a root

Let the other roots be α and β

Sum of roots = 1 + i + 1 – i + α + β = 6

2 + α + β = 6

α + β = 4

α = 4 – β …… (i)

Product of roots (1 + i)(1 – i)αβ = 26

2αβ = 26

Aβ = 13 …… (ii)

Substituting eqn.(i) into eqn. (ii)

(4 – β)β = 13

β^2 - 4β + 13 = 0

$\beta = \frac{4\pm\sqrt{16-4(1)(13)}}{2(1)} = \frac{4\pm\sqrt{-36}}{2} = \frac{4\pm6i}{2}$

= 2 ± 3i

Hence the other roots are 2 + 3i and 2 – 3i

Revision exercise 2

1. Given Z_1 = 7 – 4i and Z_2 = -1 + 3i, express in the modulus-argument form the following
   (a) Z_1Z_2
   {r = 25.45 [cos(0.437π) + isin(0.437π)]}
   (b) $\left|\frac{Z_1}{Z_2}\right|$

   $\{p = 2.55[cos(1.233\pi) + isin(1.233\pi)]$
2. Find the modulus and the principal arguments of
   (a) $\frac{1-i}{1+i}\left[1, \frac{-\pi}{3}\right]$
   (b) $\frac{-1-7i}{4+3i}\left[\sqrt{2}, \frac{-3\pi}{4}\right]$
   (c) $\frac{1+i}{2-i}\left[\frac{\sqrt{10}}{5}, 125\right]$
   (d) $\frac{(3+i)^2}{1-i}\left[2\sqrt{5}, 143\right]$
3. If z_1 and z_2 are complex numbers, solve the simultaneous equation, giving your answer in the form a + bi.
   $4z_1 + 3z_2$= 23
   $z_1 + iz_2$ = 6 [2 +3i, 5 – 4i]
4. Given that 2 + I is a root of the equation z^3 – 11z + 20 = 0. Find the remaining roots [2 – i, -4]
5. Show that 2+ i is a root of the equation $2z^3 – 9z^2 + 14z – 5 = 0$. Hence find other roots [2- i, ½]
6. Find equation whose roots are -1 ± i, where i = $\sqrt{-1}$ [z^2 +2z + 2 = 0]
7. Show that z = 1 is a root of the equation z^3 -5z^2 + 9z – 5 = 0 [2 – i, 2 + i]
8. The complex number satisfies $\frac{z}{z+2} = 2 - i$ find the real and imaginary parts of z, and the modulus and argument of z.
   [*Re*(z) = -3,*Im*(z) = -1, $\sqrt{10}$, -2.82rads]
9. One root of the equation z2 + az + b = 0 where a and b are real constants is 2 + 3i. Find the values of a and b [a = -4, b = 13]
10. If $z_1 = 2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$ and $z_2 = 6\left(cos\left(\frac{-3\pi}{4}\right) + isin\left(\frac{-3\pi}{4}\right)\right)$ find
    (a) $\left|\frac{z_1}{z_2}\right|\left[\frac{1}{3}\right]$
    (b) $arg\left(\frac{z_1}{z_2}\right)\left[\frac{17\pi}{12}\right]$
    (c) $\left|\frac{z_2}{z_1}\right|$[3]
    (d) $arg\left(\frac{z_2}{z_1}\right)\left[\frac{-17\pi}{12}\right]$
11. If z1 and z2 are two complex numbers such that $|z_1 - z_2| = |z_1 + z_2|$. Show that the difference of their arguments is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
12. Find the modulus and argument of $\frac{(2-i)^2(3i-1}{i+3}$ [5, 0.6435 rad]
13. If $z_1 = \frac{1+7i}{1-i}$ and $z_2 = \frac{17-7i}{2+2i}$ find the moduli of
    (a) z_1 [5]
    (b) z_2 [6.5]
    (c) $z_1 + z_2$ [2.061]
    (d) z_1z_2 [32.5]

Demoivre's theorem

This states that for all rational integers

r(cosθ + isinθ)^n = r(cos*n*θ + isin*n*θ)

The theorem can be proved by induction approach or otherwise

By induction

Given that (cosθ+ isinθ)^n = cos*n*θ + isin*n*θ

Suppose n = 1

cosθ+ isinθ = cos*1*θ + isin*1*θ

⇨ cosθ+ isinθ = cosθ+ isinθ i.e. the proof holds for n = 1

Suppose n = k

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

For n = k + 1

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

which is true when n = k + 1

Therefore the proof holds for all values of n

Using otherwise

Given $z = \cos\theta + i\sin\theta$

$Z^2 = (\cos\theta + i\sin\theta)^2$

$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$= \cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta$

$= \cos2\theta + i\sin2\theta$

$Z^3 = (\cos2\theta + i\sin2\theta)^2(\cos\theta + i\sin\theta)$

$= \cos2\theta\cos\theta - \sin2\theta\sin\theta + i(\sin2\theta\cos\theta + \cos2\theta\sin\theta)$

$= \cos3\theta + i\sin3\theta$

⇨ $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ hold for all positive values of n

Also, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ holds for fractional and negative indices

(a) $z^{-1} = \frac{1}{Z} = \frac{1}{\cos\theta + i\sin\theta}$

$= \left(\frac{1}{\cos\theta + i\sin\theta}\right)\left(\frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}\right)$

$= \frac{(\cos\theta - i\sin\theta)}{(\cos^2\theta - i^1\sin^2\theta)} = \cos\theta - i\sin\theta$

⇨ $z^{-1} = \cos(-1\theta) + i\sin(-1\theta)$

(b) suppose n = -m

$z^{-m} = \frac{1}{Z^m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$

$= \left(\frac{1}{\cos m\theta + i\sin m\theta}\right)\left(\frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta - i\sin m\theta)}\right)$

⇨ $(\cos\theta + i\sin\theta)^{-m}$

$= \cos(-m\theta) + i\sin(-m\theta)$

Hence the theorem holds for all values of n.

Applications of Demoivre's theorem

It is mainly employed in proving trigonometric functions and finding roots of complex function.

Proving identities

Example 19

By using Demoivre's theorem, show that

(a) $\tan5x = \frac{5tanx - 10tan^3x + tan^5x}{1 - 10tan^2x + tan4x}$

Solution

$\cos5x + i\sin5x = (\cos x + i\sin x)5$

$= \cos^5x + 5i\sin x\cos^4x - 10\sin^2x\cos^3x - 10i\sin^3x\cos^2x + \sin^4x\cos x + i\sin^5x$

Equating real parts

$\cos5x = \cos^5x - 10\cos^3x\sin^2x + \cos x\sin^4x$

Equating imaginary parts

$\sin5x = 5\cos^4x\sin x - 10\cos^2x\sin^3x + \sin^5x$

$\tan5x = \frac{\sin5x}{\cos5x} = \frac{5\cos^4x\sin x - 10\cos^2x\sin^2x + \sin^5x}{\cos^5x - 10\cos^3x\sin^2x + \cos x\sin^4x}$

Dividing terms of the R.H.S by $\cos^5x$

$\tan5x = \frac{5tanx - 10tan^3x + tan^5x}{1 - 10tan^2x + tan4x}$

(b) $\frac{\cos3x + i\sin3x}{\cos5x + i\sin5x} = \cos8x + i\sin8x$

Solution

$\frac{\cos3x + i\sin3x}{\cos5x + i\sin5x} = \frac{(\cos x + i\sin x)^3}{(\cos x + i\sin x)^5}$

$= (\cos x + i\sin x)^{3-(-5)}$

$= (\cos x + i\sin x)^8$

$= \cos8x + i\sin8x$

(c) $\frac{(\cos x + i\sin x)(\cos2x + i\sin2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}} = \cos\frac{5x}{2} + i\sin\frac{5x}{2}$

Solution

$\frac{(\cos x + i\sin x)(\cos2x + i\sin2x)}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$

$= \frac{(\cos x + i\sin x)^1(\cos x + i\sin x)^2}{(\cos x + i\sin x)^{\frac{1}{2}}}$

$=(cosx + isinx)^{1+2-\frac{1}{2}}$

$=(cosx + isinx)^{\frac{5}{2}} = cos\frac{5x}{2} + isin\frac{5x}{2}$

(d) $16\sin^5x = \sin5x - 5\sin3x + 10\sin x$

Given Z = cosx + isinx

$\bar{Z}$ = cosx – isinx

(Z – $\bar{Z}$) = 2isinx

This means that $(Z - \bar{Z})^n = 2i\sin nx$

$(Z - Z^*)^5 = 2i\sin5x = 32i\sin^5x$

$(Z - \bar{Z})^5$

$= Z^5 - 5Z^4\bar{Z} + 10Z^3\bar{Z}^2 + 10Z^2(-\bar{Z})^3 + 5Z\bar{Z}^4 - \bar{Z}^5$

$= Z^5 - 5Z^3 + 10Z - 10(\bar{Z}) + 5\bar{Z}^3 - \bar{Z}^5$

$=[Z^5 - \bar{Z}^5] - 5[Z^3 - \bar{Z}^3] + 10[Z - \bar{Z}]$

$= 2i\sin5x - 10i\sin3x + 20i\sin x$

⇨ $32i\sin^5x = 2i\sin5x - 10i\sin3x + 20i\sin x$

$\therefore 16\sin^5x = \sin5x - 5\sin3x + 10\sin x$

Finding roots

Suppose that z = r(cosθ + isinθ)

Then $z^n = r^n(\cos n\theta + i\sin n\theta)$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}\theta + isin\frac{1}{n}\theta\right)$$

The general expression for finding the roots of a complex function is given by

$z = r[cos(\theta + 2\pi k) + isin(\theta + 2\pi k)]$ where k = 0, 1, 2, …k and θ is usually in radians.

Now for n^{th} root, we have

$z^n = r^n[cosn(\theta + 2\pi k) + isin\, n(\theta + 2\pi k)]$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(cos\frac{1}{n}(\theta + 2\pi k) + isin\frac{1}{n}(\theta + 2\pi k)\right)$$

Example 20

Use Demoivre's theorem to find the square root of

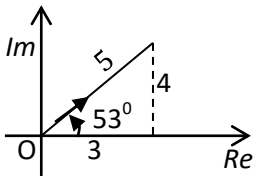
(a) 3 + 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{4}{3} = 53.1^0 = 0.295\pi$

The principal angle lies in the first quadrant



$z = 5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}}$

$= 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi k}{2}\right) + isin\left(\frac{0.295\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

$\therefore \sqrt{3 + 4i} = 2 \pm i$

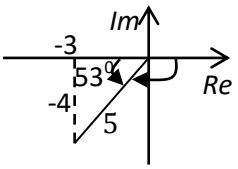
(b) -3 – 4i

Solution

$|z| = \sqrt{(9 + 16} = 5$

$\theta = \tan^{-1}\frac{-4}{-3} = -127^0 = -0.7\pi$

The principal angle lies in the third quadrant



$z = 5[cos(0.295\pi + 2\pi k) + isin(0.295\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi k}{2}\right) + isin\left(\frac{0.295\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1^{\frac{1}{2}} = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi}{2}\right) + isin\left(\frac{0.295\pi}{2}\right)\right]$= -2 - i

Substituting for k = 1

$z_2 = 5^{\frac{1}{2}}\left[cos\left(\frac{0.295\pi+2\pi}{2}\right) + isin\left(\frac{0.295\pi+2\pi}{2}\right)\right]$

= ±(1-2i)

$\therefore \sqrt{-3 - 4i} = \pm(1 - 2i)$

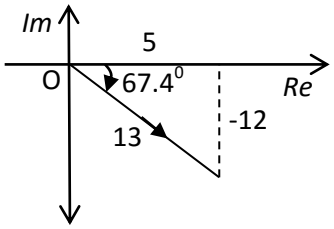
(c) 5 – 12i

Solution

$|z| = \sqrt{(25 + 144} = 13$

$\theta = \tan^{-1}\frac{-12}{5} = -67.4^0 = -0.374\pi$

The principal angle lies in the third quadrant



z = 13$[cos(-0.374\pi + 2\pi k) + isin(-0.374\pi + 2\pi k)]$

$z^{\frac{1}{2}} = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi k}{2}\right) + isin\left(\frac{-0.374\pi+2\pi k}{2}\right)\right]$

Substituting for k = 0

$z_1 = 13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi}{2}\right) + isin\left(\frac{-0.374\pi}{2}\right)\right]$

= 3 - 2i

Substituting for k = 1

$z_2 =$
$13^{\frac{1}{2}}\left[cos\left(\frac{-0.374\pi+2\pi}{2}\right) + isin\left(\frac{-0.374\pi+2\pi}{2}\right)\right]$

= -3 + 2i

$\therefore \sqrt{5 - 12i} = \pm$(3 – 2i)

Example 21

(a) Use roots 1, α and α^2 of unity. Hence show that $1 + \alpha + \alpha^2 = 0$

Solution

Let $\sqrt[1]{1} = z$

z = $1^{\frac{1}{3}} = (cos0^0 + isin0^0)$ (In mod-Arg form

=$[cos(0 + 2\pi k) + isin\,(0 + 2\pi k)]^{\frac{1}{3}}$, k = 0,1,2

= $cos(\frac{2\pi k}{3}) + isin\,(\frac{2\pi k}{3})$

When k = 0

$z_1 = cos0^0 + isin\,0^0 = 1$

When k = 1

$z_2 = cos(\frac{2\pi}{3}) + isin\,(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

When k = 2

$z_3 = cos(\frac{4\pi}{3}) + isin\,(\frac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Using otherwise

Taking

z = $1^{\frac{1}{3}}$, then $z^3 - 1 = 0$

Let $f(z) = z^3 - 1$

$f(1) = 1^3 - 1 = 0$

∴ z – 1 is a factor.

Using long division

$$z-1\overline{)\,z^3 - 1}\quad \Big|\quad z^2 + z + 1$$

$z^3 - z^2$

$z^2 - 1$

$z^2 - 1$

$z^3 - 1 = (z - 1)(z^2 + z + 1)$

Either z – 1 = 0, z = 1

Or $z^2 + z + 1 = 0$

$z = \frac{-1\pm\sqrt{1^2-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

∴the cube roots are 1, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Hence

Letting $\alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the

$\alpha^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Letting $\alpha = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, the

$\alpha^2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Hence the roots can be written as $1, \alpha + \alpha^2$

Adding these roots

$1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2} = 0$

(b) Evaluate $(-16)^{\frac{1}{4}}$

Solution

Let z = $(-16)^{\frac{1}{4}} = 16^{\frac{1}{4}}(-1)^{\frac{1}{4}} = 2(-1)^{\frac{1}{4}}$

$= [cos180^0 + isin180^0]^{\frac{1}{4}}$

$=2\left[cos\left(\frac{\pi+2\pi k}{4}\right) + isin\left(\frac{\pi+2\pi k}{4}\right)\right]$, k = 0, 1, 2, 3

When k = 0

$z_1 = [cos45^0 + isin45^0] = \sqrt{2} + i\sqrt{2}$

When k = 1

$Z_2 = [cos135^0 + isin135^0] = -\sqrt{2} + i\sqrt{2}$

When k = 2

$Z_3 = [cos225^0 + isin225^0] = \sqrt{2} - i\sqrt{2}$

When k = 3

$Z_3 = [cos315^0 + isin315^0] = -\sqrt{2} - i\sqrt{2}$

$\therefore (-16)^{\frac{1}{4}} = \pm(\sqrt{2} \pm i\sqrt{2})$

Example 23

Evaluate $\left(1 + \sqrt{3}i\right)^{\frac{2}{3}}$

Solution

$$\left(1 + \sqrt{3}i\right)^{\frac{2}{3}} = \left[\left(1 + \sqrt{3}i\right)^2\right]^{\frac{1}{3}}$$

$$= \left[\left(1 + \sqrt{3}i\right)\left(1 + \sqrt{3}i\right)\right]^{\frac{1}{3}}$$

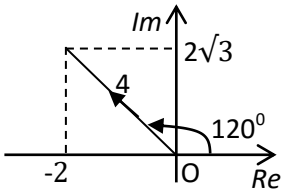
$$= \left[-2 + 2\sqrt{3}i\right]^{\frac{1}{3}}$$

Let z = -2 + 2√3i

$$|z| = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2} = 4$$

Arg(z) = $\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -60^0 = 120^0 = \frac{2\pi}{3}$

The principal angle lies in the second quadrant



$z = 4\left[cos\left(\frac{2\pi}{3} + 2\pi k\right) + isin\left(\frac{2\pi}{3} + 2\pi k\right)\right]$

$z^{\frac{1}{3}} = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi+6\pi k}{9}\right) + isin\left(\frac{2\pi+6\pi k}{9}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{3}}\left[cos\left(\frac{2\pi}{9}\right) + isin\left(\frac{2\pi}{9}\right)\right]$

=1.2160 +1.0204i

When k = 1

$z_2 = 4^{\frac{1}{3}}\left[cos\left(\frac{8\pi}{9}\right) + isin\left(\frac{8\pi}{9}\right)\right]$

= -1.4917 + 0.5429i

When k = 2

$z_3 = 4^{\frac{1}{3}}\left[cos\left(\frac{14\pi}{9}\right) + isin\left(\frac{14\pi}{9}\right)\right]$

= 0.2756 – 1.5633i

$\therefore \left(1 + \sqrt{3}i\right)^{\frac{2}{3}}$ =1.2160 +1.0204i,
-1.4917 + 0.5429i, 0.2756 – 1.5633i

Example 24

Find the root of $z^4 + 4 = 0$ using Demoivre's theorem

$z^4 = -4 = 4(-1 + 0i)$

arg(z) =$\tan^{-1}\left(\frac{0}{-1}\right) = 180^0 = \pi$

⇨ $z^4 = 4[cos180^0 + sin180^0]$

$z = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi+2\pi k}{4}\right) + sin\left(\frac{\pi+2\pi k}{4}\right)\right]$

When k = 0

$z_1 = 4^{\frac{1}{4}}\left[cos\left(\frac{\pi}{4}\right) + sin\left(\frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

When k = 1

$z_2 = 4^{\frac{1}{4}}\left[cos\left(\frac{3\pi}{4}\right) + sin\left(\frac{3\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$

When k = 2

$z_5 = 4^{\frac{1}{4}}\left[cos\left(\frac{5\pi}{4}\right) + sin\left(\frac{5\pi}{4}\right)\right]$

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$

When k = 3

$z_4 = 4^{\frac{1}{4}}\left[cos\left(\frac{7\pi}{4}\right) + sin\left(\frac{7\pi}{4}\right)\right]$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$

∴ the roots of $z^4 + 4 = 0$ are $1 \pm i$ and $-1 \pm i$

The locus of a complex number

This is an equation of a set of point representing a variable complex number or a path traced. It may be in form of an equation of a circle or a straight line.

Example 25

Determine the locus of z if

(a) $|z - 4| = 3$
(b) Arg(z+2) = $\frac{\pi}{6}$

Solution

Let z = x + iy

(a) $|x + iy - 4| = 3$
$|(x - 4) + iy| = 3$
$(x - 4)^2 + y^2 = 9$
$x^2 + y^2 - 8x + 7 = 0$
The locus of z is a circle with centre (4, 0)

(b) Arg(x + iy +2) = $\frac{\pi}{6}$
Arg((x+2) + iy) = $\frac{\pi}{6}$
$\tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{6}$
$\frac{y}{x+2} = tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
$\frac{y}{x+2} = \frac{\sqrt{3}}{3}$
$y = \frac{\sqrt{3}}{3}(x + 2)$

The locus of z is a straight line

Example 26

Determine the locus of z if:

(a) $|z| = 2$ and sketch it
(b) $\arg(z - 1) = \frac{\pi}{4}$ and sketch it
(c) $2|z - 1| = |z + i|$
(d) $\arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{3}$

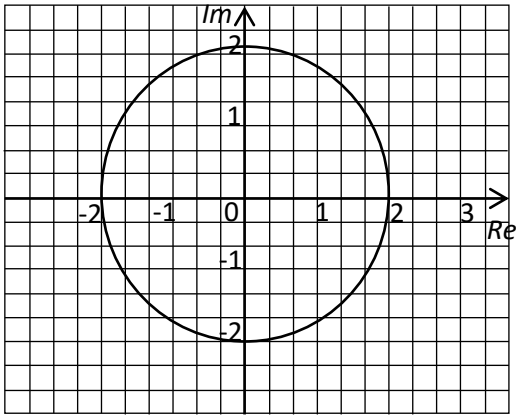
Solution

Let z = x + iy

(a) $|x + iy| = 2$ i.e. $\sqrt{x^2 + y^2} = 2$

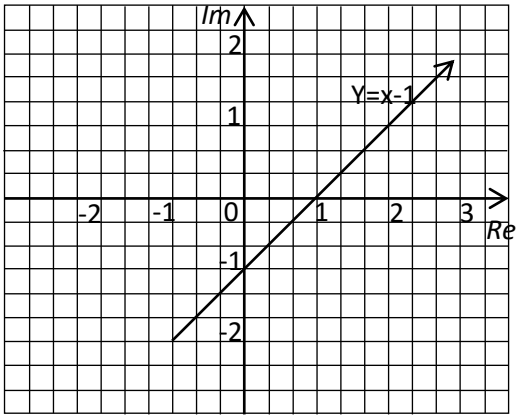
$x^2 + y^2 = 4\ or\ (x - 0)^2 + (y - 0)^2 = 2^2$

The locus of z is a circle with centre (0, 0) and radius 2units



(b) Arg[(x – 1) + iy] = $\frac{\pi}{4}$
$\tan^{-1}\left(\frac{y}{x-1}\right) = \frac{\pi}{4}$
$\frac{y}{x-1} = tan\frac{\pi}{4} = 1$
$y = x - 1$

The locus is a straight line



(c) $2|(x - 1) + iy| = |x + i(y + 1)|$

$\left[2\sqrt{(x - 1)^2 + y^2}\right]^2 = \left[\sqrt{x^2 + (y + 1)^2}\right]^2$

$3x^2 + 3y^2 - 8x - 2y + 3 = 0$

$x^2 + y^2 - \frac{8}{3}x - \frac{2}{3}y + 1 = 0$

$= \left(x-\frac{4}{3}\right)^2 + \left(y-\frac{1}{3}\right)^2 = -1+\frac{16}{9}+\frac{1}{9}$

$= \left(x-\frac{4}{3}\right)^2 + \left(y-\frac{1}{3}\right)^2 = \frac{8}{9}$

The locus of Z is a circle with the centre $\left(\frac{4}{3},\frac{1}{3}\right)$ and radius $\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

(d) $\arg\left(\frac{x+iy-i}{x+iy+1}\right) = \frac{\pi}{3}$

$\arg\left(\frac{x+i(y-1)}{(x+1)+iy}\right) = \frac{\pi}{3}$

$\arg[x+i(y-1)] - \arg[x+1)+iy] = \frac{\pi}{3}$

$\tan^{-1}\left(\frac{y-1}{x}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{3}$

$\frac{\left(\frac{y-1}{x}\right)-\left(\frac{y}{x+1}\right)}{1+\left(\frac{y-1}{x}\right)\left(\frac{y}{x+1}\right)} = \sqrt{3}$

$\frac{y-x-1}{x^2+y^2+x-y} = \sqrt{3}$

$\sqrt{3}x^2+\sqrt{3}y^2+(\sqrt{3}+1)x-(\sqrt{3}+1)y+\frac{\sqrt{3}}{3}=0$

$x^2+y^2+\left(\frac{3+\sqrt{3}}{3}\right)x-\left(\frac{3+\sqrt{3}}{3}\right)y+\frac{\sqrt{3}}{3}=0$

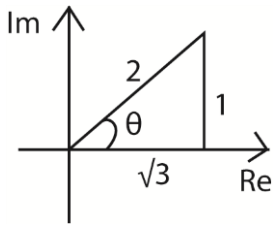
The locus is a circle with centre at $\left[-\frac{1}{6}(3+\sqrt{3})-\frac{1}{6}(3+\sqrt{3})\right]$ and radius $\sqrt{\frac{2}{3}}$

Example 27

The complex number z = $\sqrt{3}+i$.
$\bar{Z}$ *is the conjugate of* Z.

(a) Express Z in the modulus argument form

$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = 2$



$\tan\theta = \frac{1}{\sqrt{3}}$

θ = Arg (z) = $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Hence z = $2\left[\cos\left(\frac{\pi}{6}\right)+isin\left(\frac{\pi}{6}\right)\right]$

or z = $2[\cos 30^0 + isin30^0]$

(i) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z}+3i$

$\bar{z} = \sqrt{3}-i$

$|z| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = 2$

$2\bar{Z}+3i = 2(\sqrt{3}-i)+3i = 2\sqrt{3}+i$

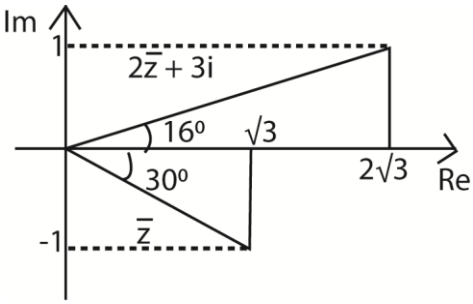
$|2\bar{Z}+3i| = \sqrt{\left(2\sqrt{3}\right)^2 + 1^2} = \sqrt{13}$

Finding arg($\bar{z}$):

Arg($\bar{z}$) = $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30^0$

Finding arg($2\bar{Z}+3i$):

Arg($2\bar{Z}+3i$) = $\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.2^0$



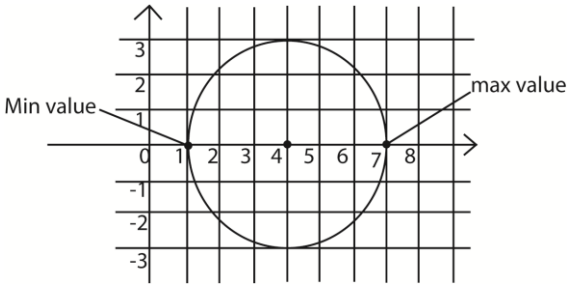
(b) What are the greatest and least values of $|Z|$ *if* $|Z-4| \le 3$?

$|z-2| \le 3$

$|x-4+iy| \le 3$

$(x-4)^2+y^2 \le 3$

This is an equation of the circle with centre (4, 0) and radius 3.



Greatest value of $|z| = 4 + 3 = 7$

Lowest value of $|z| = 4 - 3 = 1$

Example 28

(a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation

$Z\bar{Z} + 2iZ = 12 + 6i$. Find Z.

Let z = x + iy then $\bar{z} = x - iy$

(x + iy)(x – iy) +2i(x + iy) = 12 + 6i

$x^2 + y^2 - 2y + 2xi = 12 + 6i$

Equating imaginary parts

2xi = 6i

x = 3

Equating real parts

$x^2 + y^2 - 2y = 12$

By substituting for x

$9 + y^2 - 2y = 12$

$y^2 - 2y - 3 = 0$

(y – 3)(y+1) = 0

y = 3 or y = -1

when y = 3; z = 3 + 3i

when y = -1; z = 3 - i

∴ the possible values of z are 3 +3i and 3- i.

(b) One root of the equation

$Z^3 - 3Z^2 + 9Z + 13 = 0\ is\ 2 + 3i.$

Determine other roots

Solution

Given root 2 + 3i; its conjugate is also a root of the equation

The equation of these two is

z^2 – (sum of roots)z + product of roots = 0

Sum = 2 + 3i + 2 – 3i = 4

Product of roots =(2 + 3i)(2 – 3i) = 2 + 9 = 13

$$z^2 - 4z + 13 = 0$$

$$\begin{array}{r} z + 1 \\ z^2 - 4z + 13 \overline{)\ z^3 - 3z^2 - 9z + 13} \\ \underline{-\ z^3 - 4z^2 - 13z} \\ z^2 - 4z + 13 \\ \underline{-z^2 - 4z + 13} \end{array}$$

$$(z^2 - 4z + 13)(z + 1) = 0$$

z + 1= 0

z = -1

So the roots are 2±3i and -1

Example 29

(a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$

Solution

$z_1 - z_2 = \frac{2i}{1+3i} - \frac{3+2i}{5}$,

$= \frac{10i-(1+3i)(3+2i)}{5(1+3i)} = \frac{10i-[3+2i+9i-6]}{5(1+3i)}$

$= \frac{10i-11i+3}{5(1+3i)} = \frac{3-i}{5(1+3i)}$

$= \frac{(3-i)(3-i)}{5(1+3i)(3-i)} = \frac{3-9i-i-3}{5(1+9)} = \frac{-i}{5}$

$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$

Alterative 2

$z_1 = \frac{2i}{1+3i} = \frac{2i\,(1-3i)}{(1+3i)(1-3i)} = \frac{2i+6}{1+9}$

$= \frac{2i+6}{10} = \frac{3+2i}{5}$

$z_1 - z_2 = \frac{3+2i}{5} - \frac{3+2i}{5} = \frac{-i}{5}$

$|z_1 - z_2| = \sqrt{0^2 - \left(-\frac{1}{5}\right)^2} = \frac{1}{5}$

(b) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$

$\frac{z+i}{z+2} = \frac{x+i(1+y)}{(x+2)+iy} = \frac{[x+i(1+y)][(x+2)-iy]}{[(x+2)+iy][(x+2)-iy]}$

$= \frac{x[(x+2)-iy]-i(1+y)[(x+2)-iy]}{(x+2)^2+y^2}$

$= \frac{x^2+2x-ixy+i(x+2+xy+2y)+y+y^2}{(x+2)^2+y^2}$

$= \frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line.

If imaginary part is zero

$(2 + x + 2y) = 0$

$2y = -x - 2$

y = $-\frac{1}{2}x + 1$

Comparing with y = mx + c

The gradient = $-\frac{1}{2}$

Example 30

(a) Given that a complex number Z* and its conjugate satisfy the equation

ZZ* + 2iZ = 12 + 6i, find the possible values of Z.

Solution

Let z* = x + iy, then z = x –iy

(x + iy)(x – iy) + 2i(x – iy) + 6i = 12 + 6i

$x^2 + y^2 + 2y + 2xi = 12 + 6i$

Equating real parts

$x^2 + y^2 + 2y = 12$

Equating imaginary parts

2x = 6 => x = 3

$9 + y^2 + 2y = 12$

$y^2 + 2y - 3 = 0$

(y – 1)(y + 3) = 0 => y = 1 or y = -3

When y = -3; z = 3 + 3i and

When y = 1, z = 3 –i

The possible values of z are 3 + 3i and 3 - i

(b) Find the Cartesian equation, in its simplest form of the curve described by $|z - 3 + 6i| = 2|z|$ where z is the complex number x + iy.

Hence sketch an Argand diagram the region satisfying $|z - 3 + 6i| \leq 2|z|$

Solution

Substituting z = x = iy

$|(x - 3) + i(y - 6)| = 2|x + iy|$

$(x - 3)^2 + (y + 6)^2 = 4(x^2 + y^2)$

$x^2 + y^2 - 6x + 12y + 45 = 4x^2 + 4y^2$

$x^2 + y^2 + 2x - 4y - 15 = 0$

Hence sketching the region satisfying $|z - 3 + 6i| \leq 2|z|$

i.e. $x^2 + y^2 + 2x - 4y - 15 \geq 0$

substituting (0, 0)

LHS = - 15 ; RHS = 0

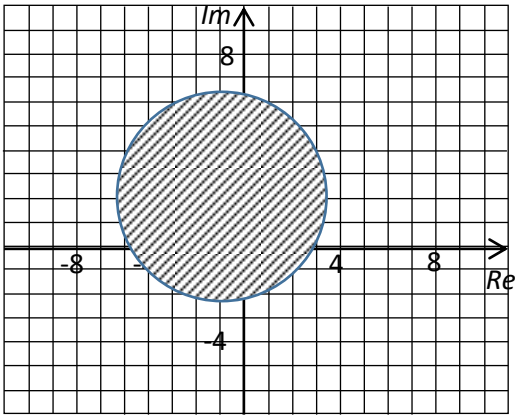
Since LHS< RHS, the point (0, 0) does not lie in the region$|z - 3 + 6i| \leq 2|z|$ therefore it lies in unwanted region.

Now $x^2 + y^2 + 2x - 4y - 15 \geq 0$

= $(x + 1)^2(y - 2)^2 \geq 20$

The locus is a circle centre (-1, 2) and radius $\sqrt{20}$

The plot of $|z - 3 + 6i| \leq 2|z|$



Example 31

If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of $|z|$

Solution

Let z = x = iy

⇨ x + iy = 4n +3i(1 – n)

Equating real parts

x = 4n => n = $\frac{x}{4}$

Equating imaginary parts

y = 3(1 – n)

⇨ y = $3\left(1-\frac{x}{4}\right)$

y =$-\frac{3x}{4}+3$ which is a straight line with gradient$-\frac{3}{4}$ and intercept 3

Given z = 4n + 3i(1 – n)

$|z|^2 = 16n^2 + 9(1-n)^2$

$|z|^2 = 25n^2 - 18n + 9$

The minimum value can be obtained by completing squares or differentiation

By completing squares

$|z|^2 = 25(n^2 - \frac{18}{25}n + \frac{9}{25})$

$= 25\left[\left(n-\frac{9}{25}\right)^2 + \frac{9}{25} - \frac{81}{625}\right]$

$= 25\left[\left(n-\frac{9}{25}\right)^2 + \frac{144}{625}\right]$

$= 25\left(n-\frac{9}{25}\right)^2 + \frac{144}{25}$

$|z|^2$ is minimum when $n = \frac{9}{25}$

$|z|^2_{min}$ = $\frac{144}{25}$

⇨ $|z|_{min} = \frac{12}{5}$

By differentiation

$2|z|\left|\frac{dz}{dn}\right| = 50n - 18 = 0$

n = $\frac{18}{50} = \frac{9}{25}$

$|z_{min}|^2 = 25\left(\frac{9}{25}\right)^2 - 18\left(\frac{9}{25}\right) + 9 = \frac{3600}{625}$

⇨ $|z|_{min} = \frac{60}{25} = \frac{12}{5}$

Example 32

Show that the locus of z if $\frac{z-3}{z+i}$ is wholly imaginary is a circle with centre at $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Solution

Let z = x + iy

$\frac{z-3}{z+i} = \frac{(x-3+iy)}{(x+i(y+1))}.\frac{(x-i(y+1))}{(x-i(y+1))}$

= $\frac{x(x-3)+y(y+1)+i[(xy-(x-3)(y+1)}{x^2+(y+1)^2}$

If the above expression is wholly imaginary, then the real part must be zero

i.e. $\frac{x(x-3)+y(y+1)}{x^2+(y+1)^2} = 0$

=> $x^2 + y^2 - 3x + y = 0$

$\left(x-\frac{3}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{9}{4}+\frac{1}{4} = \frac{10}{4}$

The locus is a circle with centre with $\frac{3}{2}-\frac{1}{2}i$ and radius $\frac{1}{2}\sqrt{10}$

Revision exercise 4

1. Use Demoivre's theorem to show that

(a) $\cos 6x = \cos^6 x - 15\cos^4 x \sin x + 15\cos x \sin^4 x - \sin^6 x$

(b) $\tan 3x = \frac{3tanx - tan^3 x}{1-3tan^2 x}$

2. (a) Evaluate $\left(1+i\sqrt{3}\right)^{\frac{2}{3}}$
   [1.2160 + 1.0204i, -1.4917 +0.5429i]
   (b) Find the roots of $z^4 + 4 = 0$ using Demoivre's theorem
   [1± i, -1± i]
3. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation $Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. (06marks)
   [Z = $\pm 2 + 1$]
   (b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. (06marks) [y = $\frac{3}{4}x$ + 6]
4. (a)Express 5 + 12i in polar form
   [5 + 12i = 13(cos 67.38^0 + isin67.38^0)]
   Hence, evaluate$\sqrt[3]{5+12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[z = 2.17 + 0.90i, -1.86 + 1.43i, -0.31 – 2.33i]

5. (a) The total impedance z in an electric circuit with two branches z_1 and z_2 is given by $\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$. Given that z_1= 3 + 4i and z_2

= 5 + 5i where i =$\sqrt{-1}$, calculate the total impedance z in form a + bi
$\left[z = \frac{275}{145} + i\frac{325}{145}\right]$

(b) If n is a variable and z = 4n + 3i(1 – n), show that the locus of Z is a straight line. Determine the minimum value of |z| $\left[|z|_{min} = \frac{12}{5}\right]$

6. Given that z_1 = 3 + 4i and z_2 = -1 + 2i
   (a) Express z_1, z_2, $z_1 + z_2$ and $z_1 – z_2$ in the form r(cosθ + isinθ)
   [z_1= 5(scos53.1^0+ isin53.1^0),
   z_2 =$\sqrt{5}$(cos116.6^0 + isin116.6^0),
   $z_1 + z_2$ = 2$\sqrt{10}$(cos71.6^0 + isin71.6^0)
   $z_1 – z_2$ = 2$\sqrt{5}$(cos-153.4^0 + isin-153.4^0)]
   (b) Determine the angle between $z_1 + z_2$ and z_1- z_2 [225^0]
7. Solve the simultaneous equation
   $z_1 + z_2$ = 8
   $4z_1 – 3iz_2$ = 26 + 8i
   Using the values of z_1 and z_2, find the modulus and argument of $z_1 + z_2 – z_1z_2$.
   [z_1 = 8 +2i, z_2 = -2i; 6.49, 75.96^0]
8. (a) Given that z = 3 + 4i, find the value of expression z + $\frac{25}{z}$ [6]
   Given that $\left|\frac{z-1}{z+1}\right| = 2$, show that the locus of the complex number is
   $x^2 + y^2 + \frac{10x}{3} + 1 = 0$. Sketch the locus
   [The centre of the circle is $\left(\frac{-5}{3}, 0\right)$ it passes through -3 but does not touch the y-axis]
9. (a)(i) Express each of the following complex numbers in the form a + bi
   z1 = (1 –i)(1 + 2i) [3 + i]
   z2 = $\frac{2+6i}{3-i}$ [2i]
   z3 = $\frac{-4i}{1-i}$ [2 – 2i]
   (b) Find the modulus and argument of $z_1z_2z_3$ [17.889, 63.4349^0]
   (c) Find the square root of 12i – 5
   [1 + 3i or -2 – 3i]
10. (a) given that z = $\sqrt{3}$+ i, find the modulus and argument
    (i) z^2 $\left[4, \frac{\pi}{3}\right]$
    (ii) $\frac{1}{z}\left[\frac{1}{2}, -\frac{\pi}{6}\right]$
    (iii) Show in an Argand diagram the points representing complex numbers z, z^2 and $\frac{1}{z}$.
    (b) In an Argand diagram, P represents a complex number z such that $2|z - 2| = |z - 6i|$ show that P lies on a circle, find
    (i) The radius of this circle[4.2164]
    (ii) The complex number represented by it centre
    $\left[z = \frac{8}{3} - 2i\right]$
11. (a) given the complex number z_1 = 1 – i, z_2 = 7 + I represent z_1z_2 and $z_1 – z_2$ on the Argand diagram.
    Determine the modulus and argument of $\frac{z_1 - z_2}{z_1 z_2}$ [0.6325, -124.7^0]
    (b) If z is a complex number of form a + bi, solve $\left(\frac{z-1}{z+1}\right)^2 = i$ [z = 1 ±i$\sqrt{2}$]xz
12. (a) show the region represented by $|z - 2 + i| < 1$ on an Argand diagram
    [it is a circle of centre(2, -1) and radius 1 unit, the circle is drawn with dotted line since the equation is less than, and the unwanted region is the area outside the circle]
    (b) Express the complex number z = 1 - i$\sqrt{3}$ in modulus argument form and hence find z^2 and $\frac{1}{z}$ in the form a + bi
    $\left[z = 2\left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right), \quad z^2 = -2 - i2\sqrt{3}, \frac{1}{z} = \frac{1}{4} + i\frac{\sqrt{3}}{4}\right]$
13. (a)Given that z_1 = -i + 1, z_2 = 2 + i, and z_3 = 1 + 5i, represent z_2z_3, z_2-$z_1\frac{1}{z_1}$and $\frac{z_2z_3}{z_2 - z_1} + \frac{1}{z_1}$ o, Argand diagram
    (b) Prove that for positive integers $(cos\theta + sin\theta)^n = cosn\theta + sinn\theta$. Deduce that this formula is also true for negative values of n.
14. If z is a complex number, describe and illustrate on the Argand diagram the locus given by each of the following

(a) $\left|\frac{z+i}{z-2}\right| = 3$

[it is a circle centre $\left(\frac{9}{4}, \frac{1}{8}\right)$ radius 0. 8385]

(ii) Arg (z + 3) = $\frac{\pi}{6}$ [is it a straight line represented by equation y = $\frac{\sqrt{3}}{3}x + 3$]

15. (a) Use Demoivre's theorem or otherwise to simplify $\frac{(cos\theta+isin\theta)(cos2\theta+isin2\theta}{cos\frac{\theta}{2}+isin\frac{\theta}{2}}$

$\left[cos\left(\frac{5\theta}{2}\right) + isin\left(\frac{5\theta}{2}\right)\right]$

(b) Express $\frac{i}{4+6i}$ in modulus-argument form.

{z = 0.1387[cos(0.187π)+ isin(0.187π)]}

(c) Solve (z + zz*)z = 5 + 2z where z* is the complex conjugate of z.
[z = 1 + 2i, z* = 1 -2i]

16. Show that 2 + I is a root of the equation $2z^3 – 9z^2 + 14 – 5 = 0$. Hence find the other roots 2 – i and ½

17. (a) find the equation whose root are -1 ± i where i = $\sqrt{-1}$ [$z^2 + 2z + 2 = 0$]

(b) Find the sum of the first 10 terms of the series 1 + 2i+ -4 – 8i + 16 + ….
[205 + 410i]

(c) Prove by induction that $(cos\theta + sin\theta)^n = cosn\theta + sinn\theta$.

18. Show that z = 1 is a root of the equation $z^3 – 5z2 + 9z – 5 = 0$.
Hence solve the equation for other roots
[2 ± i]

19. (a) Use Demoivre's theorem to express tan5θ in terms of tanθ.

(b) Solve the equation $z^3 + 1 = 0$

$\left[-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

20. (a) Without using tables or calculators, simplify $\frac{\left(cos\frac{\pi}{17}+isin\frac{\pi}{17}\right)^8}{\left(cos\frac{\pi}{17}-isin\frac{\pi}{17}\right)^9}$ [-1]

(b) Given that x and y are real, find the values of x and y which satisfy the equation: $\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$ [x = -1 when y = -2 and x = 1 when y = 2]

21. Given the complex number z = $\frac{(3i+1)(i-2)^2}{i-3}$

(i) Determine z in form a + bi where a and b are constant [-4,- 3i]

(ii) arg(z) [-143.13^0]

22. (a) Express the complex numbers z_1 =4i and z_2 = 2 – 2i in trigonometric form r(cosθ + isinθ).
[$z_1 = 4(cos90^0 + isin90^0)$,
$z_2 = 2\sqrt{2}(cos\text{-}45^0 + isin\text{-}45^0)$]
Hence or otherwise evaluate $\frac{z_1}{z_2}\left[\frac{1}{2}\right]$

(b) Find the values of x and y in $\frac{x}{3+3i} - \frac{y}{(2-3i)} = \frac{(6+2i)}{(1+8i)}$ [x = 2.8, y =0.4]

23. Find the fourth root of 4 + 3i

[±(1.4760 + 0.23971i), ±(0.2397 – 1.4760i)]

24. (a) Given that $\frac{ix}{1+iy} = \frac{3x+4i}{x+3y}$; find the values of x and y
[x = 2, y= 1.5 or x = -2 and y = -1.5]

(b) If z = x +iy, find the equation of the locus $\left|\frac{z+3}{z-1}\right| = 4$
$\left[x^2 + y^2 - \frac{38}{15}x + \frac{7}{15} = 0\right]$

25. (a) given that the complex number z and its conjugate Z*, satisfy the equation zz*+ 3z* = 34 – 12i, find the value of z.
[z = 3 - 4i and z = -6 + 4i]

(b) Find the Cartesian equation of the locus of a point P represented by equation $\left|\frac{z+3}{z+2-4i}\right| = i$
[the locus P is an equation of line 8y + 2x = 11]

26. (a) form a quadratic equation having -3 + 4i as one of its roots. [$z^2 + 6z + 25$]

(b) Given z_1 = -1 + i$\sqrt{3}$ and z_2 = -1 - i$\sqrt{3}$

(i) Express $\frac{z_1}{z_2}$ in form a + i$\sqrt{b}$, where a and b are real number
$\left[\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right]$

(ii) Represent $\frac{z_1}{z_2}$ on an Argand diagram

(iii) find $\left|\frac{z_1}{z_2}\right|$ [1]

27. If z = $\frac{(2-i)(5+12i)}{(1+2i)^2}$

(a) Find the

(i) Modulus of z [5.814]
(ii) Argument z [-86.055^0]

(b) Represent z on a complex plane

(c) Write z in the polar form
[z = 5.814(cos(-86.055^0) + isin(-86.055^0) or z = 5.814(cos (0.47π) + isin(0.47π)]

28. (a) the complex number z = $\sqrt{3} + i$. $\bar{Z}$ *is the conjugate of* Z.

(i) Express Z in the modulus argument form
[z = $2\left[\cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right]$ or z = $2[\cos 30^0 + isin30^0]$]

(ii) On the same Argand diagram plot $\bar{Z}$ and $2\bar{Z} + 3i$

(b) What are the greatest and least values of $|Z|$ *if* $|Z - 4| \leq 3$?

[Greatest value of $|z| = 7$, Lowest value of $|z| =$ = 1]

29. (a) Given that the complex number Z and its conjugate $\bar{Z}$ satisfy the equation
$Z\bar{Z} + 2iZ = 12 + 6i$. Find Z.
[possible values of z are 3 +3i and 3- i]

(b)One root of the equation $Z^3 - 3Z^2 + 9Z + 13 = 0$ *is* $2 + 3i$. Determine other roots
[other roots are 2-3i and -1]

30. (a) If $z_1 = \frac{2i}{1+3i}$ and $z_2 = \frac{3+2i}{5}$, find $|z_1 - z_2|$
$\left[\frac{1}{5}\right]$

(b) Given the complex number z = x + iy

(i) Find $\frac{z+i}{z+2}$ $\left[\frac{x^2+2x+y^2+y+i(2+x+2y)}{(x+2)^2+y^2}\right]$

(ii) Show that the locus of $\frac{z+i}{z+2}$ is a straight line when its imaginary part is zero. State the gradient of the line. [y = $-\frac{1}{2}x$ + 1]

31. (a) Given that the complex number Z and its conjugate $\overline{Z}$ satisfy the equation
$Z\overline{Z} - 2Z + 2\overline{Z} = 5 - 4i$. Find possible values of Z. [Z = $\pm 2 + 1$]

(b) Prove that if $\frac{Z-6i}{Z+8}$ is real, then the locus of the point representing the complex number Z is a straight line. [y = $\frac{3}{4}x$ + 6]

32. (a) Express 5 + 12i in polar form
Hence, evaluate $\sqrt[3]{5 + 12i}$, giving your answer in the form a + ib where a and b are corrected to two decimal places. (12marks)

[$\sqrt[3]{5 + 12i} = \sqrt[3]{13}\left[\cos\left(\frac{67.38+2\pi k}{3}\right) + isin\left(\frac{67.38+2\pi k}{3}\right)\right]$ *taking* $k = 0, 1, 2$;
2.17 + 0.90i; -1.86 + 1.43i; -0.31 – 2.33i]

33. Show that the modulus of $\frac{(1-i)^6}{(1+i)}$=$4\sqrt{2}$

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Binomial theorem

Pascal's triangle

The Pascal's triangle below and its further extension is used to determine the coefficients of the expansion of $(p + q)^n$

1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Observations

- The coefficients are symmetrical; they are the same irrespective of which side they are read from, for instance the coefficients of $(p + q)^6$ are
  1 6 15 20 15 6 1
- The coefficient of the 2nd term in the expansion is the index of a given expansion. E.g. in the expansion of $(p + q)^6$, the coefficient of the 2nd term is 6.
- The number of terms in the expansion exceeds the index by one, e.g. $(p + q)^4$ with index 4, has 5 terms
- The index of the first term of the expansion decreases by one, from the index given till zero, whereas, the index of the second term increases by one from zero to the given index
  For $(p + q)^3 = 1p^3q^0 + 3p^2q^1 + 3p^1q^2 + 1p^0q^3$
  $=p^3 + 3p^2q + 3pq^2 + q^3$
- The sum of indices in each term s constant and equal to the index of the expansion.

Example 1

Use Pascal's triangle to expand

(a) $(p + q)^4$

The coefficients are 1 4 6 4 1

Terms are $1p^4q^0 + 4p^3q^1 + 6p^2q^2 + 4p^1q^3 + 1q^4$

$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

(b) $(2x + 3y)^3$

The coefficients are 1 3 3 1

Terms are

$1(2x)^3(3y)^0 + 3(2x)^2(3y)^1 + 3(2x)^1(3y)^2 + 1(2x)^0(3y)^3$

$= 8x^3 + 36x^2y + 54xy^2 + 27y^3$

(c) $(a + b + c)^2$

The terms can be grouped into two ways:

Either a and $(b + c)$ or $(a + b)$ and c

The coefficients are 1 2 1

Either

Terms are: $1a^2(b + c)^0 + 2a^1(b + c)^1 + 1a^0(b + c)^2$

$= a^2 + 2a(b + c) + (b + c)^2$

$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

Or

$(a + b + c)^2$

$= 1(a + b)^2c^0 + 2(a + b)_1c^1 + 1c^2$

$= (a + b)^2 + 2c(a + b) + c^2$

$=a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

Example 2

Expand $(a + b)^4$ using Pascal's triangle. Hence find $(1.996)^4$ correct to 3 decimal places

Solution

From Pascal's triangle the coefficients are
1 4 6 4 1

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(1.996)^4 = (2- 0.004)^4$

Substituting a = 2 and b = -0.004

$24 + 4(2)^3(-0.004) + 6(2)^2(-0.004)^2 + 4(2)(-0.004)^3 + (-0.004)^4$

16 – 0.128 + 0.000384 = 15.872 (3D)

Example 3

(a) Expand $(2 – 3x)^4$ using Pascal's triangle. Hence evaluate $(1.97)^4$ correct to 3 decimal places.

Solution

Coefficients: 1 4 6 4 1

$(2 – 3x)^4 = 1 (2)^4(-3x)^0 + 4(2)^3(-3x)^1 + 6(2)^2(-3x)^2 + 4(2)^1(-3x)^3 + 1(2)^0(-3x)^4$

$= 16 – 96x + 216x^2 – 216x^3 + 81x^4$

Now $(1.97)^4 = (2 – 0.03)^4 = (2 – 3(0.01))^4$

⇨ x = 0.01

$(1.97)^4 = 16 – 96 (0.01) + 216(0.01)^2 - 216(0.01)^3 + 81(0.01)^4 = 15.0613848$

= 15.061 (3d.p)

(b) Use Pascal's triangle to evaluate $(1.02)^3$ correct 5 significant figures

Solutions

$(1.02)^3 = (1 + 02)^3$

Coefficients are : 1 3 3 1

$(1 + 02)^3 = 1^3 + 3(1)^2(0.02) + 3(1)(0.02)^2 + (0.02)^3 = 1.061208$

= 1.0612 (5 significant figures)

The idea of factorial and combination can also be used to determine the coefficients of the expansions

Example 3

Expand

(a) $(p + 3q)^3$

Solution

The coefficients are ${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$

Terms are p^3 $p^2(3q)$ $p(3q)^2$ $(3q)^3$

$(p + 3q)^3 = {}^3C_0p^3 + {}^3C_1p^2(3q) + {}^3C_2p(3q)^2 + {}^3C_3(3q)^3$

$= P^3 + 9P^23q + 27pq^2 + 27q^3$

(b) $(1 – x)^4$. Hence evaluate $(0.99)^4$ correct to four decimal places

Solution

Coefficients are ${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

Or simply $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

$1 – x)^4 = {}^4C_01^4(-x)^0 + {}^4C_11^3(-x)^1 + {}^4C_21^2(-x)^2 + {}^4C_31^1(-x)^3 + {}^4C_41^0(-x)4^0$

$= 1 – 4x + 6x^2 – 4x^3 + x^4$

Now 0.99 = 1 – 0.01) = x = 0.01

$(0.99)^4$

$= 1 – 4(0.01) + 6(0.01)^2 – 4(0.01)^3 + (0.01)^4$

= 0.96059601

= 0.9606 (4d.p)

The binomial theorem for positive integral index

Consider the expansion of $(1 + x)^4$, the result is

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Observations

(i) The indices of x increase by 1 from term to term
(ii) The index of the last term being the same as the power to which (1 + x) s raised
(iii) The coefficients of the terms of expansion are $^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

Hence the expansion of $(1 + x)^n$ is

$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots. x^n$

The expansion of $(a + x)^n$ is

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n$$

$$= a^n\left[1 + \binom{n}{1}\left(\frac{x}{a}\right) + \binom{n}{2}\left(\frac{x}{a}\right)^2 + \cdots x\left(\frac{x}{a}\right)^n\right]$$

$$= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots x^n$$

Example 4

Expand $(1 + 4x)^{14}$ in ascending power of x up to and include the 4th term. Hence evaluate $(1.0004)^{14}$ correct to four decimal places

Solution

$(1 + 4x)^{14}$

$$= 1 + \binom{14}{1}(4x) + \binom{14}{2}(4x)^2 + \binom{14}{3}(4x)^3$$

$= 1 + 56x + 1456x^2 + 23296x^3$

Now $(1.0004)^{14} = (1 + 0004)^{14}$ => x = 0.0001

$(1 + 4x)^{14}$

$= 1 + 56(0.0001) + 1456(0.0001)^2 +$

$23296(0.0001)^3$

= 1.005614583

= 1.0056 (4d.p)

Note the next term in the expansion is

$\binom{14}{4}(4x)^4 = 256256x^4$

Its value = $256256(0.0001)^4 = 2.56256 \times 10^{-11}$

Which when added to the above answer there will negligible change in value

Example 5

Expand $(3 – 2x)^{12}$ in ascending powers of x up to and including the term x^3. Hence evaluate $(2.998)^{12}$ correct to the nearest whole number.

Solution

$$(3 – 2x)^{12} = 3^{12} + \binom{12}{1}(3)^{11}(-2x)^1 + \binom{12}{2}(3)^{10}(-2x)^2 + \ldots$$

$= 531441 – 4251528x + 1588936x^2$

Now $(2.998)^{12} = (3 – 0.002)^{12}$ => x = 0.001

$(2.998)^{12} = 531441 – 4251528(0.001) + 1588936(0.001)^2$

= 527205.0609

=527205 (nearest whole number)

Particular terms of binomial expansion

As earlier seen, the expansion of

$(a + x)^5 = a^5x^0 + 5a^4x^1 + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$

In general if r is the power of x in the expansion $(a + x)^n$, then

the U_{r+1} term of x = $^nC_ra^{n-r}x^r$

Example 6

Find the term of x^4 in the expression of

(i) $(1 + x)^9$

By using U_{n+1}; term of x = $^nC_ra^{n-r}x^r$

n = 9, r = 4, a = 1

$U_5 = {}^9C_4 1^{9-4}x^4 = {}^9C_4x^4 = 126x^4$

(ii) (3 + x)7

n = 7, a = 3, r = 4

$U_5 = {}^7C_4 3^{7-4}x^4 = {}^7C_4(3)^3x^4 = 945x^4$

(iii) $\left(2 - \frac{x}{2}\right)^{12}$

$n = 12, a = 2$

$U_5 = {}^{12}C_4 2^{12-4}\left(-\frac{x}{2}\right)^4 = {}^{12}C_4(2)^8\left(-\frac{x}{2}\right)^4 = 7920x^4$

Example 7

Find the term indicated in expansion of the following expression

(i) $\left(3x-\frac{2}{x}\right)^5$ [x^3]

$$\left(3x-\frac{2}{x}\right)^5 = 3^5x^5\left(1-\frac{2}{3x^2}\right)^5$$

The term in $x^3 = 3^5x^5.{}^5C_1\left[1^4\left(-\frac{2}{3x^2}\right)^1\right]$

$= 3^5x^5.{}^5C_1\left(-\frac{2}{3x^2}\right) = -810x^3$

(ii) $\left(2x+\frac{5}{x}\right)^6$ [x^4]

$$\left(2x+\frac{5}{x}\right)^5 = 2^6x^6\left(1+\frac{5}{2x^2}\right)^6$$

The term in $x^3 = 2^6x^6.{}^6C_1\left[1^5\left(\frac{5}{2x^2}\right)^1\right]$

$= 2^6x^6.{}^6C_1\left(\frac{5}{2x^2}\right)\ 960x^4$

Finding terms independent of x

The term is said to be independent of x if the power of x is zero

Example 8

Find the term independent of x in the following expansion

(a) $\left(2x+\frac{1}{x^2}\right)^{12}$

Solution

$$\left(2x+\frac{1}{x^2}\right)^{12} = 2^{12}x^{12}\left(1+\frac{1}{2x^3}\right)^{12}$$

The term independent of $x = 2^{12}x^{12}$ multiplied by the term in x^{-12} in the expansion $\left(2x+\frac{1}{x^2}\right)^{12}$

⇨ The term independent of x

$= 2^{12}x^{12}.{}^{12}C_4\left(\frac{1}{2x^3}\right)^4 = 2^{12}x^{12} \times {}^{12}C_4 \times \frac{1}{2^4x^{12}}^4$

$=2^8 \times 495 = 126720$

Alternatively

By using $U_{r+1} = {}^nC_ra^{n-r}x^r$

The term independent of x is got by equating the index of x to zero

$U_{r+1} = {}^{12}C_r(2x)^{12-r}\left(\frac{1}{x^2}\right)^r$

$= {}^{12}C_r(2x)^{12-r}x^{-2r}$

$= {}^{12}C_r.\ 2^{12-r}.x^{12-3r}$

Equating the $x^{12-3r} = 1$

⇨ $12 - 3r = 0$

$r = 4$

Term independent of $x = {}^{12}C_4.\ 2^{12-4}$

$= {}^{12}C_4.\ 2^8$

$= 126720$

(b) $\left(2x^2+\frac{3}{x}\right)^{12}$

Solution

$$\left(2x+\frac{3}{x}\right)^{12} = 2^{12}x^{12}\left(1+\frac{3}{2x^2}\right)^{12}$$

$Un+r = {}^{12}C_r(2x^2)^{12-r}(3x^{-1})^r$

$= {}^{12}C_r(2x^2)^{12-r}(3^r)x^{-r}$

$= {}^{12}C_r.\ 2^{12-r}(3^r)x^{24-3r}$

Equating the $x^{12-3r} = 1$

⇨ $24 - 3r = 0$

$r = 8$

Term independent of $x = {}^{12}C_8.\ 2^{12-4}.3^8$

$= {}^{12}C_8.\ 2^4.3^8$

$= 51,963,120$

Binomial expansion of terms with fractional or negative powers

As noted earlier

$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots. x^n$

The following is noted

(i) For positive integral value of n, i.e. $n \geq 1$, the series above terminates at the term x^n and its sum is $(1+x)^n$.

(ii) For fractional or negative values of n, the series above does not terminate but instead converges to $(1 + x)^n$ as the limit of its sum only -1 < x< 1 or $|x| < 1$

Example 9

Expand $\sqrt{(1-2x)}$ up to the term x^3. Hence evaluate $\sqrt{0.98}$ correct to four decimal places.

Solution

$$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}}$$

Using

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots. x^n$$

Comparing with $(1-2x)^{\frac{1}{2}}$ => x ≡ -2x and n = ½

$$\sqrt{(1-2x)} = 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!}$$

$$=1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$$

Now $\sqrt{0.98} = \sqrt{(1-0.02)}$

Comparing with $\sqrt{(1-2x)}$; x = 0.01

Substituting

$$\sqrt{0.98} = 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$$

$$= 0.9899495$$

$\therefore \sqrt{0.98} = 0.9899$ (4d.p)

Example 10

Given that x is very small that its cube and higher powers can be neglected, show that

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = 1 - x + \frac{1}{2}x^2$$

By putting x = $\frac{1}{8}$ show that $\sqrt{7} = \frac{339}{128}$

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1-x}{1+x}\right)\left(\frac{1-x}{1-x}\right)} = \sqrt{\frac{(1-x)^2}{(1-x^2)}}$$

$$= (1-x)(1-x^2)^{\frac{1}{2}}$$

But $(1-x^2)^{\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x^2) + \ldots..$

$$= 1 + \frac{1}{2}x^2$$

$$\sqrt{\left(\frac{1-x}{1+x}\right)} = (1-x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 - x + \frac{1}{2}x^2$$

Putting x = $\frac{1}{8}$

$$\sqrt{\left(\frac{1-\frac{1}{8}}{1+\frac{1}{8}}\right)} = 1 - \frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)^2$$

$$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3} = 1 - \frac{1}{8} + \frac{1}{128} = \frac{113}{128}$$

$$\sqrt{7} = \frac{339}{128}$$

Example 11

Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ up to the term x^3

Use your expansion to evaluate $\sqrt{23}$ correct to 3 decimal places taking x = $\frac{1}{24}$.

Solution

By rationalizing the denominator

$$\sqrt{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x}{1+x}\right)} = (1+x)(1-x^2)^{\frac{1}{2}}$$

But$(1-x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots$

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = (1+x)\left(1 + \frac{1}{2}x^2\right)$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3$$

Hence putting x = $\frac{1}{24}$

$$\sqrt{\left(\frac{1+\frac{1}{24}}{1-\frac{1}{24}}\right)} = 1 + \frac{1}{24} + \frac{1}{2}\left(\frac{1}{24}\right)^2 + \frac{1}{2}\left(\frac{1}{24}\right)^3$$

$\sqrt{\frac{25}{23}} = \frac{5}{\sqrt{23}} = \frac{28825}{27648}$

$\sqrt{23} = \frac{5\ x\ 27648}{28825} = 4.7958$

$\therefore \sqrt{23} = 4.7958$ (4d.p)

Example 12

Use the binomial theorem to expand $\sqrt[3]{(1-x)}$ up to x^3. Use your expansion to evaluate $\sqrt[3]{7}$ correct to four decimal places

Solution

$\sqrt[3]{(1-x)} = (1-x)^{\frac{1}{3}}$

$= 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)(-x)^2}{2!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-x)^3}{3!} + \ldots$

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{1}{81}x^3$

Hence $\sqrt[3]{7} = \sqrt[3]{(8-1)} = \sqrt[3]{8\left(1-\frac{1}{8}\right)}$

$= 2\left(1-\frac{1}{8}\right)^{\frac{1}{3}}$ => x = $\frac{1}{8}$

$\sqrt[3]{7} = 2\left[1 + \frac{1}{3}\left(\frac{1}{8}\right) - \frac{1}{9}\left(\frac{1}{8}\right)^2 + \frac{1}{81}\left(\frac{1}{8}\right)^3\right]$

=1.9130(4d.p)

Example 13

Write down the expansion of $\sqrt{(1-x)}$ in ascending powers of x as far as the term x^4. Use your expansion to find $\sqrt{80}$ correct to four significant figures.

Solution

$\sqrt{(1-x)} = (1-x)^{\frac{1}{2}}$

For binomial expansion

$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots. x^n$

Comparing $(1-x)^{\frac{1}{2}}$ with $(1+x)^n$

n = $\frac{1}{2}$ and x = -x

$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(-x)^2}{2!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^3}{3!} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^4}{4!}$

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{6} - \frac{x^4}{24}$

Now $\sqrt{80} = \sqrt{81-1} = \sqrt{81\left(1-\frac{1}{81}\right)}$

$= 9\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

Comparing $\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$ with $(1-x)^{\frac{1}{2}}$

⇨ x = $\frac{1}{81}$

Substituting for x

$\left(1-\frac{1}{81}\right)^{\frac{1}{2}}$

$= 1 - \frac{1}{2}\left(\frac{1}{81}\right) - \frac{\left(\frac{1}{81}\right)^2}{8} - \frac{\left(\frac{1}{81}\right)^3}{6} - \frac{\left(\frac{1}{81}\right)^4}{24}$

$= \frac{52163}{52488}$

$\sqrt{80} = 8.944$ (4 S.F)

Not the termx3 has been neglected as it does not affect the answer to 4 significant figured

Example 14

John operates an account with a certain bank which pays a compound interest rate of 13.5% per annum. He opened the account at the beginning of the year with sh. 500,000 and deposits the same amount of money at the beginning of every year. Calculate how much he will accumulate at the end of 9 years. After how long will the money have accumulated to sh. 3.32 millions?

Solution

The 1st deposit will grow to

$500000\left(1+\frac{13.5}{100}\right)$= 500000 x 1.135

2nd deposit will grow to 500000 x 1.135^2

nth deposit will grow to 500000 x 1.135^n

9th deposit will grow to 500000 x 1.135^9

The total = 500,000[1.135 +1.135^2 + … + 1.135^9]

$=500000\left[a\left(\frac{r^n-1}{r-1}\right)\right]$

$=500000\ \left[1.135\left(\frac{1.135^9-1}{1.135-1}\right)\right]$

=8,936,381

Finding how long it will take the money to accumulate to sh. 3,320,000

$500000\ \left[1.135\left(\frac{1.135^n-1}{1.135-1}\right)\right]=3320000$

n = 4.6years

Example 15

Expand $(1+x)^{-2}$ in descending powers of x including the term x^{-4}. If x = 9 find the percentage error in using the first two terms of the expression.

Solution

From

$$(1+x)^n=1+nx+\frac{2(n-1)x^2}{2!}+\cdots+x^n$$

Now $(1+x)^{-2}=x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$x^{-2}\left(1+\frac{1}{x}\right)^{-2}$

$=x^{-2}\left[1+(-2)\frac{1}{x}+\frac{(-2)(-3)}{2!}\left(\frac{1}{x}\right)^2\right]$

$=x^{-2}\left[1-\frac{2}{x}+\frac{3}{x^2}\right]$

$=x^{-2}[1-2x^{-1}+3x^{-2}]$

$=x^{-2}-2x^{-3}+3x^{-4}$

If x = 9

$(1+x)^{-2}=9^{-2}-2(9)^{-3}+3(9)^{-4}$

$=\frac{1}{81}-\frac{2}{729}$ (using the first 2 terms)

$=\frac{7}{729}$

The exact value is $(1+9)^{-2}=\frac{1}{100}$

Error = $\frac{1}{100}-\frac{7}{729}=\frac{29}{72900}$

%error = $\frac{29}{72900}\ x100\ x100$= 3.978%(3d.p)

Example 16

(a) Find the three terms of the expansion $(2-x)^6$ and use it to find $(1.998)^6$ correct to two decimal places (06 marks)

$(2-x)^6=2^6+\binom{6}{1}2^5(-x)^1+\binom{6}{2}4(-x)^2$

$=64-192x+240x^2$

$(1.998)^2=(2-0.002)^2$

$$=64-192(0.002)+240(0.002)^2$$
$$=64-0.384+0.00096$$
$$=63.61696$$
$$=63.62\ (2D)$$

Example 17

(a) Prove by induction

1.3 + 2.4 + … + n(n + 2) = $\frac{1}{6}n(n+1)(2n+7)$ for all values of n.

Suppose n = 1

L.H.S = 1 x 3 = 3

R.H.S = $\frac{1}{6}x1(1+1)(2+7)=3$

L.H.S = R.H.S, hence the series holds for n =1

Suppose n = 2

L.H.S= 1 x 3 + 2 x 4 = 11

R.H.S = $\frac{1}{6}\ x\ 2(2+1)(4+7) = 11$

L.H.S = R.H.S, hence the series holds for n =2

Suppose n = k

1.3 + 2.4 + … + k(k + 2) = $\frac{1}{6}k(k+1)(2k+7)$

For n = k + 1

1.3 + 2.4 + … + k(k + 2), (k + 1)(k + 3) = $\frac{1}{6}k(k+1)(2k+7)$+(k +1)(k + 3)

=(k + 1)$\left[\frac{1}{6}k(2k+7)+(k+3\right]$

= $\frac{1}{6}(k+1)(2k^2+13k+18)$

= $\frac{1}{6}(k+1)(2k^2+4k+9k+18)$

= $\frac{1}{6}(k+1(k+2)(2k+9)$

= $\frac{1}{6}(k+1)(k+2)[2(k+1)+7]$

Which is equal to R.H.S when n = k + 1

It holds for n = 1, 2, 3 …, hence it holds for all integral values of n.

(b) A man deposits Shs. 150,000 at the beginning of every year in a micro finance bank with the understanding that at the end of the seven years he is paid back his money with 5% per annum compound interest. How much does he receive?

Using amount, A = P$\left(1+\frac{r}{100}\right)^n$

=150000$\left(1+\frac{5}{100}\right)^7$=211,065.06

Alternatively

1st year

P = 150,000

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 150,000$\left(1+\frac{5}{100}\right)$=157,500

2nd year

P = 157, 500

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 157500$\left(1+\frac{5}{100}\right)$=165375

3rd year

P = 165375

Interest = $\frac{5}{10}\ x\ 165375 = 8268.75$

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 165375 $\left(1+\frac{5}{100}\right)$=173643.75

4th year

P=173643.75

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 173643.75$\left(1+\frac{5}{100}\right)$=182325.94

5th year

P =182325.94

His paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 182325.94$\left(1+\frac{5}{100}\right)$=191442.23

6th year

P = 191442.23

He is paid back principal plus interest; P$\left(1+\frac{5}{100}\right)$ = 191442.23$\left(1+\frac{5}{100}\right)$=201014.35

7^{th} year

P = 201014.35

He is paid back principal plus interest; $P\left(1 + \frac{5}{100}\right)$ = $201014.35\left(1 + \frac{5}{100}\right)$=211,065.06

∴ by the 7^{th} year he has accumulated shs. **211,065.06**

Example 18

Expand $\sqrt{\left(\frac{1+2x}{1-x}\right)}$ up to the term x^2. Hence find the value of $\sqrt{\left(\frac{1.04}{0.98}\right)}$ to four significant figures. (12marks)

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = (1+2x)^{\frac{1}{2}}(1-x)^{\frac{-1}{2}}$$

$$using\ (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

$$\sqrt{\left(\frac{1+2x}{1-x}\right)} = \left(1 + x - \frac{1}{2}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + x + \frac{1}{2}x^2 - \frac{1}{2}x^2$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$\therefore \sqrt{\left(\frac{1+2x}{1-x}\right)} \approx 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

$$substituting\ for\ x = 0.02$$

$$\sqrt{\left(\frac{1.04}{0.98}\right)} = \sqrt{\frac{1+2(0.02)}{1-0.02}}$$

$$= 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 = 1.030$$

Revision exercise

1. Given that the ratio of the 3^{rd} to the 4^{th} term of the expansion (2+3x)n is 5:14, find the value of n when x = $\frac{2}{5}$. [n= 16]
2. Expand $(3 - 4x)^5$ in ascending order of x up to and including the term x^3. Hence evaluate $(4.96)^5$ correct to 2 d.p. [3001.98]
3. (a) Find the coefficient of x2 in the expansion of $(1 - 2x)^n$ is 24 [4]
   (b) Find the term independent of x in the expansion of $\left(2x^3 - \frac{1}{x}\right)^{20}$ [-496128]
   (c) Use the binomial expansion to expand $\sqrt[4]{(1+2x)}$ up to the term x^3. Hence evaluate $\sqrt[4]{83}$ correct to three decimal places [3.018]
4. Expand $\sqrt{\left(\frac{1+2x}{1-2x}\right)}$ up to and including the term x3. Hence find the value of $\sqrt{\frac{1.02}{0.98}}$ to four significant figures. Deduce the value of $\sqrt{51}$ to 3 significant figures [1.0202, 7.14]
5. Five millions shillings are invested each year at a rate of 15%. In how many years will it accumulate to more than 50millions? [6years]
6. Expand $\left(1 - \frac{x}{3}\right)^{\frac{1}{2}}$ as far as x3. Hence evaluate $\sqrt{8}$ correct to 3 decimal places [2.829]
7. A man deposits sh. 800,000 into his saving account on which interest is 15% per annum. If he makes no withdraws, after how many years will his balance exceed sh. 8millions? [16.5 years]
8. Determine the binomial expansion of $\left(1 + \frac{x}{2}\right)^4$. Hence evaluate $(2.1)^4$ correct to 2 decimal places.[19.45]
9. Determine the binomial expansion of $\left(1 - \frac{x}{2}\right)^5$. Hence evaluate $(0.875)^5$ correct to four decimal places [0.5129]

10. A financial credit society give a compound interest of 2% per annum to its members. If Bbosa deposits sh. 10000 at the beginning of every year. How much would he accumulate at the end of the fifth year if no withdraws within this period [sh. 530812]
11. Expand $\sqrt{\left(\frac{1+x}{1-x}\right)}$ in ascending powers of x up to a term x^2. $\left[1 + x + \frac{x^2}{x} + \cdots\right]$
12. (a) Using the expansion $(1 + x)^{\frac{1}{2}}$ up to the term x^3, find the value of $\sqrt{1.08}$ to four decimal places [1.0392]
    (b) Express $\sqrt{1.08}$ in the form $\frac{a}{b}\sqrt{c}$. Hence evaluate $\sqrt{3}$ correct to 3 significant figure $\left[\frac{3}{5}\sqrt{3}, 1.73\right]$
13. (a) obtain the first four non – zero terms of the binomial expansion in ascending powers of x of $(1 - x^2)^{-\frac{1}{2}}$ give that$|x| < 1$ $\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16}\right]$
    (b) show that, when x = $\frac{1}{3}$; $(1 - x^2)^{-\frac{1}{2}} = \frac{3\sqrt{2}}{4}$
    (c) substitute x = $\frac{1}{3}$ into your expansion and hence obtain an approximation of $\sqrt{2}$, give your answer to five decimal places [1.41415]
14. (a) show that $\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$
    (c) Write the first three terms in the binomial expansion of $\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$ in ascending power of x stating the range for which this expansion is valid. $\left[1 + \frac{x}{8} + \frac{3x^2}{128};\ |x| < 4\right]$
    (d) Find the first three terms in the expansion of $\frac{2(1+x)}{\sqrt{4-x}}$ in ascending power of x for small values of x.$\left[1 + \frac{9x}{8} + \frac{19x^2}{128};\right]$

Thank you

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Polynomials

A polynomial in x is a function in the form

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$ where $a \neq 0$

(a) Quadratics

If α and β are the two roots, then, $x = \alpha$ and $x = \beta$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

(b) Cubic

If α, β and λ are the three roots, then,

$x = \alpha$, $x = \beta$ and $x = \lambda$

⇨ $(x - \alpha)(x - \beta)(x - \lambda) = 0$

$[x^2 - (\alpha + \beta)x + \alpha\beta](x - \lambda) = 0$

$x^3 - \lambda x^2 - (\alpha + \beta)x^2 + (\alpha + \beta)\lambda x + \alpha\beta x - \alpha\beta\lambda = 0$

$x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda = 0$

(c) Qurtic

If α, β, λ and ρ are the four roots, then

$x = \alpha$, $x = \beta$, $x = \lambda$ and $x = \rho$

⇨ $(x - \alpha)(x - \beta)(x - \lambda)(x - \rho) = 0$

$[x^3 - (\alpha + \beta + \lambda)x^2 + (\alpha\beta + \alpha\lambda + \beta\lambda)x - \alpha\beta\lambda](x - \rho) = 0$

$x^4 - (\alpha + \beta + \lambda + \rho)x^3 + (\alpha\beta + \alpha\lambda + \alpha\rho + \beta\lambda + \beta\rho + \lambda\rho)x^2 - (\alpha\beta\lambda + \alpha\beta\rho + \alpha\lambda\rho + \beta\lambda\rho)x + \alpha\beta\lambda\rho = 0$

The above illustration shows that:

(i) The signs of the terms alternate in the order: +, - , + ….. starting with the first term.

(ii) The coefficient of the second term is – (sum of the roots) and the last term with its appropriate sign is the product of the roots

Example 1

Find the sums and products of the roots of the following equations

(a) $5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6 = 0$

Solution

Dividing through by 5

$x^5 + \frac{4}{5}x^4 - \frac{3}{5}x^3 + \frac{2}{5}x^2 - \frac{1}{5}x + \frac{6}{5} = 0$

sum of roots = $-\frac{4}{5}$

product of roots = $\frac{6}{5}$

(b) $3x^4 - 5x^3 + 2x^2 + 0x + 9 = 0$

Dividing through by 3

$x^4 - \frac{5}{3}x^3 + \frac{2}{3}x^2 + 0x + 3 = 0$

Sum of roots = $\frac{5}{3}$

Product of roots = 3

Addition and subtraction of polynomial

Polynomial are added or subtracted if they are of the same degree. This is done by adding or subtracting the coefficients of the corresponding term

Example 2

(a) Given the polynomial

$f(x) = 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$ and

$g(x) = 3x^4 - 5x^3 + 2x^2 + 9$

Find (i) f(x) + g (x) (ii) f(x) – g(x)

Solution

(i) f(x) + g(x)

$5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$

$+ \quad 3x^4 - 5x^3 + 2x^2 + 0x + 9$

$= \quad 5x^5 + 7x^4 - 8x^3 + 2x^2 - x + 15$

f(x) - g(x)

$5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 6$

$- \quad 3x^4 - 5x^3 + 2x^2 + 0x + 9$

$= \quad 5x^5 + x^4 + 2x^3 + 0x^2 - x - 3$

(b) Given polynomials

$f(x) = 2x^3 + 4x^2 - 2x - 8$ and

$g(x) = x^3 - 2x^2 + 3x + 5$

Find (i) f(x) + g (x) (ii) f(x) – g(x)

Solution

(i) f(x) + g (x)

$= (2 + 1)x^3 + (4 - 2)x^2 + (-2 + 3)x + (-8 + 5)$

$= 3x^3 + 2x^2 + x - 3$

(ii) f(x) – g(x)

$= (2 - 1)x^3 + (4 - -2)x^2 + (-2 - 3)x + (-8 - 5)$

$= x^3 + 6x^2 - 5x - 13$

Multiplication of polynomials

When multiplying two functions together, the terms of the first function are multiplied by the terms of the second function

Example 3

(a) Given the polynomial

$f(x) = 5x^3 + 2x^2 + 9$ and

$g(x) = 4x^4 - 3x^3 + 2x^2 - x + 6$

Find f(x) x g (x)

Solution

$= 5x^3 (4x^4 - 3x^3 + 2x^2 - x + 6)$

$+ 2x^2 (4x^4 - 3x^3 + 2x^2 - x + 6)$

$+ 9 (4x^4 - 3x^3 + 2x^2 - x + 6)$

$= (20x^7 - 15x^6 + 10x^5 - 5x^4 + 30x^3)$

$+ (8x^6 - 6x^5 + 4x^4 - 2x^3 + 12x^2)$

$+ (36x^4 - 27x^3 + 18x^2 - 9x + 54)$

$= 20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

Hence f(x) .g(x) =

$20x^7 - 7x^6 + 4x^5 + 35x^4 + x^3 + 30x^2 - 9x + 54$

Example 4

Find coefficient of x^3 in the expansion

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

Solution

$(2x^3 + x^2 - 5x + 6)(2x + 4)$

$= 2x(2x^3 + x^2 - 5x + 6) + 4(2x^3 + x^2 - 5x + 6)$

$= (2x^4 + 2x^3 - 10x^2 + 12x) + 8x^3 + 4x^2 - 20x + 16$

$= 2x^4 + 10x^3 - 6x^2 - 8x + 16$

The coefficient of x^3 is 10

Division of polynomials

The division of polynomial may be done by long division as follows.

Example 5

(a) Divide $x^3 - 7x - 6$ by $(x + 1)$

Solution

$$
\begin{array}{r}
x^2 - x - 6 \\
(x + 1)\overline{)\,x^3 - 7x - 6} \\
-\,(x^3 + x^2) \\
\hline
x^2 - 7x - 6 \\
-\,(x^2 + x) \\
\hline
-6x - 6 \\
-\,(-6x - 6) \\
\hline
0 + 0
\end{array}
$$

Example 6

Find the remainder when $2x^3 + x^2 + 5x - 4$ is divided by $2x - 1$

$$
\begin{array}{r}
x^2 + x + 3 \\
(2x - 1)\overline{)\,2x^3 + x^2 + 5x - 4} \\
-\,(2x^3 - x^2) \\
\hline
2x^2 + 5x - 4 \\
-\,(2x^2 - x) \\
\hline
6x - 4 \\
-\,(6x - 3) \\
\hline
-1
\end{array}
$$

Hence the remainder is -1

Example 7

Show that x = -2 is a root of the equation $2x^3 – x2 – 8x + 4 = 0$. Hence find the other roots

Solution

If x =-2 is a root of the function, then its

Remainder must be equal to zero

Hence x = -2 is a root of $2x^3 – x^2 – 8x+ 4 = 0$

x = -2 => x + 2 =0

$$
\begin{array}{r}
2x^2 - 5x + 2 \\
(x - 2)\overline{)2x^3 - x^2 - 8x + 4} \\
-\ 2x^3 + 4x^2 \\
\hline
-5x^2 - 8x + 4 \\
-\ 5x^2 - 10x \\
\hline
2x + 4 \\
-\ \ 2x + 4 \\
\hline
0
\end{array}
$$

Since a cubic equation has at most three roots; the remaining two roots are obtained by solving the quadratic equation by factorization

2x2 – 5x + 2 = 0

2x2 – 4x – 2x + 2 =0

2x(x – 2) – 1(x – 2) = 0

(2x – 1)(x – 2) = 0

2x – 2 = 0 or x – 2 = 0

x = $\frac{1}{2}$ or x = 2

∴ the other roots are x = $\frac{1}{2}$ and x = 2

Revision exercise 1

1. Find the degree of each of the following polynomials
   (a) $x^6 + 4x^4 – 2$ [6]
   (b) $2x^3 – x^2 – 8x+ 4$ [3]
   (c) $5x^3 + 2x^2 + 9$ [3]
   (d) $2x^4 + 10x^3 – 6x^2 – 8x + 16$ [4]
2. Given that
   (a) $f(x) = x^3 + 2x^2 – 3x + 2$ and $g(x) = 2x^3 – x^2 + 5x – 4$, find f(x) – g(x) $[-x^3 + 3x^2 – 8x + 6]$
   (b) $f(x) = x^3 + 2x^2 – 3x + 2$ and $g(x) = 2x^3 – x^2 + 5x – 4$, find g(x) – f(x) $[x^3 - 3x^2 + 8x - 6]$
   (c) $f(x) = 2x^3 – 5x^2 + 6x$ and $g(x) = x^3 – 6x^2 + 5x + 1$, find f(x) – g(x) $[x^3 + x^2 + x – 1]$
   (d) $f(x) = 2x^3 – 5x^2 + 6x$ and $g(x) = x^3 – 6x^2 + 5x + 1$, find 2f(x) + g(x) $[5x^3 – 16x^2 + 7x + 1]$
3. Given that
   (a) $f(x) = x^2 – 2x + 5$ and $g(x) = x^3 + 6x – 4$, find xf(x) + 3g(x) $[4x^3 – 2x^2 + 23x – 12]$
   (b) $f(x) = x^3 + 6x – 4$ and $g(x) = x^2 – 2x + 5$, find 3f(x) + xg(x) $[2x^3 + 2x^2 + 13x – 12]$

Other methods of finding the remainder

Apart from using long division, the remainder when a function is divided by a certain factor can be obtained by **remainder theorem** and **synthetic approach**.

(a) The remainder and factor theorems

When a number say 186 is divided by 4, this can be represented simply as follow

$$
\begin{array}{r}
46 \\
4\overline{)186} \\
-\ 16 \\
\hline
26 \\
-\ 24 \\
\hline
2
\end{array}
$$

∴ the quotient is 46 and the remainder is 2
The above alogarithm can be written as
$\frac{186}{4} = 46 + \frac{2}{4}$
Or simply 186 = 4Q + R, where the quotient Q = 46 and the remainder, R = 2. This is referred to as the remainder theorem

The **remainder theorem** states that, when a function f(x) is divided by by (x – a) and leaves a remainder, then the remainder of the function is f(a)
From $f(x) = (x – a)Q(x) + R$
When we substitute for x = a; f(a) = R

When R = 0, => f(a) = 0. This is referred to as the **factor theorem**,

Example 8

Find the remainder when

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by x – 1
Solution
Let $x^3 + 3x^2 – 4x + 2 = (x – 1)Q(x) + R$
Putting x = 1: 1 + 3 – 4 + 2 = R
2 = R

(b) $f(x) = 3x^3 + 2x – 4$ is divided by x – 2
Solution
Let $3x^3 + 2x – 4 = (x – 2)Q(x) + R$
Putting 2: 24 + 4 – 4 = R
R = 24

(c) $f(x) = 2x^3 + 4x^2 – 6x + 5$ is divided by x – 1
Let $2x^3 + 4x^2 – 6x + 5 = (x – 1)Q(x) + R$
Putting 1: 2 + 4 – 6 + 5 = R
R = 5

(d) $8x^3 + 4x + 3$ is divided by 2x -1
Let $8x^3 + 4x + 3 = (2x – 1)Q(x) + R$
Putting $\frac{1}{2}$: 1 + 2 + 3 = R
R = 6

Synthetic approach for finding the remainder

The synthetic approach can be illustrated as follows
$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + …. + a_n$ is divided by x – a, then the following steps are taken using a table

(i) The factor in question must be linear
(ii) The 1st row of the table contains the coefficients of x in f(x) in descending order.
(iii) The at most left hand side of the 3rd row contains a value a from the division, x – a i.e. if x – a = 0, then x = a
(iv) The first digit in the second row comes under the second digit in the first row and this digit is equal to a_0 x a, where a_0 is the coefficient of x^n.
(v) The second digit in the 3rd row is the same as the sum of the digits in the 1st and 2nd rows
(vi) The next digit in the 2nd row is equal to a_0 x the 2nd digit in the 3rd row
(vii) The corresponding numbers in the 1st and 2nd row are added to give the digit in the 3rd row and the process continues
(viii) The last digit in the 3rd row is the remainder of the polynomial and the digit to the left of the remainder are coefficients of the **quotient** provided the divisor is in the form tx + b, where t = 1. If t ≠ 0, then we divide the digit to left of the remainder by t to obtain the coefficients of the quotient.

Example 9

Find the remainder and quotient when the function

(a) $f(x) = x^3 + 3x^2 – 4x + 2$ is divided by x – 1
from $f(x) = x^3 + 3x^2 – 4x + 2$, the coefficients of x in a descending order are 1, 3, -4 and 2
From x – 1 = 0, => x = a = 1

1st row	1	3	-4	2	
2nd row		1	4	0	+ ; a=1
3rd row	1	4	0	(2)	←remainder

The remainder is 2
Quotient $x^2 + 4x$

(b) $x^5 + x – 9$ is divided by x + 1

$x^5 + x – 9 \equiv x^5 + 0x^4 + 0x^3 + 0x^2 + x - 9$

1st row	1	0	0	0	1	-9	
2nd row		-1	1	-1	1	-2	+ ; x = -1
3rd row	1	-1	1	-1	2	(-11)	←remainder

Remainder: -11

Quotient: $x^4 – x^3 + x^2 – x + 2$

(c) $4x^5 – 3x^3 + 2x + 7$is divided by 2x – 1

$4x^5 - 3x^3 + 2x + 7 \equiv 4x^5 + 0x^4 - 3x^3 + 0x^2 + 2x + 7$

1st row	4	0	-3	0	2	7
2nd row		2	1	-1	$-\frac{1}{2}$	$\frac{3}{4}$
3rd row	4	2	-2	-1	$\frac{3}{2}$	$\left(\frac{31}{4}\right)$ ←remainder

+; $x = \frac{1}{2}$

The quotient is $\frac{1}{2}[4x^4 + 2x^3 - 2x^2 - x + \frac{3}{2}] =$
$2x^4 + x^3 - x^2 - \frac{1}{2}x + \frac{3}{4}$ and remainder is $\frac{31}{4}$

Example 10

The polynomial $x^4 + px^3 - x2 + qx - 12$ has factors $x + 1$ and $x + 2$.

Find the values of p and q, hence factorize the polynomial completely.

Solution

Let $f(x) = x^4 + px^3 - x^2 + qx - 12$

By factor theorem

Putting $x = -1$

$f(-1) = 1 - p - 1 - q - 12 = 0$

$p + q = -12$ ..... (i)

Putting $x = -2$

$f(-2) = 16 - 8p - 4 - 2q - 12 = 0$

$4p + q = 0$ .....(ii)

Eqn. (ii) – eqn. (i)

$3p = 12$

$p = 4$

substituting for p into eqn. (i)

$4 + q = -12$

$q = -16$

$\therefore f(x) = x^4 + 4x^3 - x^2 + 16x - 12$

Now $(x + 1)(x + 2) = x^2 + 3x + 2$

Since $(x + 1)(x + 2)$ is a factor, then $x^2 + 3x + 2$ is also a factor of f(x)

By long division to find other factors

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 + 3x + 2 \overline{)\, x^4 + 4x^3 - x^2 - 16x - 12} \\
\underline{-\ x^4 + 3x^3 + 2x^2} \\
x^3 - 3x^2 - 16x \\
\underline{-\ x^3 + 3x^2 + 2x} \\
-6x^2 - 18x - 12 \\
\underline{-\quad -6x^2 - 18x - 12} \\
0 \quad + 0 \quad + 0
\end{array}
$$

$\therefore x^4 + 4x^3 - x^2 + 16x - 12$

$= (x^2 + 3x + 2)(x^2 + x - 6)$

Now $x^2 + x - 6 = (x + 3)(x - 2)$

Hence $x^4 + 4x^3 - x^2 + 16x - 12$

$= (x + 1)(x + 2)(x + 3)(x - 2)$

Example 11

When $f(x) = x^3 - ax + b$ is divided by $x + 1$, the remainder is 2 and $x + 2$ is a factor. Find a and b.

Solution

By substitution for $x = -1$ in the function

$(-1)3 - a(-1) + b = 0$

$-1 + a + b = 2$

$a + b = 3$ ..... (i)

since $x + 2$ is a factor, substituting for $x = -2$ in the function gives zero

$(-2)^3 - a(-2) + b = 0$

$-8 + 2a + b = 0$

$2a + b = 8$ .....(ii)

Eqn. (ii) – eqn. (i)

$a = 5$

substituting for a in eqn. (i)

$5 + b = 3 \Rightarrow b = -2$

Example 12

The function $f(x) = x^3 + px^2 – 5x + q$ has a factor $x – 2$ and has a value of 5 when $x = -3$. Find the values of p and q

Solution

By substitution for $x = 2$ in the function

$2^3 + p(2)^2 – 5(2) + q = 0$

$4p + q = 2$ ……(i)

By substitution for $x = -3$ in the function

$(-3)^3 + p(-3)^2 – 5(-3) + q = 5$

$9p + q = 17$ ……. (ii)

Eqn. (ii) – eqn.(i)

$5p = 15$

$p = 3$

From eqn. (i)

$4 \times 3 + q = 2$

$q = -10$

Hence $p = 3$ and $q = -10$

Example 13

The polynomial $x^4 + px^3 – x^2 + qx – 12$ has a factor $x^2 + 3x + 2$.

Find the value of p and q and hence factorise completely

Solution

$x^2 + 3x + 2 = (x + 1)(x + 2)$

Let $f(x) = x^4 + px^3 – x^2 + qx – 12$

$f(-1) = (-1)^4 + p(-1)^3 – (-1)^2 + q(-1) – 12 = 0$

$1 – p – 1 + q – 12 = 0$

$-p + q = 12$……..(i)

$f(-2) =(-2)^4 + p(-2)^3 – (-2)^2 + q(-2) – 12 = 0$

$16 – 8p – 4 – 2q – 12 = 0$

$-8p – 2q = 0$

$4p – q = 0$ ……..(ii)

Eqn. (i) + eqn. (ii)

$3p = 12$

$p = 4$

From eqn. (i)

$q = -4 – 12 = -16$

Hence $p = 4$ and $q = -16$

$\therefore x^4 + px^3 – x^2 + qx – 12 = x^4 + 4x^3 – x^2 – 16x – 12$

By long division to find other factors

$$
\begin{array}{r}
x^2 + x - 6 \\
x^2 + 3x + 2 \overline{)\, x^4 + 4x^3 - x^2 - 16x - 12} \\
\underline{-\, x^4 + 3x^3 + 2x^2} \\
x^3 - 3x^2 - 16x \\
\underline{-\, x^3 + 3x^2 + 2x} \\
-6x^2 - 18x - 12 \\
\underline{-\quad -6x^2 - 18x - 12} \\
0 \quad + 0 \quad + 0
\end{array}
$$

$\therefore x^4 + 4x^3 – x^2 + 16x – 12$

$= (x^2 + 3x + 2)(x^2 + x – 6)$

Now $x^2 + x – 6 = (x + 3)(x – 2)$

Hence $x^4 + 4x^3 – x^2 + 16x – 12$

$= (x + 1)(x + 2)(x + 3)(x – 2)$

Example 14

If $4x^3 + ax^2 + bx + 2$ is divisible by $x^2 + k^2$, show that $ab = 8$

Solution

By long division

$$\begin{array}{r} 4x + a \\ (x^2 + k^2)\overline{)4x^3 + ax^2 + bx + 2} \\ \underline{-4x^3 + 4k^2} \\ ax^2 + (b - 4k^2)x + 2 \\ \underline{-ax^2 + ak^2} \\ (b - 4k^2)x + 2 - ak^2 \end{array}$$

Since the remainder is zero,

$(\lambda - 4k^2)x + 2 - ak^2 = 0$

Comparing coefficients of x:

$b - 4k^2 = 0$ ..... (i)

$2 - ak^2 = 0$

$k^2 = \frac{2}{\mu}$..... (ii)

Eqn. (ii) into eqn. (i)

$b - 4\left(\frac{2}{a}\right) = 0$

⇨ $ab - 8 = 0$

∴ $ab = 8$

Revision exercise 2

1. Given that f(x) $3x^3 - 4x^2 - 5x + 2$, factorize f(x) completely and hence solve the equation f(x) = 0 $\left[x = -1, 2 \; or \; \frac{1}{3}\right]$
2. Prove that a – b is a factor of $a^2(b - c) + b^2(c - a) + c^2(a - b)$ and write down two other factors of the expression. Hence or otherwise factorize the expression completely
   [-(a – b)(b – c)(c – a) or (b – a))(b – c)(c – a)]
3. The polynomial $ax^3 + bx^2 - cx - 2$ is divisible by x +2. When divided by x – 1 it leaves a remainder of 18 and when divided by x + 3, it leaves a remainder -50. Determine the values of a, b, and c. Hence factorize the polynomial completely
   [a = 6, b = 13, c = -1; (x + 2)(2x +1)(3x-1)]
4. Factorize completely
   $f(xyz) = (x + y)^3(x - y) + (y + z)^3(y - z) + (z + x)^3(z - x)$
   [(x + y + z)(x – y)(y – z)(z – x)
5. When the quadratic expression ax2 + bx + c is divided by x – 1, x – 2 and x + 1, the remainders are 1, 1, and 25 respectively, determine the factors of the expression
   [(2x – 3) and (2x – 3)]
6. The remainder when $px^3 + 2x^2 - 5x + 7$ is divided by x – 2 is equal to the remainder when the same expression is divided by x + 1. Find the value of p [p = 1]
7. Given that x – 4 is a factor of $2x^3 - 3x^2 - 7x + k$, where k is a constant, find the remainder when the expression is divided by 2x -1 [k = - 52, remainder = -56]
8. The expression $px^3 + qx^2 + 3x + 8$ leaves a remainder of -6 when divided by x – 2 and a remainder of -34 when divided by x + 2. Find the values of constants p and q. [p = 1, q = -7]
9. Show that x – 2 is a factor of $x^3 - 9x^2 + 26x - 24$. Find the set of values of x for which $x^3 - 9x^2 + 26x - 24 < 0$
   [x < 2 and 3< x < 4]
10. The remainder when $x^3 - 2x^2 + kx + 5$ is divided by x – 3 is twice when the same expression is divided by x + 1. Find the value of the constant k [k = -2]

Thank you

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Quadratic equations

These are equations expressed in the form $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$

They have at most two roots which may be real or complex.

The latter roots are handled exclusively under complex numbers

Example of quadratic equations

$2y^2 + 3y + 5 = 0$; a = 2, b = 3 and c = 5

$x^2 + 4x - 10 = 0$; a = 1, b = 4, c = -10

Forming quadratic equations

Suppose that the roots of a quadratic equation are α and β,

then $x - \alpha = 0$ and $x - \beta = 0$

When forming a quadratic equation

$(x - \alpha)(x - \beta) = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

This means that if the roots of a quadratic equation are given, its equation in terms of x is x^2-(sum of roots)x + product of roots = 0

Example 1

Form quadratic equation I terms of x with roots

(a) (2, 3)
Solution
Let x = 2 or x = 3
$x - 2 = 0$ $x - 3 = 0$
⇨ $(x - 2)(x - 3) = 0$
$x^2 - 5x - 6 = 0$

(b) (p, q)
Solution
Let x = p or x = q
$x - p = 0$ $x - q = 0$
⇨ $(x - p)(x - q) = 0$
$x^2 - (p + q)x - pq = 0$

(c) (-3, -2)
Solution
Let x = -3 or x = -2
$x + 3 = 0$ $x + 2 = 0$
⇨ $(x + 3)(x + 2) = 0$
$x^2 + 5x + 6 = 0$

Example 2

Given that the root of the equation $x^2 + px + q = 0$ are α and β, express $(\alpha - \beta^2)(\beta - \alpha^2)$ in terms of p and q. Deduce that for one root to be the square of another $p^3 - 3pq + q^2 + q = o$ must hold

Solution

For $x^2 + px + q = 0$

$\alpha + \beta = -p$

$\alpha\beta = q$

$(\alpha - \beta^2)(\beta - \alpha^2) = \alpha\beta - \alpha^3 - \beta^3 + (\alpha\beta)^2$

$= \alpha\beta - (\alpha^3 + \beta^3) + (\alpha\beta)^2$

But $(\alpha^3 + \beta^3) = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$(\alpha - \beta^2)(\beta - \alpha^2)$

$= \alpha\beta - [(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)] + (\alpha\beta)^2$

$= q - [(-p)^3 - 3q(-p)] + q^2$

$= q + p^3 - 3pq + q^2$

$(\alpha - \beta^2)(\beta - \alpha^2) = p^3 - 3pq + q^2 + q$

If $\alpha = \beta^2$

$(\beta^2 - \beta^2)(\beta - \beta^4) = p^3 - 3pq + q^2 + q$

$0 = p^3 - 3pq + q^2 + q$

Or

$p^3 - 3pq + q^2 + q = 0$ As required

Example 3

Given that the root of the equation $x^2 + px + q = 0$ are α and β, Form quadratic equations with roots

(a) $\frac{\alpha}{\beta} and \frac{\beta}{\alpha}$

Solution

Sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

Product of roots $= \frac{\alpha}{\beta} x \frac{\beta}{\alpha} = 1$

Required equation

$x^2 - \left(\frac{p^2-2q}{q}\right)x + 1 = 0$

Or $qx^2 - (p^2 - 2q)x + q = 0$

(b) pα + qβ and qα + pβ

Solution

Sum of roots

pα + qβ + qα + pβ = p(α +β) + q(α +β)

=(p + q)(α +β)

=p(p + q)

Product of roots= (pα + qβ)(qα + pβ)

$= pq\alpha^2 + p^2\alpha\beta + q^2\alpha\beta + pq\beta^2$

$= pq(\alpha^2 + \beta^2) + \alpha\beta(p^2 + q^2)$

$= pq[(\alpha + \beta)^2 - 2\alpha\beta] + \alpha\beta(p^2 + q^2)$

$= pq(p^2-2q) + q(p^2 + q^2)$

Required equation

$x^2 - p(p + q)x + pq(p^2 - 2q) + q(p^2 + q^2) = 0$

Example 4

Given the equation $x^3 + x - 10 = 0$.

(a) Show that x = 2 is a root of the equation

Let $f(x) = x^3 + x - 10$

Substituting for x = 2

$f(2) = 2^3 + x - 10$

$= 8 + 2 - 10$

$= 10 - 10 = 0$

Hence x = 2 is a root of $x^3 + x - 10 = 0$

(b) Deduce the values of α + β and αβ where α and β are roots of the equation.

Hence form a quadratic equation whose roots are α^2 and β^2.

⇨ $x^3 + x - 10 = (x-2)(x^2 + 2x + 5)$

Either x – 2 = 0

Or $(x^2 + 2x + 5) = 0$

α + β = 2

αβ = 5

Sum of roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-2)^2 - 2(5) = 4 - 10 = -6$

Product= $\alpha^2\beta^2 = (\alpha\beta)^2 = 5^2 = 25$

The equation become

$x^2 - (-6)x + 25 = 0$

$x^2 + 6x + 25 = 0$

Solving quadratic equations

Quadratic equations may be solved by

(a) Factorization method
(b) Completing square method
(c) Graphical method

(a) Factorization method

It is used for quadratic equations that are easy to factorise

Example 5

(a) $4x^2 + 7x + 3 = 0$

Solution	side work
$4x(x + 1)+ 3(x + 1) = 0$	Product = 4 x 3 = 12
$(x + 1)(4x + 3) = 0$	Sum = 7
Either $x + 1 = 0$; $x = -1$	Factor = (4, 3)
Or $4x + 3 = 0$; $x = -\frac{3}{4}$	

(b) $2x^2 + 5x + 3 = 0$

Solution	side work
$2x(x + 1)+ 3(x + 1) = 0$	Product = 3 x 2 = 6
$(x + 1)(2x + 3) = 0$	Sum = 5
Either $x + 1 = 0$; $x = -1$	Factors 3, 2
Or $2x + 3 = 0$; $x = -\frac{3}{2}$	

(c) $x^2 + x – 20 = 0$

Solution	Side work
$x(x – 4) + 5(x – 4) = 0$	Product = -20
$(x – 4)(x + 5) = 0$	Sum = 1
Either $x – 4 = 0$; $x = 4$	Factors (5, - 4)
Or $x + 5 = 0$; $x = -5$	

(d) $10x^2 + x – 3 = 0$

Solution	Side work
$5x(2x -1) + 3(2x – 1) = 0$	product $10 x – 3 = -30$
$(5x + 3)(2x – 1) = 0$	sum = 1
Either $5x + 3 = 0$; $x = -\frac{3}{5}$	Factors (6, -5)
Or $2x – 1 = 0$; $x = \frac{1}{2}$	

(b) Method II: Completing squares approach

The idea is to create a perfect square on one side of the equation:

Given the equation $ax^2 + bx + c = 0$

- Dividing the equation by and transposing the constant term to the RHS
  $x^2 + \frac{b}{a}x = -\frac{c}{a}$
- Marking the LHS a perfect square, add a half the coefficient of x squared on both sides
  $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$
- Factorise the terms on the LHS
- $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$
- Taking square root on both sides of the equation
- $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
- Solving $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
- $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This is the general quadratic equation formula for finding the square root of any quadratic equation. This formula is locally known as ***bull dozer formula***

Example 6

Solve the following equations by completing squares

(a) $2x^2 – x – 3 = 0$

Solution

$2x^2 – x – 3 = 0$

$x^2 - \frac{1}{2}x = \frac{3}{2}$

$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = \frac{3}{2} + \left(-\frac{1}{4}\right)^2$

$\left(x - \frac{1}{4}\right)^2 = \frac{25}{16}$

$x - \frac{1}{4} = \sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Either $x = \frac{5}{4} + \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Or $x = -\frac{5}{4} + \frac{1}{4} = \frac{-4}{4} = -1$

Hence x = -1 and $x = \frac{3}{2}$

(b) $18x^2 + 7x – 1 = 0$

Solution

$18x^2 + 7x – 1 = 0$

$x^2 + \frac{7}{18}x = \frac{1}{18}$

$x^2 + \frac{7}{18}x + \left(\frac{7}{36}\right)^2 = \frac{1}{18} + \left(-\frac{7}{36}\right)^2$

$\left(x + \frac{7}{36}\right)^2 = \frac{1}{18} + \frac{49}{1296} = \frac{121}{1296}$

$x + \frac{7}{36} = \sqrt{\frac{121}{1296}} = \pm \frac{11}{36}$

Either $x = \frac{11}{36} - \frac{7}{36} = \frac{4}{36} = \frac{1}{9}$

Or $x = -\frac{11}{36} - \frac{7}{36} = -\frac{18}{36} = -\frac{1}{2}$

Hence x = $\frac{1}{9}$ or x =$-\frac{1}{2}$

(c) $3x^2 + 7x + 2 = 0$

Solution

$3x^2 + 7x + 2 = 0$

$x^2 + \frac{7}{3}x = -\frac{2}{3}$

$x^2 + \frac{7}{3}x + \left(\frac{7}{6}\right)^2 = -\frac{2}{3} + \left(\frac{7}{6}\right)^2$

$\left(x + \frac{7}{6}\right)^2 = -\frac{2}{3} + \frac{49}{36} = \frac{25}{36}$

$x + \frac{7}{6} = \sqrt{\frac{25}{36}} = \pm\frac{5}{6}$

Either x =$\frac{5}{6} - \frac{7}{6} = -\frac{2}{6} = -\frac{1}{3}$

Or x =$-\frac{5}{6} - \frac{7}{6} = -\frac{12}{6} = -2$

Hence x = -2 and x =$-\frac{1}{3}$

Example 7

Solve the following equations by using the quadratic formula

(a) $7x^2 - 5x - 2 = 0$

Solution

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{-5^2-4\ x\ 7\ x-2}}{2\ x\ 7} = \frac{5 \pm 9}{14}$

Either $x = \frac{14}{14} = 1$ or $x = \frac{-4}{14} = -\frac{2}{7}$

Hence $x = 1\ and\ x = -\frac{2}{7}$

(b) $3x^2 - 7x - 6 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{7 \pm \sqrt{-7^2-4\ x\ 3\ x-6}}{2\ x\ 3} = \frac{7 \pm 11}{6}$

Either $x = \frac{18}{6} = 3$ or $x = \frac{-4}{6} = -\frac{2}{3}$

Hence $x = 3\ and\ x = -\frac{2}{3}$

(c) $6x^2 - 5x - 6 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{-5^2-4\ x\ 6\ x-6}}{2\ x\ 6} = \frac{5 \pm 13}{12}$

Either $x = \frac{18}{12} = \frac{3}{2}$ or $x = \frac{-8}{12} = -\frac{2}{3}$

Hence $x = \frac{3}{2}\ and\ x = -\frac{2}{3}$

Method 3: Graphical approach

Here, the roots of the equations are established by plotting suitable graphs. It may done by:-

(i) Single graph plot
(ii) Two graph plot

Single graph method

Given the function y = f(x), tabulate the selected values of x closer to zero within a given range which must be substituted in the function given to obtain the corresponding values of y. These values are plotted on the same x and y axes and joined by a curve. The roots of the function f(x) = 0 are the values of x where the function y = f(x) crosses the x-axis.

Two graphs method

Here the function y = f(x) is split into two and the equations plotted separately on the same axes. The points of intersection of the two graphs are noted and the corresponding values of x are read off on the x – axis.

Example 8

Solve the following equation by using graphical approach

(a) $2x^2 + 3x - 3 = 0$

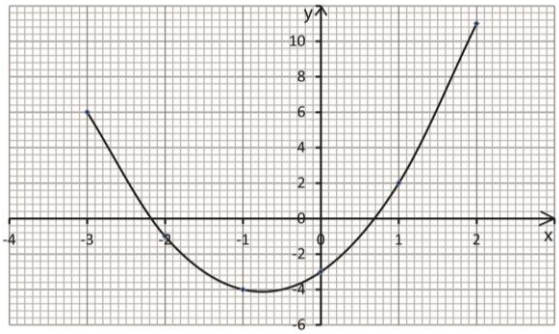
Solution

Using a single graphs

Let y = $2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



From the graph, the roots of the equation are x = -2.2 and x = 0.7

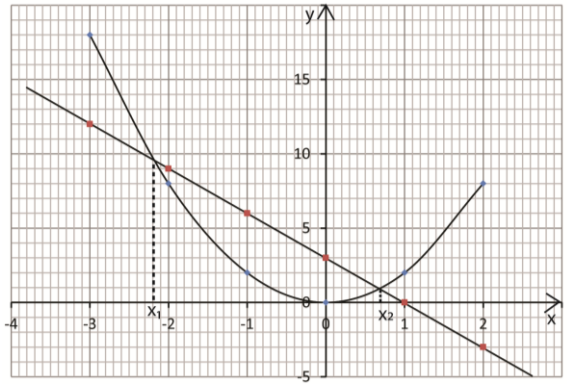
Using two graphs

By splitting the equation $2x^2 + 3x – 3 = 0$ into two we have $2x^2 = -3x + 3$

Let $y_1 = 2x^2$ and $y_2 = 3 - 3x$

Table of results

X	-3	-2	-1	0	1	2
y_1	18	8	2	0	2	8
y_1	12	9	6	3	0	-3



From the graph, the roots of the equation are x = -2.2 and x = 0.7

(b) $x^2 + x – 6 = 0$

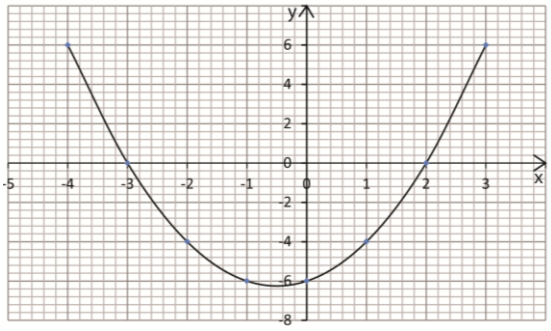
Solution

Using a single graphs

Let $y = x^2 + x – 6 = 0$, taking $-4 \leq x \leq 3$

Table of values

X	-4	-3	-2	-1	0	1	2	3
y	6	0	-4	-6	-6	-4	0	6



From the graph, the roots of the equation are x = -3 and x = 2

Using two graphs

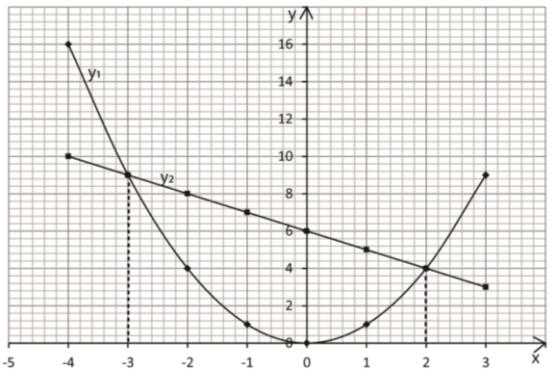
By splitting the equation $x^2 + x – 6 = 0$

into two we have $x^2 = 6 - x$

Let $y_1 = x^2$ and $y_2 = 6 - x$

Table of results

X	-4	-3	-2	-1	0	1	2	3
y_1	16	9	4	1	0	1	4	9
y_1	10	9	8	7	6	5	4	3



From the graph, the roots of the equation are x = -3 and x = 2

Minimum and maximum values of quadratic expression

The methods of completing squares and graphing can be used to obtain minimum and maximum values of quadratic expression

Completing square method

The general form of quadratic equation $y = ax^2 + bx + c$ can be expressed as

$y = ax^2 + bx + c$

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac-b^2}{4a}\right)$$

The minimum/maximum value is the constant $\frac{4ac-b^2}{4a}$ which is attained when $x + \frac{b}{2a} = 0$

Note

- if a < 0, the value of the function is maximum
- if a > 0, the value of the function is minimum

Example 9

Determine the minimum or maximum values of the following expressions using completing squares

(a) $2x^2 - x - 3 = 0$

Solution

Let $y = 2x^2 - x - 3 = 0$

$$y = 2\left(x^2 + \frac{2}{2}x - \frac{3}{2}\right)$$

$$y = 2\left(x^2 + x + \left(\frac{1}{2}\right)^2 - \frac{3}{2} - \left(\frac{1}{2}\right)^2\right)$$

$$y = 2\left(\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}\right)$$

$$y = 2\left(x + \frac{1}{2}\right)^2 - \frac{7}{4}$$

Since a > 0, the function has got a minimum value at $x = -\frac{1}{2}$. Hence $y_{min} = -\frac{7}{4}$

(b) $-4 + 6x - x^2$

Let $y = -4 + 6x - x^2$

$y = -(x^2 - 6x + 4)$

$y = -(x^2 - 6x + (-3)^2 + 4 - (-3)^2)$

$y = -((x - 3)^2 + 4 - 9)$

$y = -((x - 3)^2 + -5)$

Since a = -1 < 0, the expression has got a maximum value when x = 3. Hence $y_{max} = 5$

Using graphical method

After graphing, y =f(x) the minimum value of the expression is lowest point if a trough/valley like or curving upwards and the maximum value is the maximum point if downwards

Example 10

Determine the maximum or minimum values of the following expression using graphical method

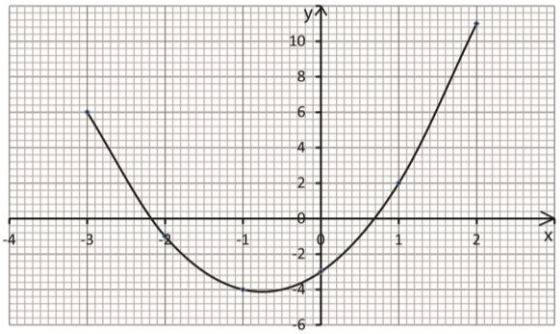
(a) $2x^2 + 3x - 3 = 0$

Solution

Let $y = 2x^2 + 3x - 3$ taking $-3 \leq x \leq 2$

Table of values

X	-3	-2	-1	0	1	2
y	6	-1	-4	-3	2	11



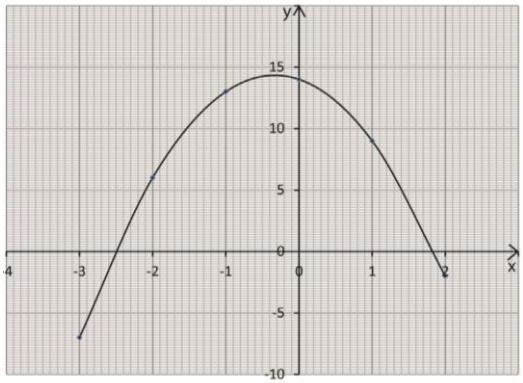
y_{min} = - 4.1 at -0.75

(b) $14 - 2x - 3x^2$

Ley $y = 14 - 2x - 3x^2$; $-3 \leq x \leq 2$

Table of results

x	-3	-2	-1	0	1	2
y	-7	6	13	14	9	-2



y_{max} = 14.3 at x =-0.3

Revision exercise

1. Given that the roots of the equation $x^2 + px + q = 0$ are α and β, find the values of:
   (a) $\alpha^2 + \beta^2 [p^2 - 2q]$
   (b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \left[\frac{p^2-2q}{q}\right]$
   (c) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} \left[\frac{p}{q}(3q - p^2)\right]$
   (d) $\alpha - \beta \left[\pm\sqrt{p^2 - 4q}\right]$
   (e) $\alpha^2 - \beta^2 \left[-p\sqrt{(p^2 - 4q)}\right]$
   (f) $\alpha^3 + \beta^3 [p(3q - p^2)]$
2. If the roots of the equation $x^2 + 2x + 3 = 0$ are α and β, form an equation whose roots are $\alpha^2 - \beta$ and $\beta^2 - \alpha$. [$x^2 + 2 = 0$]
3. Given that the roots of the equation $x^2 - 2x + 10 = 0$ are α and β, form an equation whose roots are

   $\frac{1}{(2+\alpha)^2}$ and $\frac{1}{(2+\beta)^2}$ $[324x^2 + 1 = 0]$
4. If α and β are roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\frac{\alpha^3-1}{\alpha}$ and $\frac{\beta^3-1}{\beta}$
   $[qx^2 - (p^2q - 2q^2 - p)x + (q^3 - p^3 + 3pq) + 1 = 0]$
5. The roots of the equation $3x^2 - ax + 6b = 0$ are α and β. Find the condition for one root to be
   (a) Twice the other [$81b = a^2$)
   (b) The cube of the other
   $[a^4 - 648(b - 1)b^2 - 18(4a^2 + 9)b = 0]$
6. By Factorization method solve the following quadratic equations
   (a) $x^2 + 9x + 14$ [x = -7, x= -2]
   (b) $x^2 + 2x - 8 = 0$ [x = -4, x = 2)
   (c) $2x^2 + x - 10 = 0$ [x= 2, x = $-\frac{5}{2}$]
   (d) $6x^2 - 19x + 10 \left[x = \frac{5}{2},\ x = \frac{2}{3}\right]$
7. Solve the following equations by completing squares
   (a) $2x^2 + 5x + 3 = 0 \left[x = -1\ and\ x = -\frac{3}{2}\right]$
   (b) $x^2 + 9x + 20 = 0$ [x = -5 and x = -4]
   (c) $x^2 + x - 20 = 0$ [x = -5 and x = 4]
   (d) $x^2 - x - 20 = 0$ [x = 4 and x = 5]
8. Solve the following equations using the quadratic formula
   (a) $2x2 + 5x + 3 = 0 \left[\frac{-3}{2}, -1\right]$
   (b) $x^2 + 9x + 20$ [-5, -4]
   (c) $x^2 + x - 20$ [-5, 4]
9. Determine the maximum or minimum values of the following expression
   (a) $3x^2 - 2x + 1 \left[y_{min} = \frac{2}{3} at\ x = \frac{1}{3}\right]$
   (b) $4 - x - x^2 \left[y_{max} = \frac{33}{8} at\ x = -\frac{1}{4}\right]$
10. Determine the maximum or minimum values of the following expression using graphical method
    (a) $14 - 2x - 2x^2$

Thank you

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Surds

These are irrational numbers which cannot be expressed in terms of $\frac{a}{b}$ where a and b are rational. Irrational numbers may be defined as square roots of prime numbers.

Examples are $\sqrt{2}, \sqrt{3}, \sqrt{5}$ ….

Expression of root of numbers in surd form

Example 1

Write the following as the simplest surds

(i) $\sqrt{32}$ (ii) $\sqrt{50}$ $(iii)\sqrt{8}$ $(iv)\sqrt{27}$

Solution

(i) $\sqrt{32} = \sqrt{16\ x\ 2} = \sqrt{16}\ x\ \sqrt{2} = 4\sqrt{2}$
(ii) $\sqrt{50} = \sqrt{25\ x\ 2} = \sqrt{25}\ x\ \sqrt{2} = 5\sqrt{2}$
(iii) $\sqrt{8} = \sqrt{4\ x\ 2} = \sqrt{4}\ x\ \sqrt{2} = 2\sqrt{2}$
(iv) $\sqrt{27} = \sqrt{9\ x\ 3} = \sqrt{9}\ x\ \sqrt{3} = 3\sqrt{3}$

Addition and subtraction of surds

This is done by expressing the surds in their simplest form

Example 2

(i) $\sqrt{75} - 3\sqrt{27} + 2\sqrt{12}$

$= \sqrt{25\ x\ 3} - 3\sqrt{9\ x\ 3} + 2\sqrt{4\ x\ 3}$

$= \sqrt{25}\ x\ \sqrt{3} - 3x\sqrt{9}x\ \sqrt{3} + 2\ x\ \sqrt{4}x\sqrt{3}$

$= 5\ x\ \sqrt{3} - 3\ x\ 3\ x\ \sqrt{3} + 2\ x\ 2\ x\sqrt{3}$

$(5 - 9 + 4)\sqrt{3} = 0$

(ii) $\sqrt{50} + \sqrt{2} - 3\sqrt{18} + 2\sqrt{8}$
$= \sqrt{25\ x\ 2} + \sqrt{2} - 3\sqrt{9x\ 2} + 2\sqrt{4\ x\ 2}$
$=5\sqrt{2} + \sqrt{2} - 3\ x\ 3\sqrt{2} + 2x\ 2\sqrt{2}$

$= (5 + 1 - 9 + 4)\sqrt{2} = \sqrt{2}$

Multiplication of surds

Finding the product of two surd numbers is the same as finding the root of the product of two numbers.

i.e. $\sqrt{a}\ x\ \sqrt{b} = \sqrt{ab}$

Example 3

Find the value of the following and give your answers in the simplest form

(i) $\sqrt{2}\ x\ \sqrt{2}$
(ii) $\sqrt{2}\ x\ \sqrt{20}$
(iii) $\left(3\sqrt{2} - 2\sqrt{3}\right)^3$

Solution

(i) $\sqrt{2}\ x\ \sqrt{2} = \sqrt{2x2} = \sqrt{4} = 2$
(ii) $\sqrt{2}\ x\ \sqrt{30} = \sqrt{2x30} = 2\sqrt{15}$
(iii) $\left(3\sqrt{2} - 2\sqrt{3}\right)^3$

Using Pascal's triangle; the coefficients of the terms in the expansion $(a + b)^3$ are 1 3 3 1

$\left(3\sqrt{2} - 2\sqrt{3}\right)^3 =$

$(3\sqrt{2})^3 + 3(3\sqrt{2})^2(-2\sqrt{3}) + 3(2\sqrt{3})(-2\sqrt{3})^2 + (-2\sqrt{3})^3$

$=(54\sqrt{2}) + 3(18)(-2\sqrt{3}) + 3(12)(3\sqrt{2}) - 24\sqrt{3}$

$= 162\sqrt{2} - 132\sqrt{3}$

Division of surds

There are two types of division of surds;

- A fraction whose denominators has a single term such as $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}$, etc.

- A fraction whose denominator has double terms, such as $\frac{1}{\sqrt{2}+\sqrt{3}}, \frac{2+\sqrt{3}}{2-\sqrt{3}}, \frac{1}{1+\sqrt{5}}$, etc.
- In both cases, first eliminate the surds from the denominator. The process of eliminating surds from the denominator is called rationalization.
- In the first case rationalize the fraction by multiplying the numerator and denominator by the surd term of the denominator.

Example 4

Rationalize the following

(i) $\frac{2}{3\sqrt{5}}$
(ii) $\frac{1}{\sqrt{2}}$
(iii) $\frac{5}{\sqrt{3}}$

Solution

(i) $\frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{3\sqrt{5}\,x\,\sqrt{5}} = \frac{2\sqrt{5}}{3\,x\,5} = \frac{2\sqrt{5}}{15}$
(ii) $\frac{1}{\sqrt{2}} = \frac{1\,x\sqrt{2}}{\sqrt{2}\,x\,\sqrt{2}} = \frac{\sqrt{2}}{2}$
(iii) $\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3}\,x\,\sqrt{3}} = \frac{5\sqrt{3}}{3}$

In the second case rationalize by multiplying the numerator and denominator by the conjugate of the denominator.

Note

(i) The conjugate a + $\sqrt{b}$ is a - $\sqrt{b}$ and that a - $\sqrt{b}$ is a + $\sqrt{b}$
(ii) The product of a surd function and its conjugate is equal to the difference of two squares. i.e. (a + $\sqrt{b}$)(a - $\sqrt{b}$) = $\left(a^2 - \left(\sqrt{b}\right)^2\right)$

Example 5

Rationalize the following

(i) $\frac{2}{2-\sqrt{2}}$
(ii) $\frac{1}{3-\sqrt{5}}$
(iii) $\frac{3-\sqrt{5}}{\sqrt{5-3}}$

solution

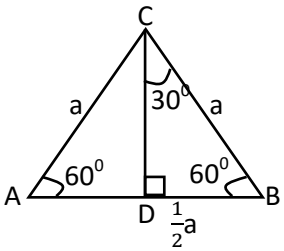
(i) $\frac{2}{2-\sqrt{2}} = \frac{2(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{2(2+\sqrt{2})}{4-2} = (2+\sqrt{2})$

(ii) $\frac{1}{3-\sqrt{5}} = \frac{1(3+\sqrt{5})}{(3-\sqrt{5})\left((3+\sqrt{5})\right)} = \frac{(3+\sqrt{5})}{9-5} = \frac{(3+\sqrt{5})}{4}$

(iv) $\frac{3-\sqrt{5}}{\sqrt{5-3}}x\ \frac{\sqrt{5+3}}{\sqrt{5+3}} = \frac{(3)^2-\left(\sqrt{5}\right)^2}{\left(\sqrt{5}\right)^2-(3)^2} = \frac{9-5}{5-9} = -1$

Set square angle (30^0, 45^0 and 60^0)

(a) Consider an equilateral triangle ABC of each side = a units



$ED^2 = a^2 - \left(\frac{1}{2}a\right)^2 = \frac{3a^2}{4}$

EC=$\frac{a\sqrt{3}}{2}$

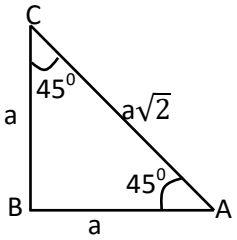
cos 60^0 = $\frac{a}{2a} = \frac{1}{2}$; sin$60^0$ =$\frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$

tan 60^0= $\frac{\sqrt{3}}{2}\ x\ \frac{2}{1} = \sqrt{3}$

cos 30^0 = $\frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$; sin$60^0$ =$\frac{a}{2a} = \frac{1}{2}$

tan 60^0= $\frac{1}{2}\ x\ \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

(b) Given a right angled isosceles triangle ABC with equal perpendicular sides each of length a units



$AC^2 = a^2 + a^2 = 2a^2$
$AC = a\sqrt{2}$
cos 45^0 = sin45^0 = $\frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$
tan 45^0 = $\frac{a}{a} = 1$

Example 6

Express without a surd in the denominator each of the following

(a) $\frac{1+\tan 30^0}{1-\tan 30^0}$

(b) $\left(\frac{1+\cos 45^0}{2-\sin 60^0}\right)^2$

Solution

(a) $\frac{1+\tan 30^0}{1-\tan 30^0} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$

$= \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$

(b) $\left(\frac{1+\cos 45^0}{2-\sin 60^0}\right)^2 = \left(\frac{1+\frac{\sqrt{2}}{2}}{2-\frac{\sqrt{3}}{2}}\right)^2 = \left(\frac{2+\sqrt{2}}{4-\sqrt{3}}\right)^2$

$= \frac{4+4\sqrt{2}+2}{16-8\sqrt{3}+3}$

$= \left(\frac{6+4\sqrt{2}}{19-8\sqrt{3}}\right)\left(\frac{19+8\sqrt{3}}{19+8\sqrt{3}}\right)$

$= \frac{114+48\sqrt{3}+76\sqrt{2}+32\sqrt{6}}{169}$

Revision exercise

1. Simplify
   (a) $\sqrt{48}$ [$4\sqrt{3}$]
   (b) $\sqrt{162}$ [$9\sqrt{2}$]
   (c) $\sqrt{28}$ [$2\sqrt{7}$]
   (d) $\sqrt{45}$ [$3\sqrt{5}$]
   (e) $\sqrt{125}$ [$5\sqrt{5}$]
   (f) $\sqrt{147}$ [$7\sqrt{3}$]
2. Simplify the following suds
   (a) $\sqrt{8}+\sqrt{200}-4\sqrt{18}$ [$2\sqrt{2}$]
   (b) $5\sqrt{20}+2\sqrt{45}+2\sqrt{5}$ [$18\sqrt{5}$]
   (c) $3\sqrt{50}+2\sqrt{32}-2\sqrt{75}+2\sqrt{12}-\sqrt{27}$ [$23\sqrt{2}-7\sqrt{3}$]
3. Given that $4\sqrt{20}+3\sqrt{5}-5\sqrt{125}=x\sqrt{5}$, find the value of x [-14]
4. Find the value of the following simplifying the answer as much as possible
   (a) $(5\sqrt{2}-\sqrt{5})(3\sqrt{5}-2\sqrt{2})$[$5+13\sqrt{10}$]
5. Express each of the following in the form $\frac{a\sqrt{b}}{c}$ where a, b and c are integers
   (a) $\frac{2}{\sqrt{7}}$ $\left[\frac{2\sqrt{7}}{7}\right]$
   (b) $\frac{3}{\sqrt{2}}$ $\left[\frac{3\sqrt{2}}{2}\right]$
   (c) $\frac{14\sqrt{5}}{\sqrt{7}}$ $\left[2\sqrt{35}\right]$
   (d) $\frac{3\sqrt{50}}{5\sqrt{27}}$ $\left[\frac{\sqrt{6}}{3}\right]$
   (e) $\frac{4\sqrt{45}}{5\sqrt{8}}$ $\left[\frac{3\sqrt{10}}{5}\right]$
6. Express the following in the form a +b$\sqrt{c}$
   (a) $\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$ $\left[-5+2\sqrt{6}\right]$
   (b) $\frac{4+3\sqrt{2}}{4-3\sqrt{2}}$ $\left[-17-12\sqrt{2}\right]$
   (c) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}+1}$ $\left[\frac{1}{2}-\frac{1}{4}\sqrt{2}+\frac{1}{4}\sqrt{6}\right]$
7. Show that $\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)^2+\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^2$=194
8. Express without a surd in the denominator each of the following
   (a) $\frac{1}{1-\sin 30^0}$ [2]
   (b) $\frac{1+\tan 60^0}{1-\tan 60^0}$ [-2-$\sqrt{3}$]
9. Express each of the following in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c and d are integers
   (a) $\frac{3}{6+\sqrt{3}}$ $\left[\frac{6-\sqrt{3}}{11}\right]$
   (b) $\frac{3+\sqrt{2}}{5-\sqrt{2}}$ $\left[\frac{17+8\sqrt{2}}{23}\right]$
   (c) $\frac{3+\sqrt{24}}{2+\sqrt{6}}$ $\left[\frac{6-\sqrt{6}}{2}\right]$
10. Solve the equation $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{x-2}$ [3]
11. Without using mathematical tables or calculators, find the value of $\frac{(\sqrt{5}+2)^2-(\sqrt{5}-2)^2}{8\sqrt{5}}$ [1]

Thank you

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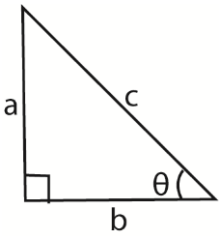
Trigonometry

The word 'trigonometry' suggests 'tri'-three, 'gono'-angle, 'metry'-measurement. As such, trigonometry is basically about triangles, most especially right-angled triangles.

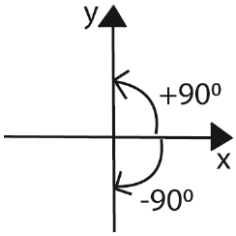
Important to note

(a) For a right angled triangle below

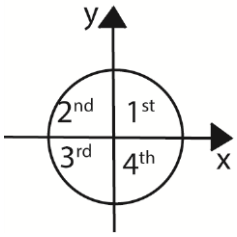


- $\sin\theta = \frac{a}{c}$ $\quad \text{cosec}\theta = \frac{1}{\sin\theta} = \frac{c}{a}$
- $\cos\theta = \frac{b}{c}$ $\quad \sec\theta = \frac{1}{\cos\theta} = \frac{c}{b}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}\frac{a}{b}$ • $\quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{b}{a}$

(b) All positive angles are measured anticlockwise from positive x-axis

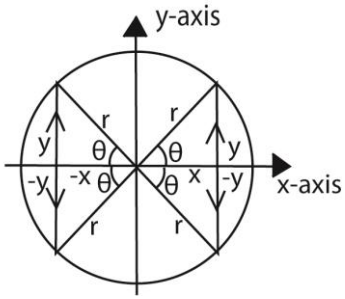


(c) A circle drawn with the centre O, divides the co-ordinate axis into four equal parts called quadrants



The quadrants are also labelled anti-clockwise from the positive x – axis.

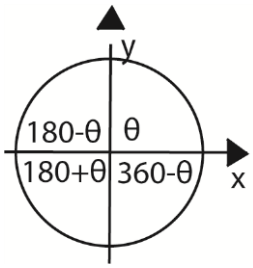
The signs the trigonometric ratios in the quadrants are given below



Ratio	Quadrant			
	1st	2nd	3rd	4th
cosθ	$\frac{+x}{r}$	$\frac{-x}{r}$	$\frac{-x}{r}$	$\frac{+x}{r}$
sinθ	$\frac{+y}{r}$	$\frac{+y}{r}$	$\frac{-y}{r}$	$\frac{-y}{r}$
tanθ	$\frac{+y}{x}$	$\frac{y}{-x}$	$\frac{y}{x}$	$\frac{-y}{x}$
secθ	$\frac{+r}{x}$	$\frac{-r}{x}$	$\frac{-r}{x}$	$\frac{+r}{x}$
cosecθ	$\frac{+r}{y}$	$\frac{+r}{y}$	$\frac{-r}{y}$	$\frac{-r}{y}$
cotθ	$\frac{+x}{y}$	$\frac{-x}{y}$	$\frac{+x}{y}$	$\frac{-x}{y}$

Note

- If θ is the angle in the 1st quadrat
- In the 2nd quadrat the angle is (180 – θ)
- In the 3rd quadrat the angle is (180 + θ)
- In the 4th quadrat the angle is (360 –θ)



Solving equations

We make use of the quadrants to find the ranges of values within which the angle follows

Example 1

Solve the following equations for $0^0 \leq \theta \leq 360^0$

(i) $3\cos\theta + 2 = 0$

Solution

$\cos\theta = -\frac{2}{3}$

But cos is negative in the 2nd and 3rd quadrants.

Ignoring the negative sign, the angle obtained is referred to as the key or principle angle, i.e.

key angle = $\cos^{-1}\frac{2}{3} = 48.2^0$ (1d.p)

In the 2nd quadrant, $\theta = 180 - 48.2 = 131.8^0$

In the 3rd quadrant, $\theta = 180 + 48.2 = 228.2^0$

$\therefore\{\theta: \theta = 131.8^0, 228.2^0\}$

Note that: the key angle s not part of the solution but only a guide.

(ii) $4\cos^2\theta - 1 = 0$

Solution

$\cos\theta = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

Key angle, $\theta = \cos^{-1}\frac{1}{2} = 60^0$

When $\cos\theta$ = ½ (positive is 1st and 4th quadrants)

1st quadrant $\theta = 60^0$

4th quadrant $\theta = 360 - 60 = 300^0$

When $\cos\theta$ = -½ (positive is 2nd and 3rd quadrants)

3rd quadrant $\theta = 180 - 60 = 120^0$

4th quadrant $\theta = 180 + 60 = 240^0$

$\therefore\{\theta: \theta = 60^0, 120^0, 240^0, 300^0\}$

(iii) $\text{cosec}\theta + 2 = 0$

Solution

$\text{cosec}\theta = -2 \Rightarrow \sin\theta = -\frac{1}{2}$ (taking reciprocal)

Key angle = $\sin^{-1}\frac{1}{2} = 30^0$

In the 3rd quadrant $\theta = 180 + 30 = 210^0$

In the 4th quadrant $\theta = 360 - 30 = 330^0$

$\therefore\{\theta: \theta = 210^0, 330^0\}$

(iv) $3\sec^2\theta - 4 = 0$

Solution

$\sec\theta = \pm\frac{2}{\sqrt{3}} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$

Key angle = $\cos^{-1}\frac{\sqrt{3}}{2} = 30^0$

For $\cos\theta = \frac{\sqrt{3}}{2}$; $\theta = 30^0, 330^0$

For $\cos\theta = -\frac{\sqrt{3}}{2}$; $\theta = 120^0, 210^0$

$\therefore\{\theta: \theta = 30^0, 120^0, 210^0, 330^0\}$

(d) Definitions of angle

(i) **Acute angle** is an angle between 0^0 and 90^0. It lies in the 1st quadrant

(ii) **Right angle** is an angle = 90^0

(iii) **Obtuse angle** is an angle between 90^0 and 180^0. It lies in the 2nd quadrant

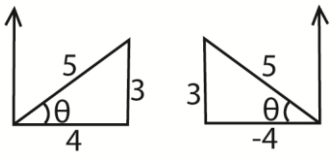
(iv) **Reflex angle** is an angle between 180^0 and 360^0. It lies in the 3rd and 4th quadrant

Example 2

(a) If $\sin\theta = \frac{3}{5}$ and $0^0 \leq \theta \leq 360^0$. Find the possible values of $3\tan\theta - \cot\theta$

Solution

If $\sin\theta = \frac{3}{5}$; θ lies in 1st or 2nd quadrants



In 1st quadrant

$3\tan\theta - \cot\theta = 3\left(\frac{3}{4}\right) - \left(\frac{4}{3}\right) = \frac{11}{12}$

In 2nd quadrant

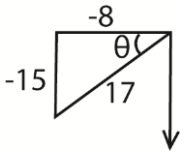
$3\tan\theta - \cot\theta = 3\left(-\frac{3}{4}\right) - \left(-\frac{4}{3}\right) = -\frac{11}{12}$

∴ the possible values are $\pm\frac{11}{12}$

(b) If $\cos\theta = -\frac{8}{17}$ and θ is reflex, find the value of $4\sec^2\theta + \tan\theta$

Solution

If $\cos\theta = -\frac{8}{17}$ and θ is reflex, θ lies in the 3rd quadrant



$4\sec^2\theta + \tan\theta = 4\left(-\frac{17}{8}\right)^2 + \frac{15}{8} = \frac{319}{16}$

Example 3

Solve for θ, where $\theta^0 \le \theta \le 360^0$

(i) $3\tan^2 3\theta = 1$

Solution

$\tan 3\theta = \pm\frac{1}{\sqrt{3}}$

taking $\tan 3\theta = \frac{1}{\sqrt{3}}$

⇨ $3\theta = 30^0, 210^0, 390^0, 570^0, 750^0, 930^0$
$\theta = 10^0, 70^0, 130^0, 190^0, 250^0, 310^0$

taking $\tan 3\theta = -\frac{1}{\sqrt{3}}$

⇨ $3\theta = 150^0, 330^0, 510^0, 690^0, 870^0, 1050^0$
$\theta = 50^0, 110^0, 170^0, 230^0, 290^0, 350^0$

∴{θ: $\theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, 190^0, 230^0, 250^0, 290^0, 310^0, 350^0$}

Note

- If $\theta^0 \le \theta \le 360^0$ then $\theta^0 \le 3\theta \le 1080^0$
  [multiply the interval through by 3]

(ii) $2\cos 2\theta + \sqrt{3} = 0$

Solution

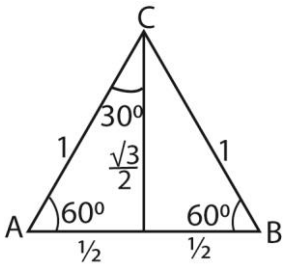
$\cos 2\theta = -\frac{\sqrt{3}}{2}$ and $\theta^0 \le 2\theta \le 720^0$

$2\theta = 150^0, 210^0, 510^0, 570^0$

∴{θ: $\theta = 75^0, 105^0, 255^0, 285^0$}

Set square angles: 30^0, 45^0, and 60^0

(i) From equilateral triangle ABC with side equal to 1 unit



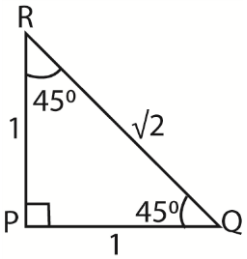
$\cos 60^0 = \sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \cot 60^0 = \frac{1}{\sqrt{3}}$

$\tan 60^0 = \cot 30^0 = \sqrt{3}$

(ii) From the right angled triangle PQR below



$\cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}$

$\tan 45^0 = 1$

Example 4

Without using tables or calculators find the value of

(i) $\cos 240^0$

Solution

$\cos 240^0 = -\cos(240 - 180)^0 = -\cos 60^0 = -\frac{1}{2}$

(ii) $\tan 3990^0$

Solution

$\tan 3990^0 = \tan[(360 \times 11) + 30]^0 = \tan 30^0 = \frac{1}{\sqrt{3}}$

(iii) $\sin 570^0$

Solution

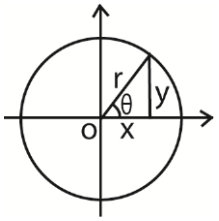
$\sin 570^0 = \sin\{(360 \times 1) + 210]^0 = -\sin 30 = -\frac{1}{2}$

(iv) $\sec 225^0$

Solution

$\sec 225^0 = \sec(225 - 180)0 = \sec 45^0 = -\sqrt{2}$

The Pythagoras theorem



For any acute angle θ

$x = r\cos\theta$ and $y = r\sin\theta$

By Pythagoras theorem

$x^2 + y^2 = r^2$

Substituting for x and y

$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\therefore \cos^2\theta + \sin^2\theta = 1$

Now $\tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$

$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$

Identities

$\cos^2\theta + \sin^2\theta = 1$ ……(i)

Identity (i) ÷ $\cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ …… (ii)

Identiy (i) ÷ $\sin^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$ ……(iii)

Example 5

Show that

(i) $\sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

Solution

$\sin^2\theta + (1 + \cos\theta)^2$

$= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta$

$= \sin^2\theta + \cos^2\theta + 1 + 2\cos\theta$

$= 1 + 1 + 2\cos\theta$ (Recall that $\sin^2\theta + \cos^2\theta = 1$)
$= 2 + 2\cos\theta = 2(1 + \cos\theta)$

$\therefore \sin^2\theta + (1 + \cos\theta)^2 = 2(1 + \cos\theta)$

(ii) $\frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

Solution

$$\frac{1+sin\theta}{1+cos\theta} \cdot \frac{1+sec\theta}{1+cosec\theta} = \frac{1+sin\theta}{1+cos\theta} \cdot \frac{1+\frac{1}{cos\theta}}{1+\frac{1}{sin\theta}}$$

$$= \frac{1+sin\theta}{1+cos\theta} \cdot \frac{\frac{cos\theta+1}{cos\theta}}{\frac{sin\theta+1}{sin\theta}}$$

$$= \frac{1+sin\theta}{1+cos\theta} \cdot \frac{cos\theta+1}{cos\theta} \div \frac{sin\theta+1}{sin\theta}$$

$$= \frac{1+sin\theta}{1+cos\theta} \cdot \frac{cos\theta+1}{cos\theta} x \frac{sin\theta}{sin\theta+1}$$

$$= \frac{sin\theta}{cos\theta} = tan\theta$$

$\therefore \frac{1+\sin\theta}{1+\cos\theta} \cdot \frac{1+\sec\theta}{1+\csc\theta} = \tan\theta$

(iii) $(\tan\theta + \sec\theta)^2 = \frac{1+sin\theta}{1-sin\theta}$

Solution

$$(\tan\theta + \sec\theta)^2 = \left(\frac{sin\theta}{cos\theta} + \frac{1}{cos\theta}\right)^2 = \left(\frac{sin\theta+1}{cos\theta}\right)^2$$

$$= \frac{(1+sin\theta)^2}{cos^2\theta} = \frac{(1+sin\theta)^2}{1-sin^2\theta}$$

$= \frac{(1+sin\theta)(1+sin\theta)}{(1+sin\theta)(1-sin\theta)} = \frac{1+sin\theta}{1-sin\theta}$

$\therefore(\tan\theta + \sec\theta)^2 = \frac{1+sin\theta}{1-sin\theta}$

Example 6

Solve the following equations for
$-180^0 \leq x \leq 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

$[\theta: \theta = -150^0, -30^0, 90^0$ for given range]

(ii) $\cos\theta + \sqrt{3}\sin\theta = 1$

Solution

1st approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\text{Cos}\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

2nd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\cos\theta$

$\sqrt{3}\tan\theta = \sec\theta - 1$

Squaring both sides

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3\tan^2\theta = \sec^2\theta - 2\sec\theta + 1$

$3[\sec^2\theta - 1] = \sec^2\theta - 2\sec\theta + 1$

$2\sec^2\theta + 2\sec\theta - 4 = 0$

$\sec^2\theta + \sec\theta - 2 = 0$

$(\sec\theta + 2)(\sec\theta - 1) = 0$

$\sec\theta = -2$ or $\sec\theta = 1$

$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\therefore[\theta: \theta = 0^0, \pm 120^0]$

3rd approach

$\sqrt{3}\sin\theta = 1 - \cos\theta$

Dividing through by $\sin\theta$

$\sqrt{3} = \text{cosec}\theta - \cot\theta$

Rearranging

$\sqrt{3} + \cot\theta = \text{cosec}\theta$

Squaring both sides

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = \text{cosec}^2\theta$

$3 + 2\sqrt{3}\cot\theta + \cot^2\theta = 1 + \cot^2\theta$

$\cot\theta = -\frac{1}{\sqrt{3}}$; $\Rightarrow \tan\theta = -\sqrt{3}$

$\therefore[\theta: \theta = -60^0, 120^0]$

Example 7

(a) Given that $7\tan\theta + \cot\theta = 5\sec\theta$, derive a quadratic equation for $\sin\theta$. Hence or otherwise, find all values of θ in the interval $0^0 \leq \theta \leq 180^0$ which satisfy the equation, giving your answer to the nearest 0.10 where necessary

Solution

$7\tan\theta + \cot\theta = 5\sec\theta$

$7\frac{sin\theta}{cos\theta} + \frac{cos\theta}{sin\theta} = \frac{5}{cos\theta}$

$7\frac{sin^2\theta + cos^2\theta}{cos\theta sin\theta} = \frac{5}{cos\theta}$

$7\sin^2\theta + \cos^2\theta = 5\sin\theta$

$7\sin^2\theta + (1 - \sin^2\theta) = 5\sin\theta$

$6\sin^2\theta - 5\sin\theta + 1 = 0$

$(3\sin\theta - 1)(2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{3}$	$\sin\theta = \frac{1}{2}$
$\theta = 19.5^0, 160.5^0$	$\theta = 30^0, 150^0$

$\therefore[\theta: \theta = 19.5^0, 30^0, 150^0, 160.5^0]$

Example 8

Find the solution of $3\cot\theta + \operatorname{cosec}\theta = 2$ for $0^0 \le \theta \le 180^0$.

Solution

$3\cot\theta + \operatorname{cosec}\theta = 2$

$3\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = 2$

$(3\cos\theta + 1)^2 = (2\sin\theta)^2$

$9\cos^2\theta + 6\cos\theta + 1 = 4\sin^2\theta$

$9\cos^2\theta + 6\cos\theta + 1 = 4(1 - \cos^2\theta)$

$13\cos^2\theta + 6\cos\theta - 3 = 0$

$\cos\theta = \frac{-6 \pm \sqrt{6^2 + 4x3x13}}{2x13}$

$\cos\theta = 0.3021$	$\cos\theta = 0.7637$
$\theta = 72.40$	$\theta = 40.2$

$\therefore[\theta: \theta = 72.4^0, 40.2^0]$

Elimination of trigonometric parameter

This involves the use of identities to eliminate the trigonometric values in equation

Example 9

(a) If $x = \tan\theta + \sec\theta$ and $y = \tan\theta - \sec\theta$; show that $xy + 1 = 0$

Solution

$x + y = \tan\theta$

$x - y = 2\sec\theta$

$\sec\theta = \frac{1}{2}(x - y)$

Using identity: $1 + \tan^2\theta = \sec^2\theta$

$1 + (x + y)\ 2 = \left[\frac{1}{2}(x - y)\right]^2$

$4 + x^2 + 2xy + y^2 = x^2 - 2xy + y^2$

$4xy + 4 = 0$

$xy + 1 = 0$ as required

(b) $x = 2 + 3\sin\theta$ and $y = 3 + 2\cos\theta$ show that $4(x - 2)^2 + (y - 3)^2 = 36$

Solution

$x = 2 + 3\sin\theta \Rightarrow \sin\theta = \frac{x-2}{3}$

$y = 3 + 2\cos\theta \Rightarrow \cos\theta = \frac{y-3}{2}$

Using identity $\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$

$4(x - 2)^2 + (y - 3)^2 = 36$ as reqyured

(c) $x = 2\sin\theta$ and $y = \tan\theta$, prove that $x = \pm\frac{2y}{\sqrt{(1+y^2)}}$

Solution

$x = 2\sin\theta; \Rightarrow \operatorname{cosec}\theta = \frac{2}{x}$

$y = \tan\theta; \Rightarrow \cot\theta = \frac{1}{y}$

Using identity: $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 + \left(\frac{1}{y}\right)^2 = \left(\frac{2}{x}\right)^2$

$x = \pm\frac{2y}{\sqrt{(1+y^2)}}$

Revision exercise 1

1. Solve for θ, where $\theta^0 \le \theta \le 360^0$

(a) $\sec\theta\operatorname{cosec}\theta + 2\sec\theta - 2\operatorname{cosec}\theta - 4 = 0$
[$\theta: \theta = 60^0, 210^0, 300^0, 330^0$]

(b) $\tan^2\theta - (\sqrt{3} + 1)\tan\theta + \sqrt{3} = 0$
[$\theta: \theta = 45^0, 60^0, 225^0, 240^0$]

2. Show that

(a) $\frac{1-\cos\theta+\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta+\sin\theta}{\sin\theta}$

(b) $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$

(c) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

(d) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

(e) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

3. Solve the following equations for $-180^0 \le x \le 180^0$

(i) $2\cos^2\theta + \sin\theta - 1 = 0$
[$\theta: \theta = -150^0, -30^0, 90^0$]

(ii) $\sin 2\theta + 5\cos 2\theta = 3$
[$\theta: \theta = \pm 45^0, \pm 135^0$]

(iii) $4\cot^2\theta + 24\operatorname{cosec}\theta + 39 = 0$

[θ: θ = 16.6^0, 23.6^0, 156.4^0, 163.4^0]

4. Solve each of the following equations in the stated range

(a) $4\cos^2\theta + 2\sin\theta = 4 \quad 0^0 \le \theta \le 360^0$
[θ: θ = 0^0, 48.6^0, 131.4^0, 180^0, 360^0]

(b) $2\sec^2\theta - 4\tan\theta - 2 = \quad -180^0 \le \theta \le 360^0$
[θ: θ = -135^0, -161.6^0, 18.4^0, 45^0]

(c) $5\cos^2 3\theta = 3(1 + \sin 3\theta), \quad 0^0 \le \theta \le 360^0$
[θ: θ = 7.9^0, 52.1^0, 90^0, 127.9^0, 172.1^0]

5. Solve for θ; $00 \le \theta \le 3600$

(a) $\tan\theta + 3\cot\theta = 4$
[θ: θ = 45^0, 71.6^0, 225^0,251.6^0]

(b) $4\cos\theta - 3\sin\theta = 2$
[θ: θ = 29.50, 256.70]

6. Solve

(a) $\cos\theta + \sqrt{3}\sin\theta = 2 \quad 0 \le \theta \le \pi$
$\left[\theta = \frac{\pi}{3}\right]$

(b) $2\cos\theta - \operatorname{cosec}\theta = 0 \quad 0^0 \le \theta \le 270^0$
[θ: θ = 45^0, 225^0]

(c) $2\sin^2\theta + 3\cos\theta = 0 \quad 0^0 \le \theta \le 360^0$
[θ: θ = 240^0, 120^0]

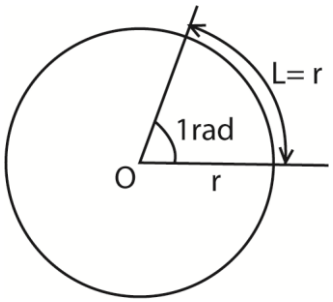
(d) $3\sin\theta + 4\cos\theta = 2 \quad -180^0 \le \theta \le 180^0$
[θ: θ = - 29.55^0, 103.29^0]

(e) $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$
[θ: θ = 38.66^0, 116.57^0]

7. Without using a tables or calculator , show that $\tan 15^0 = 2 - \sqrt{3}$
8. Solve equation
$8\cos^4\theta - 10\cos^2\theta + 3$ for $0^0 \le \theta \le 180^0$
[θ: θ = 30^0, 45^0, 135^0, 150^0]
9. Eliminate θ from the following equation
   (a) x = asecθ and y = b + ccosθ
   [ac = x(y – b)]
   (b) x = secθ +tanθ and y = secθ – tanθ
   [xy = 1]
10. Solve the simultaneous equation
Cos x + 4siny = 1
4secx – 3cosecy = 5 for values of x and y between 0^0 and 360^0
[x = 78.8^0, 281.5^0; y = 11.5^0, 168.5^0]
11. Prove each of the following identities
   (a) Sinxtanx + cosx =secx
   (b) Cosecx + tanxsecx = $\operatorname{cosecx}\sec^2 x$
   (c) Cosecx – sinx = cotxcosx
   (d) $(\sin x + \cos x)^2 - 1 = 2\sin x\cos x$
12. Eliminate θ from each of the following pairs of relationships
   (a) x =3sinθ, y = cosecθ [xy = 3]
   (b) 5x = sinθ, y = 2cosθ [$100x^2 + y^2 - 4 = 0$]
   (c) x = 3 + sinθ, y = cosθ [$(x-3)^2 + y^2 = 1$]
   (d) x = 2 + sinθ, cosθ = 1+y
   [$(x-2)^2 + (y+1)^2 = 1$]

Measuring angles in radians

A radian is defined as an angle subtended at the centre of a circle by an arc that is equal to the radius of the circle. One radian is represented by π, where $\pi = \frac{22}{7}$



How to convert between degrees and radians

1 revolution = circumference of a circle

But circumference of a circle subtends an angle 2π at the centre.

⇨ 1 revolution = $2\pi = 360^0$
$\pi = 180^0$
$1^0 = \frac{\pi}{180}$ radians
$x^0 = \frac{\pi}{180} x$ radians

Example 10

Convert the following angles to radians

(a) 330^0
(b) 90^0
(c) 30^0

Solution

(a) $330^0 = \frac{\pi}{180} x\ 330 = \frac{11\pi}{6}$ radians

(b) $90^0 = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ radians

(c) $30^0 = \frac{\pi}{180} x\ 30 = \frac{\pi}{6}$ radians

Converting radians to degrees

2π radians = 360^0

1 radian = $\frac{180^0}{\pi}$

x radians = $\frac{180^0}{\pi} x$

Example 11

Convert each of the following radians to degrees

(i) $\frac{\pi}{3}$ radians

(ii) $\frac{2\pi}{5}$ radians

(iii) π radians

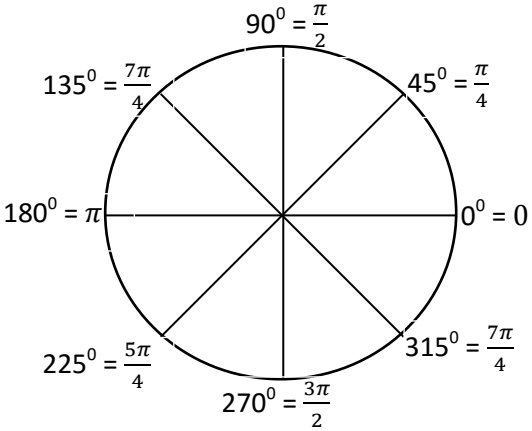
Solution

(i) $\frac{\pi}{3}$ radians = $\frac{180^0}{\pi}\ x \frac{\pi}{3} = 60^0$

(ii) $\frac{2\pi}{5}$ radians= $\frac{180^0}{\pi}\ x \frac{2\pi}{5} = 72^0$

(iii) π radians = $\frac{180^0}{\pi}\ x\ \pi = 180^0$

Some equivalent angles in degrees and radians



Example 12

Find each of the following values

(a) $\sin\left(\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{4\pi}{3}\right)$

(c) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Convert the angles from radian to degrees

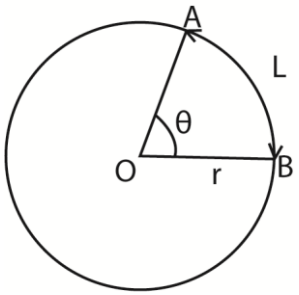
(a) $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\ x\ 180}{3}\right) = \sin 120^0 = \frac{\sqrt{3}}{2}$

(b) $\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{4\ x\ 180}{3}\right) = \cos 240^0 = -\frac{1}{2}$

(d) $tan\left(\frac{7\pi}{4}\right) = \tan\left(\frac{7x\ 180}{4}\right) = \tan 60^0 = \sqrt{3}$

Length of an arc

Suppose that the angle subtended by the length L of an arc AB of a circle is θ as shown.



$\frac{L}{\theta} = \frac{2\pi r}{2\pi}$

L = rθ where θ must be in radians

Example 13

Find the length of an arc of a circle of radius 14 if it subtends an angle

(i) $\frac{\pi}{4}$

(ii) 150^0

Solution

(i) L = rθ = $14\ x\ \frac{\pi}{4}$ = 11cm

(ii) Convert degrees to radians

$150^0 = \frac{\pi}{180} x 150 = \frac{5\pi}{6}$ radians

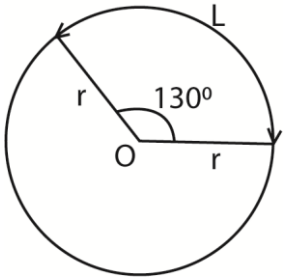
L = $14\ x\ \frac{5\pi}{6}$ = 36.67cm

Example 14

A sector was drawn which had a perimeter of 80cm, and centre angle of 130^0. Calculate the radius

Solution

The sides of a sector are composed of an arc, and two more sides which are radii of a circle.



2r + L = 80

L = 80 – 2r

Converting 130^0 to radians

$130^0 = \frac{\pi}{180} x 130 = \frac{13\pi}{18}$

But L = rθ

$80 - 2r = \frac{13\pi r}{18}$

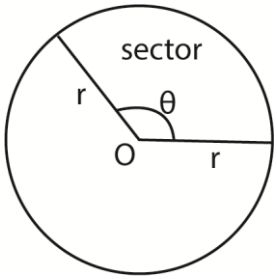
$2r + \frac{13\pi r}{18} = 80$

$\frac{(36+13\pi)r}{18} = 80$

r = 18.74cm

Area of a sector of a circle

A sector of a circle is a portion of the interior of a circle intercepted by a central angle.



The area of a sector of a circle of radius r and central angle θ is given by

$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \left(\frac{\theta}{2}\right) r^2$

Where θ must be in radians

Example 15

Find the area of a sector with radius 14cm and angle (i) $\frac{\pi}{4}$ (ii) 1200

Solution

(i) $A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{8}\right) . 14^2$=77cm^3

(ii) Converting 120^0 to radians

$120^0 = \frac{\pi}{180} x\ 120 = \frac{2\pi}{3}$

$A = \left(\frac{\theta}{2}\right) r^2 = \left(\frac{\pi}{3}\right) . 14^2$=205.25cm^3

Solving trigonometric functions whose range is in radians

When the range of the trigonometric function is in radians, the answer should be given in radians

Example 16

Solve the following equations for the ranges indicated

(i) cosθ + √3sinθ = 1 $0 \leq \theta \leq \pi$

Solution

√3sinθ = 1- cosθ

Squaring both sides

$3\sin^2\theta = 1 - 2\cos\theta + \cos^2\theta$

$3(1 - \cos^2\theta) = 1 - 2\cos\theta + \cos^2\theta$

$4\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\text{Cos}\theta = -\frac{1}{2}$	$\cos\theta = 1$
$\theta = \pm 120^0$	$\theta = 0^0$

$\pm 1200 = \pm\frac{\pi}{180} x\ 120 = \pm\frac{2\pi}{3}$ Radians

0^0 = 0 radians

$\therefore \left[\theta: \theta = 0, \pm\frac{2\pi}{3}\right]$

(ii) $2\cos^2\theta + \sin\theta - 1 = 0 \quad 0 \le \theta \le \pi$

Solution

$2(1 - \sin^2\theta) + \sin\theta - 1 = 0$

$2\sin 2\theta - \sin\theta - 1 = 0$

$(\sin\theta - 1)(2\sin\theta + 1) = 0$

Either $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$

When $\sin\theta = 1$; $\theta = 90^0$

When $\sin\theta = -\frac{1}{2}$; $\theta = -150^0, -30^0, 210^0, 330^0$

[θ: $\theta = \frac{\pi}{180} x\ 90 = \frac{\pi}{2}$ for given range]

Revision exercise 2

1. Express each of the following in radians
   (a) $30^0 \left[\frac{\pi}{6}\right]$
   (b) $45^0 \left[\frac{\pi}{4}\right]$
   (c) $120^0 \left[\frac{2\pi}{3}\right]$
   (d) $300^0 \left[\frac{5\pi}{3}\right]$

2. Express the following angle in degrees
   (a) $\frac{\pi}{3}$ rad $[60^0]$
   (b) $\frac{\pi}{8}$ rad $[22.5^0]$
   (c) 3π rad $[540^0]$
   (d) 5.2π rad $[936^0]$

3. A sector of the circle of radius 7 cm subtends an angle $\frac{\pi}{3}$ radians at the centre. Calculate the
   (a) Length of the arc $\left[6\frac{2}{3}cm\right]$
   (b) Perimeter of the sector $\left[20\frac{2}{3}cm\right]$
   (c) Area of the sector $\left[\frac{77}{3}cm^2\right]$

4. AOB is a sector of a circle, centre O, and is such that OA = OB = 7cm and angle AOB is 300. Calculate the
   (a) Perimeter of sector AOB $\left[17\frac{2}{3}cm\right]$
   (b) The area of AOB $\left[\frac{77}{6}cm^2\right]$

5. Find the value each of the following
   (a) $\sin\pi$ [0]
   (b) $\cos 3\pi$ [-1]
   (c) $\tan\frac{\pi}{3}$ $[\sqrt{3}]$

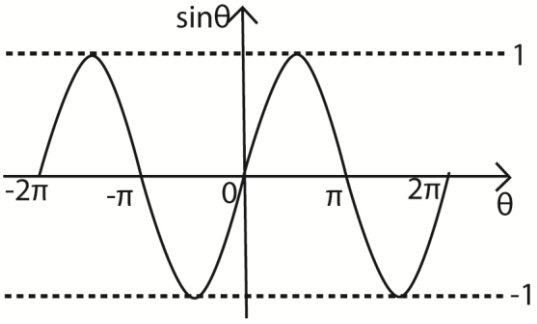
6. Solve the following equations for the ranges indicated
   (a) $2\sec^2\theta = 3 + \tan\theta$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.25\pi, 0.85\pi, 1.25\pi, 1.85\pi$]
   (b) $2\sin^2 x\cos x + \cos x - 1$ for $0 \le \theta \le 2\pi$
   [θ: $\theta = 0.38\pi, 1.62\pi, 2\pi$]
   (c) $2\tan\theta + 4\cot\theta = \operatorname{cosec}\theta$ for $-\pi \le \theta \le \pi$
   $\left[\theta: \theta = \pm\frac{1}{3}\pi, \pm 0.73\pi\right]$

Graphs of trigonometric functions

The following are the characteristic of the three major trigonometric functions

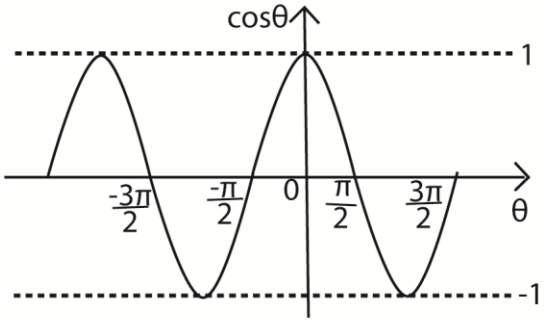
The sine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- The shape of the graph from $\theta = 0$ to $\theta = 2\pi$ is repeated every 2π radians
- This is called a periodic or cyclic function and the width of the repeating pattern that is measured on horizontal axis is called a **period**. The sine wave has a period of 2π, a maximum value of +1 and a minimum value of -1.
- The greatest value of sine wave is called the **amplitude**.



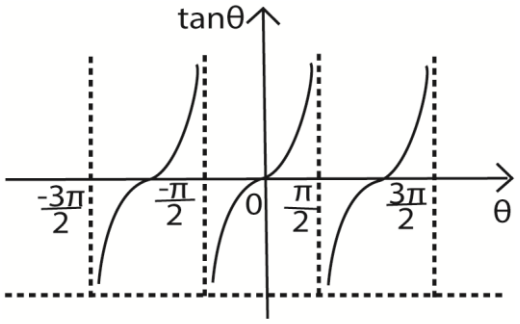
The coosine function

- It is continuous (with no breaks)
- The range $-1 \le \sin\theta \le 1$
- Has a period of 2π
- The shape is the same as the sine wave but displaced a distance $\frac{\pi}{2}$ to the left on the horizontal axis. This is called a **phase shift**



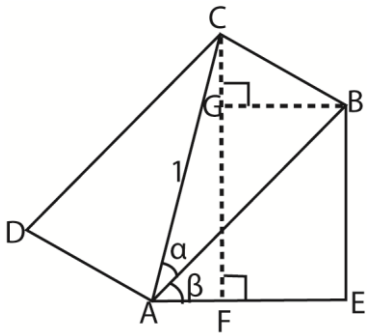
The tan function

- The tan function is found using; $tan\theta = \frac{sin\theta}{cos\theta}$. It follows that tanθ = 0 when sin θ = 0; and tanθ is undefined when cosθ = 0
- The graph is continuous, but undefined when θ= $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
- The range of values for tanθ is unlimited
- It has a period π



Compound angles

Consider a cardboard ABCD of unit diagonal that stands on the edge A, making an angle β with the horizontal ground. Let the unit diagonal AC be inclined at an angle α to the side AB (see diagram)



Angles EAB= ABG(Alternative angles)

∴ Angle ABG = β

Angle [ABG + GBC] = 90^0

∴ Angle GBC = 90 – β

From triangle GBC,

Angle BCG = 180 – (90 + 90 – β)

∴ Angle BCG = β

From

(1) Triangle ABC:
cosα = $\frac{AB}{AC} = \frac{AB}{1}$; => AB = cosα

(2) Triangle ABE:
cosβ = $\frac{AE}{AB} = \frac{AE}{cos\alpha}$; => AE = cosβcosα
sinβ = $\frac{BE}{AB} = \frac{BE}{cos\alpha}$; => BE = cosαsinβ

(3) Triangle BCG:
cosβ = $\frac{CG}{BC} = \frac{CG}{sin\alpha}$; => CG = sinαcosβ
sinβ = $\frac{BG}{BC} = \frac{BG}{sin\alpha}$; => BG = sinαsinβ

(4) Triangle ACF:
cos(α + β) = $\frac{AF}{AC} = \frac{AF-BG}{1}$ = AE –BG

∴ **cos(α + β) = cosαcosβ – sinαsinβ**

sin(α + β) = $\frac{CF}{AC} = \frac{CG-GF}{1}$ = CG + GF

∴ **sin(α + β) = sinαcosβ + cosαsinβ**

It follows that

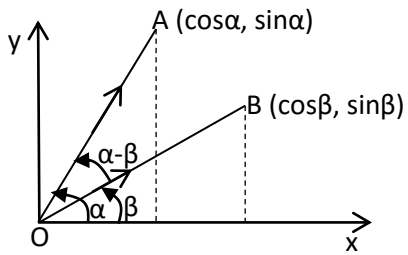
(i) cos(α + β) = cosαcosβ – sinαsinβ
(ii) cos(α - β) = cosαcosβ + sinαsinβ

[substituting –β for β)

(iii) **sin(α + β) = sinαcosβ + cosαsinβ**
(iv) **sin(α - β) = sinαcosβ - cosαsinβ**

[substituting –β for β)

These can also be derived using vector approach.
Consider two unit vectors $\underline{OA}$ and $\underline{OB}$ each inclined at angles α and β, respectively to the positive x-axis



Using the definition of a vector product:
$\underline{OA}.\underline{OB} = |\underline{OA}|.|\underline{OB}|\cos(\alpha - \beta)$

Since $\underline{OA}$ and $\underline{OB}$ are unit vectors,

$|\underline{OA}| = |\underline{OB}| = 1$

$\therefore \underline{OA}.\underline{OB} = \cos(\alpha - \beta)$

$\Rightarrow (\cos\alpha\underline{i} + \sin\alpha\underline{j}).(\cos\beta\underline{i} + \sin\beta\underline{j}) = \cos(\alpha - \beta)$

$\therefore \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

Substituting $90 - \alpha$ for α

$\cos(90 - \alpha)\cos\beta + \sin(90 - \alpha)\sin\beta$

$= \cos(90 - \alpha - \beta)$

$\therefore \sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$

Other expansions can be similar substitutions

i.e. $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$

$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

Dividing through by $\cos\alpha\cos\beta$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Similarly

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The following is a summary of compound angles

1. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
2. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
3. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
4. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
5. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
6. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Example 17

Calculate the value of $\sin15^0$ given that $\sin45^0 = \cos45 = \frac{1}{\sqrt{2}}$, $\sin30^0 = \frac{1}{2}$ and $\cos30^0 = \frac{\sqrt{3}}{2}$

$\sin15^0 = \sin(45^0 - 30^0)$

$= \sin45^0\cos30^0 - \sin30^0\cos45^0$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588$

Example 18

Prove that $\tan(45^0 + A) = \frac{1+\tan A}{1-\tan A}$

From $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan(45^0 + A) = \frac{\tan45^0 + \tan\beta}{1 - \tan45^0\tan\beta}$

$= \frac{1+\tan A}{1-\tan A}$

Example 19

Acute angles A and B are such that: $\cos A = \frac{1}{2}$, $\sin B \frac{1}{3}$. Show without using tables or calculator that $\tan(A + B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$

Solution

Using $\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{1}{2}\right)^2 + \sin^2 A = 1$

$\sin^2 A = \frac{3}{4} \Rightarrow \sin A = \frac{\sqrt{3}}{2}$

$\tan A = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$

Similarly;

$\cos^2 B + \left(\frac{1}{3}\right)^2 = 1$

$\cos B = \frac{2\sqrt{2}}{3}$

$\tan B = \frac{2\sqrt{2}}{3} \div \frac{1}{3} = \frac{1}{2\sqrt{2}}$

But

From $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\sqrt{3} + \frac{1}{2\sqrt{2}}}{1 - \sqrt{3}.\frac{1}{2\sqrt{2}}}$

$= \frac{(2\sqrt{2}\sqrt{3}+1)(2\sqrt{2}+\sqrt{3})}{(2\sqrt{2}-\sqrt{3})(2\sqrt{2}+\sqrt{3})}$

$\tan(A + B) = \frac{9\sqrt{3}+8\sqrt{2}}{5}$

Example 20

Solve $\cos(\theta + 35^0) = \sin(\theta + 25^0)$
for $0^0 \leq \theta \leq 360^0$

$\cos\theta\cos35^0 - \sin\theta\sin35^0 = \sin\theta\cos25^0 + \cos\theta\sin25^0$

Dividing through by $\cos\theta$

$\cos35^0 - \tan\theta\sin35^0 = \tan\theta\cos25^0 + \sin25^0$

$\tan\theta = \frac{cos35^0 - sin25^0}{cos35^0 + sin25^0} = \frac{0.3965337825}{1.479884223}$

$\theta = 15^0, 195^0$ for $0^0 \leq \theta \leq 360^0$

Example 21

(a) Prove than $\frac{2tan\theta}{1+tan^2\theta} = sin2\theta$

Solution

$$\frac{2tan\theta}{1+tan^2\theta} = \frac{2sin\theta}{cos\theta} \div \left(1 + \frac{sin^2\theta}{cos^2\theta}\right)$$

$$= \frac{2sin\theta}{cos\theta} \div \left(\frac{cos^2\theta + sin^2\theta}{cos^2\theta}\right)$$

$$= \frac{2sin\theta}{cos\theta} \div \left(\frac{1}{cos^2\theta}\right)$$

$$= 2sin\theta cos\theta = sin2\theta$$

(b) Solve $\sin2\theta = \cos\theta$ for $0^0 \leq \theta \leq 90^0$

Solution

$\sin2\theta = \cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$\sin\theta = \frac{1}{2}$

$\theta = 30^0$ for $0^0 \leq \theta \leq 90^0$

Example 22

Given that α, β and γ are angles of a triangle, show that $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

Hence find tanγ if tanα = 1 and tanγ = 2.

Solution

$\alpha + \beta + \gamma = 180^0$ (angle sum of a triangle)

$\tan(\alpha + \beta + \gamma) = \tan180^0 = 0$

$\tan[(\alpha + \beta) + \gamma] = 0$

$$\frac{tan(\alpha+\beta) + tan\gamma}{1 - tan(\alpha+\beta)\,tan\gamma} = 0$$

$$\Rightarrow \quad tan(\alpha + \beta) + tan\gamma = 0$$

$$\frac{tan\alpha + tan\beta}{1 - tan\alpha\, tan\beta} = -tan\gamma$$

$$tan\alpha + tan\beta = -tan\gamma + \tan\alpha\tan\beta\tan\gamma$$

$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

Example 23

In a triangle ABC, prove that

$$cot\frac{1}{2}A + cot\frac{1}{2}B + cot\frac{1}{2}C$$

$$= cot\frac{1}{2}Acot\frac{1}{2}Bcot\frac{1}{2}C$$

Solution

$$\frac{1}{2}(A + B + C) = \frac{1}{2}(180^0) = 90^0$$

$$cot\left[\frac{1}{2}(A + B + C)\right] = cot90^0 = 0$$

$$\Rightarrow \quad \frac{1 - tan\left(\frac{1}{2}A + \frac{1}{2}B\right)tan\frac{1}{2}C}{tan\left(\frac{1}{2}A + \frac{1}{2}B\right) + tan\frac{1}{2}C} = 0$$

$$1 = tan\left(\frac{1}{2}A + \frac{1}{2}B\right)tan\frac{1}{2}C$$

$$1 = \left(\frac{tan\frac{1}{2}A + tan\frac{1}{2}B}{1 - tan\frac{1}{2}Atan\frac{1}{2}B}\right)tan\frac{1}{2}C$$

$$1 - tan\frac{1}{2}Atan\frac{1}{2}B = tan\frac{1}{2}Atan\frac{1}{2}C + tan\frac{1}{2}Btan\frac{1}{2}C$$

$$1 = tan\frac{1}{2}Atan\frac{1}{2}B + tan\frac{1}{2}Atan\frac{1}{2}C + tan\frac{1}{2}Btan\frac{1}{2}C$$

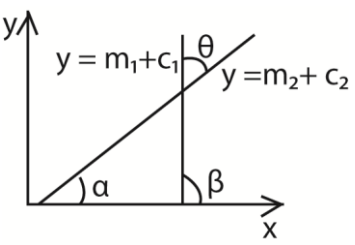
Dividing each side by $tan\frac{1}{2}Atan\frac{1}{2}Btan\frac{1}{2}C$

$$cot\frac{1}{2}Acot\frac{1}{2}Bcot\frac{1}{2}C = cot\frac{1}{2}A + cot\frac{1}{2}B + cot\frac{1}{2}C$$

Example 24

Prove that the angle θ, between the straight line $y = m_1x + c_1$ and the straight line $y = m_2x + c_2$ is given by $\tan\theta = \frac{m_2 - m_1}{1 + m_2m_1}$

Let the lines be inclines at angles α and β with the x-axis respectively



From the diagram above

θ = β – α

⇨ tanθ = tan(β – α)

$= \frac{tan\beta - tan\alpha}{1+tan\beta tan\alpha}$

$tan\theta = \frac{m_2 - m_1}{1+m_2 m_1}$

Revision exercise 3

1. (a) show that sin(α +β) - sin(α - β) = 2cosαsinβ
   (b) If sin(α +β) = 5cos(α - β) show that $tan\alpha = \frac{5-tan\beta}{1+tan\beta}$
   (c) Without using tables or calculator, show that $cos15^0 = sin75^0$
   (d) If α +β = 45^0, show that $tan\alpha = \frac{1-tan\beta}{1+tan\beta}$
2. Prove that:
   (i) $\frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} + 1 = \frac{(1+tan\beta)(1+cot\alpha)}{cos\alpha+tan\beta}$
   (ii) $tan\alpha - tan\beta = \frac{\sin(\alpha-\beta)}{cos\alpha cos\beta}$
   (iii) $cot\alpha + cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$
   (iv) $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{tan\alpha-tan\beta}{tan\alpha+tan\beta}$
   (v) $\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{cot\alpha cot\beta+1}{cot\alpha cot\beta-1}$
   (vi) $\cot(A+B) = \frac{cotAcotB-1}{cotA+cotB}$
3. (a) Determine solution of tan2x + 2sinx = 0 for $0^0 \le x \le 180^0$ [x: x= 0^0, 60^0, 120^0, 180^0]
   (vii) Show that in triangle ABC, tanA + tanB + tanC = tanAtanBtanC
4. Find the values of tan α for each of the following
   (a) sin(α - 30^0) = cos α [$\sqrt{3}$]
   (b) sin(α + 45^0) = cos α [$\sqrt{2}-1$]
   (c) cos(α + 60^0) = sin α [2 - √3]
   (d) sin(α + 60^0) = cos(α – 60^0) [1]
   (e) cos(α + 60^0) = 2cos(α + 30^0) [4 + 3√3]
   (f) sin(α + 60^0) = cos(45^0 – α)$\left[\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}\right]$
5. Given that
   (a) tan(α –β) = ½ and tanα = 3 find the value of tanβ [1]
   (b) tan(α +β) = 5 and tanβ = 2 find the value of tanα$\left[\frac{3}{11}\right]$
6. Given that
   (a) tan(θ – 45^0) = 4, find the value of θ $\left[-\frac{5}{3}\right]$
   (b) tan (θ + 60^0) find the value of cotθ $\left[8+5\sqrt{3}\right]$

Double angles and half angles

(b) From cos (θ + θ) = cosθcosθ – sinθsinθ

⇨ $cos2\theta = cos^2\theta - sin^2\theta$……. (i)

Either

$cos2\theta = cos^2\theta - 1 + cos^2\theta$ $(cos^2\theta + sin^2\theta = 1]$

$cos2\theta = 2cos^2\theta - 1$ …… (ii)

Or

$cos2\theta = 1 - sin^2\theta - sin^2\theta$

⇨ $cos2\theta = 1 - 2sin^2\theta$ ……(iii)

It follows that

$cos^2\theta = \frac{1}{2}(1 + cos2\theta)$ ……..(iv)

$sin^2\theta = \frac{1}{2}(1 - cos2\theta)$ ……..(iv)

The identities imply

$Cos6\theta = cos^2 3\theta - sin^2 3\theta$

$= 2cos^2 3\theta - 1 = 1 - 2\,sin^2 3\theta$

$Cos\,\theta = cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2}$

$= 2cos^2\frac{\theta}{2} - 1 = 1 - 2\,sin^2\frac{\theta}{2}$

(c) sin(θ + θ) = sinθcosθ + cosθsinθ

⇨ sin2θ = 2sinθcosθ

It follows that

sin6θ = 2sin3θcos3θ

$sin\theta = 2sin\frac{\theta}{2}cos\frac{\theta}{2}$

(d) tan (θ + θ)=$\frac{tan\theta+tan\theta}{1-tan\theta tan\theta}$

⇨ $tan2\theta = \frac{2tan\theta}{1-tan^2\theta}$

It follows that

$$tan\theta = \frac{2tan\frac{\theta}{2}}{1- tan^2\frac{\theta}{2}}$$

$$tan6\beta = \frac{2tan3\beta}{1- tan^23\beta}$$

Note that in all cases, the angles on the right hand side are half the angles on the left hand side [***half angle formulae]***

Example 25

Show that

(a) cosec2θ + cot2θ = cotθ

Solution

$cosec2\theta + cot2\theta = \frac{1}{sin2\theta} + \frac{cos2\theta}{sin2\theta}$

$= \frac{1+ cos2\theta}{sin2\theta}$

$= \frac{1+ 2cos^2\theta-1}{2sin\theta cos\theta}$

$= \frac{2cos^2\theta}{2sin\theta cos\theta} = cot\theta$

(b) $tan3\theta = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$

Hence deduce that if 3θ + α = 45^0, then tanα = $\frac{1-3tan\theta-3tan^2\theta+tan^3\theta}{1+2tan\theta-3tan^2\theta- tan^3\theta}$

Solution

tan3θ=tan(2θ + θ) = $\frac{tan2\theta+tan\theta}{1-tan2\theta tan\theta}$

$= \left\{\left(\frac{2tan\theta}{1-tan^2\theta}\right) + tan\theta\right\} \div \left\{1 - \left(\frac{2tan\theta}{1-tan^2\theta}\right) tan\theta\right\}$

$= \frac{2tan\theta+tan\theta-tan^3\theta}{1-tan^2\theta-2tan^2\theta} = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$

$\therefore tan3\theta = \frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}$

Hence 3θ + α = 45^0 => α = 45^0 - 3θ

Tanα = tan(45^0 - 3θ)

$= \frac{\tan 45^0-tan3\theta}{1+tan45^0 tan3\theta} = \frac{1-tan3\theta}{1+tan3\theta}$

$= \frac{1-\left(\frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}\right)}{1+\left(\frac{3tan\theta-tan^3\theta}{1-3tan^2\theta}\right)}$

$= \frac{1-3tan\theta-3tan^2\theta+tan^3\theta}{1+2tan\theta-3tan^2\theta- tan^3\theta}$

∴tanα = $\frac{1-3tan\theta-3tan^2\theta+tan^3\theta}{1+2tan\theta-3tan^2\theta- tan^3\theta}$

Example 26

If tanα = $\frac{3}{4}$ and α is acute, without using tables or calculator work out the value of

(a) tan2α

$$tan2\alpha = \frac{2tan\alpha}{1 - tan^2\alpha} = \frac{2x\frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{2}{2}}{\frac{3}{4}} = \frac{24}{7}$$

(b) tan$\frac{\alpha}{2}$

similarly $tan\alpha = \frac{2tan\frac{\alpha}{2}}{1-tan^2\frac{\alpha}{2}} = \frac{3}{4}$

⇨ $3tan^2\frac{\alpha}{2} + 8tan\frac{\alpha}{2} - 3 = 0$

$(3tan\frac{\alpha}{2} - 1)(tan\frac{\alpha}{2} + 3)$ =0

$tan\frac{\alpha}{2} = \frac{1}{2}\ or\ tan\frac{\alpha}{2} = -3$

Since α is acute, tanα cannot be negative

$\therefore tan\frac{\alpha}{2} = \frac{1}{3}$

Example 27

(a) Show that cos3α = $4cos^2$ α -3cos α. Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$ for $0^0 \le \alpha \le 180^0$

Solution

Cos3α = cos(2α +α)

= cos2αcosα – sin2αsinα

=($2cos^2$α -1)cosα – $2sin^2$αcosα

=($2cos^2$α -1)cosα – 2(1-cos^2α)cosα

= $2cos^3$α – cosα -2cosα + $2cos^3$α

=$4cos^2$ α -3cos α

Hence $4x^3 - 3x = \frac{\sqrt{3}}{3}$

i.e. $4cos^2$ α -3cos α =$\frac{\sqrt{3}}{3}$

$0^0 \le \alpha \le 180^0$; cos3α = $\frac{\sqrt{3}}{3}$

For the range $0^0 \le \alpha \le 180^0$

⇨ $0^0 \le 3\alpha \le 540^0$

$3\alpha = 54.7^0, 414.7^0$

$\alpha = 18.23^0, 138.23^0$ (2d.p)

[α: $\alpha = 18.23^0, 138.23^0$]

(b) Given that $t = \tan 22\frac{1}{2}$, show that $t^2 + 2t - 1 = 0$,

Hence show that $\tan 22\frac{1^0}{2} = -1 + \sqrt{2}$

Solution

$\tan 45^0 = \frac{2\tan 22\frac{1^0}{2}}{1-\tan^2 22\frac{1^0}{2}}$

$1 = \frac{2t}{1-t^2}$

$1 - t^2 = 2t$

$t^2 + 2t - 1 = 0$ (as required)

solving

$t = \frac{-2\pm\sqrt{2^2-(4x1\,x-1)}}{2x1}$

$t = \frac{-2\pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $22\frac{1^0}{2}$ is an acute angle,

$\tan 22\frac{1^0}{2} = -1 + \sqrt{2}$ is positive

$\therefore \tan 22\frac{1^0}{2} = -1 + \sqrt{2}$

Example 28

(a) Show that $3\sin\theta = 3\sin\theta - 4\sin^3\theta$. Hence solve the equation $\sin 3\theta + \sin\theta = 0$ for $0^0 \le \theta \le 360^0$

Solution

$\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta(1 - \sin^2\theta) + (1 - 2\sin^2\theta)\sin\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence $\sin 3\theta + \sin\theta = 0$

$3\sin\theta - 4\sin^3\theta + \sin\theta = 0$

$4\sin\theta - 4\sin^3\theta = 0$

$4\sin\theta(1 - \sin^2\theta) = 0$

$4\sin\theta(1 - \sin\theta)(1 + \sin\theta) = 0$

$\sin\theta = 0$; $\theta = 0^0, 180^0, 360^0$

$\sin\theta = 1$; $\theta = 90^0$

$\sin o = -1$; $\theta = 270^0$

$\therefore\theta$: $\theta = 0^0, 90^0, 180^0, 270^0, 360^0$

(b) Prove that $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$. Hence solve the equation $\cot 2\theta + 2\cot\theta = 2$ for $0^0 \le \theta \le 360^0$

Solution

$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$

dividing through by $\sin^2\theta$

$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

Hence, $\cot 2\theta + 2\cot\theta = 0$

$\frac{\cot^2\theta - 1}{2\cot\theta} + 2\cot\theta = 0$

$5\cot^2\theta - 4\cot\theta - 1 = 0$

$(5\cot\theta + 1)(\cot\theta - 1) = 0$

$\cot\theta = -\frac{1}{5}$ or $\cot\theta = 0$

⇨ $\tan\theta = -5$ ot $\tan\theta = 1$

When $\tan\theta = -5$; $\theta = 101.3^0, 281.3^0$

When $\tan\theta = 1$, $\theta = 45^0, 225^0$

$\therefore\{\theta$: $\theta = 45^0, 101.3^0, 225^0, 281.3^0\}$

Revison exercise 4

1. Prove that

(a) $\sin\alpha\operatorname{cosec}\beta + \cos\alpha\sec\beta = 2\sin(\alpha + \beta)\operatorname{cosec}2\beta$

(b) $\cos^6\theta + \sin^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$

(c) $\frac{\sin 3\alpha}{1+2\cos 2\alpha} = \sin\alpha$ and hence deduce that $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$

2. (a) Solve the equation for θ, $0^0 \le \theta \le 360^0$

$\sin^2\theta - 2\sin\theta\cos\theta - 3\cos^2\theta = 0$

[θ: $\theta = 71.6^0, 135^0, 251.6^0, 315^0$]

(b) show that $\frac{\cos\theta}{1+\sin\theta} = \cot(\frac{\theta}{2} + 45^0)$.

Hence or otherwise solve the equation $\frac{\cos\theta}{1+\sin\theta} = \frac{1}{2}$ for $0^0 \le \theta \le 360^0$ [$\theta = 36.8^0$]

3. (a) solve the equation $4\cos 2\theta - 2\cos\theta + 3 = 0$ for $0^0 \le \theta \le 360^0$

[θ: $\theta = 60^0, 104.5^0, 255.5^0, 300^0$]

(c) Solve the equation $\sin\theta + \sin\frac{\theta}{2} = 0$ for $0^0 \le \theta \le 360^0$

[θ: θ= -360^0, -240^0, 0^0, 2405^0, 360^0]

4. (a) Prove that $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan2\theta$
   (b) By expressing $2\sin\theta\sin(\theta+\alpha)$ as difference of cosines of two angles or otherwise, where α is constant, find its least value $\left[\frac{-a}{2}\right]$
   (c) Solve for θ in the equation $\cos\theta - \cos(\theta + 60^0) = 0.4$ for $0^0 \le \theta \le 360^0$ [θ = 126.4^0, 353.6^0]
5. (a) Show that $\cos3\theta = 4\cos^3\theta - 3\cos\theta$.
   Hence solve the equation $4x^3 - 3x - \frac{\sqrt{3}}{3} = 0$
   [x: x = -0.746, -0.204, 0.959]
   (b) Find all solutions of the equation $5\cos x - 4\sin x = 6$ in the range $-180^0 \le x \le 180^0$ [x: x = -59.1^0, -18.3^0]
6. (a) Express $\sqrt{\left(\frac{sin2\theta - cos2\theta - 1}{2 - 2sin2\theta}\right)}$ in terms of $\tan\theta\left[\frac{1}{\sqrt{(tan\theta - 1)}}\right]$
   (b) Find the solution of the equation $\sqrt{3sin\theta} - cos\theta + 1 = 0$ for $0 \le \theta \le 2\pi$
   $$\left[\theta: \theta = \frac{4}{3}\pi, 2\pi\right]$$
(c) Factorize $\cos\theta - \cos3\theta - \cos7\theta + \cos9\theta$ in form $A\cos p\theta\sin q\theta\sin r\theta$ where A, p, q and r are constants [A= -4, p =5, q = 5, r = 2]
7. (a) Given that $\sin\alpha + \sin\beta = p$ and $\cos\alpha + \cos\beta = q$ show that
   (i) $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{p}{q}$
   (ii) $\cos(\alpha + \beta) = \frac{q^2 - p^2}{q^2 + p^2}$
   (b) Solve the simultaneous equation:
   $\cos\alpha + 4\sin\beta = 1$
   $4\sec\alpha - 3\operatorname{cosec}\beta = 5$ [θ= 78.5^0, 281.5^0]
8. (a) Express $\sin\theta + \sin3\theta$ in form $p\cos\theta\sin q\theta$ where p and q are constant [p =2, q = 2]
   (b) Find the solution of $\cos7\theta + \cos5\theta = 2\cos\theta$ for $0^0 \le \theta \le 360^0$ [0^0, 60^0, 270^0, 360^0]
   (c) Prove that $\frac{sinA + sin4A + sin7A}{cosA + cos4A + cos7A} = tan4A$
9. Eliminate θ from each of the following pairs of expression
   (a) $x+1 = \cos2\theta$, $y = \sin\theta$ [$x + 2y^2 = 0$]
   (b) $x = \cos2\theta$, $y = \cos\theta - 1$ [$x = 2y^2 + 4y + 1$]
   (c) $y - 3 = \cos2\theta$, $x = 2 - \sin\theta$ [$2x^2 - 8x + y + 4 = 0$]
10. Solve the following equations for $-180^0 \le \theta \le 180^0$
   (a) $\sin2\theta + \sin\theta = 0$ [$\pm120^0$, $\pm180^0$]
   (b) $\sin2\theta - 2\cos^2\theta = 0$ [-135^0, 45^0, $\pm90^0$]
   (c) $3\cos2\theta + 2 + \cos\theta = 0$ [$\pm70.5^0$, $\pm120^0$]
   (d) $\sin2\theta = \tan\theta$ [0^0, $\pm45^0$, $\pm135^0$, $\pm180^0$]
11. Solve the following equations for $-360^0 \le \theta \le 360^0$, giving your answer correct to 1 decimal place
   (a) $\sin\theta = \sin\left(\frac{\theta}{2}\right)$ [0^0, $\pm120^0$, $\pm360^0$]
   (b) $3\cos\left(\frac{\theta}{2}\right) = 2\sin\theta$ [$\pm180^0$, 97.2^0, 262.8^0]
   (c) $2\sin\theta = \tan\left(\frac{\theta}{2}\right)$ [0^0, $\pm120^0$, $\pm240^0$, $\pm360^0$]
   (d) $2\cos\theta = 15\cos\left(\frac{\theta}{2}\right) + 2$ [$\pm209^0$]
12. Prove the following identities
   (a) $2\cos^2\theta - \cos2\theta = 1$
   (b) $2\operatorname{cosec}2\theta = \operatorname{cosec}\theta\sec\theta$
   (c) $2\cos^3\theta + \sin2\theta\sin\theta = 2\cos\theta$
   (d) $\tan\theta + \cot\theta = 2\operatorname{cosec}2\theta$
   (e) $\cos^4\theta - \sin^4\theta = \cos2\theta$
   (f) $\frac{1-cos2\theta}{1+cos2\theta} = tan^2\theta$
   (g) $\cot\theta - \tan\theta = 2\cot2\theta$
   (h) $\cot2\theta + \operatorname{cosec}\theta = \cot\theta$
   (i) $\frac{cos2\theta}{cos\theta + sin\theta} = cos\theta - sin\theta$
   (j) $\frac{sin2\theta}{1-cos2\theta} = cot\theta$
   (k) $cos2\theta = \frac{1-tan^2\theta}{1+tan^2\theta}$
   (l) $tan3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$
   (m) $tan\left(\frac{\theta}{2}\right) = \frac{sin\theta}{1+cos\theta}$
13. Express $\tan22\frac{1^0}{2}$ in the form $a + b\sqrt{2}$ where a and b are integers [a = -1, b= ±1]
14. Solve the equation
(i) $4\cos\theta - 2\cos2\theta = 3$ for $0 \le \theta \le \pi$ $\left[\frac{\pi}{3}\right]$
(ii) $\cos2\theta + \cos3\theta + \cos\theta = 0$ for $0^0 \le \theta \le 360^0$ [θ = 45^0, 120^0, 135^0]
(iii) $\cos\theta + \sin2\theta = 0$ for $0^0 \le \theta \le 360^0$ [θ = 90^0, 210^0, 270^0, 330^0]
(iv) $2\sin2\theta = 3\cos\theta$ for $-180^0 \le \theta \le 180^0$ [θ = -90^0, 48.6^0, 90^0, 132.4^0]
(v) $\sin\theta - 4\sin4\theta = \sin2\theta - \sin3\theta$ for $-\pi \le \theta \le \pi$ $\left[\frac{-\pi}{5}, \frac{-\pi}{2}, \frac{-3\pi}{5}, 0, \frac{\pi}{2}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$

Harmonic form

These are trigonometric functions expressed in the form of **Rcos(x ± α) and Rsin(x ±α).** They are in two ways

(i) solving equations in the form
acosθ + bsinθ + c = 0

(ii) determining the maximum and minimum values of the function
acosθ + bsinθ + c = 0
where a, b and c are constants

A: Solving equations

Example 29

(a) Express 3cosθ – 4sinθ in the form Rcos(θ + α), where R and α are constants
Solution
Let 3cosθ – 4sinθ = Rcos(θ + α)
= R(cosθcosα – sinθsinα)
= Rcosθcosα – Rsinθsinα
Comparing coefficient of cosθ and sinθ
Rcosα = 3 …….. (i)
Rsinα = 4 …… (ii)
Eqn (ii) ÷ eqn (i)
tanα = $\frac{4}{3}$; α = 53.1^0
Eqn. (i)^2 + eqn. (ii)^2
$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$
$R^2[\cos^2\alpha + \sin^2\alpha] = 25$
$R^2 = 25$
R = 5
∴$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

(b) Solve the equation 3cosθ – 4sinθ = 5 for $0^0 \le \theta \le 360^0$.

Solution

$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1^0)$

⇨ $5\cos(\theta + 53.1^0) = 5$
$\cos(\theta + 53.1^0) = 1$
$x + 53.1^0 = 0^0, 360^0$
$x = -53.1^0, 306.9^0$

Hence $x = 306.9^0$

Example 30

(a) Express sinθ -$\sqrt{3}$cosθ in the form Rsin(θ – α)
Solution
Let sinθ -$\sqrt{3}$cosθ = Rsin(θ – α)
=R(sinθcosα – cosθsinα)
Equating coefficients
Rcosα = 1 ……. (i)
Rsinα = $\sqrt{3}$…….. (ii)
Eqn. (ii) ÷ eqn. (i)
tanα = $\sqrt{3}$; => $\alpha = 60^0$
$R^2[\cos^2\alpha + \sin^2\alpha] = 4$
$R^2 = 4$; R = 2
∴ sinθ -$\sqrt{3}$cosθ = $2\sin(\theta - 60^0)$

(b) Solve the equation
sinθ -$\sqrt{3}$cosθ + 1 = 0 for $0^0 \le \theta \le 360^0$
sinθ -$\sqrt{3}$cosθ = $2\sin(\theta - 60^0)$
⇨ $2\sin(\theta - 60^0) + 1 = 0$

$\sin(\theta - 60^0) = -\frac{1}{2}$

$\theta - 60^0 = 210^0, 330^0$

$\theta = 270^0, 390^0$

Hence $\theta = 270^0$ for the given range

Example 31

(a) Express 4cosθ – 5sinθ in the form Acos(θ + β), where A is constant and β is an acute angle
Let 4cosθ – 5sinθ = Acos(θ + β)
= A(cosθcosβ – sinθsinβ)
= Acosθcosβ – Rsinθsinβ
Comparing coefficient of cosθ and sinθ
Acosβ = 4 …….. (i)
Asinβ = 5 …… (ii)
Eqn (ii) ÷ eqn (i)
tanα = $\frac{5}{4}$; $\alpha = 51.3^0$
Eqn. (i)^2 + eqn. (ii)^2
$A^2\cos^2\beta + A^2\sin^2\beta = 4^2 + 5^2 = 41$
$A^2[\cos^2\alpha + \sin^2\alpha] = 41$
$A^2 = 41$
A = $\sqrt{41}$
∴$3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

(b) Solve the equation 3cosθ – 4sinθ = 2.2 for $0^0 \le \theta \le 360^0$
Solution
$3\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 51.3^0)$

⇨ $\sqrt{41}\cos(\theta + 51.3^0) = 2.2$

$\cos(\theta + 51.3^0) = \frac{2.2}{\sqrt{41}} = 0.3436$

$(\theta + 51.3^0) = 69.9^0,\ 290.1^0$

$\therefore \theta = 18.6^0, 238.3^0$

B: Maximum and minimum values

The maximum and minimum values of a circular function may be obtained using three methods

(i) Express the given function either in for Rcos (θ ± α) or Rsin(θ ± α) if possible, where R and α are constants.
(ii) Differentiating the given function with respect to the given function say θ
(iii) Sketching the graphs of the function given and noting their maximum and minimum points.

In this chapter approach I will be considered.

Example 32

Determine the maximum and minimum values of the following, stating the value of θ for which they occur

(a) $\sqrt{3}\sin\theta + \cos\theta + 7$

Let $\sqrt{3}\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$= R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

Equating coefficients

$R\sin\alpha = 1$ ……. (i)

$R\cos\alpha = \sqrt{3}$…….. (ii)

Eqn. (i) ÷ eqn. (ii)

$\tan\alpha = \frac{1}{\sqrt{3}}$; => $\alpha = 30^0$

$R^2[\cos^2\alpha + \sin^2\alpha] = \left[1^2 + \left(\sqrt{3}\right)^2\right] = 2$

$R^2 = 4$; $R = 2$

$\therefore \sqrt{3}\sin\theta + \cos\theta = 2\sin(\theta + 30^0)$

⇨ $\sqrt{3}\sin\theta + \cos\theta + 7 = 2\sin(\theta + 30^0) + 7$

The minimum value occurs when

$\sin(\theta + 30^0) = -1$

⇨ Minimum value = 2(-1) + 7 = 5

Now for $\sin(\theta + 30^0) = -1$

$\theta + 30^0 = 270^0$

$\theta = 240^0 = \frac{4\pi}{3}$

The minimum value is $\left(\frac{4\pi}{3}, 5\right)$

And maximum value occurs when

$\sin(\theta + 30^0) = 1$

⇨ Minimum value = 2(1) + 7 = 9

Now for $\sin(\theta + 30^0) = 1$

$\theta + 30^0 = 90^0$

$\theta = 60^0 = \frac{\pi}{3}$

The maximum value is $\left(\frac{\pi}{3}, 9\right)$

(b) $5\cos\theta - 12\sin\theta - 13$

Solution

Let $5\cos\theta - 12\sin\theta = R\cos(\theta - \alpha)$

$= R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 5$ …….. (i)

$R\sin\alpha = 12$ …… (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{12}{5}$; $\alpha = 67.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 5^2 + 12^2 = 169$

$R^2[\cos^2\alpha + \sin^2\alpha] = 169$

$R^2 = 169$

$R = 13$

$\therefore 2\cos\theta - 12\sin\theta = 13\cos(\theta - 67.4^0)$

⇨ $5\cos\theta - 12\sin\theta - 13 = 13\cos(\theta - 67.4^0) - 13$

The minimum value occurs when

$\cos(\theta - 67.4^0) = -1$

⇨ Minimum value = 13(-1) -13 = -26

Now for $\cos(\theta - 67.4^0) = -1$

$\theta - 67.4^0 = 180^0$

$\theta = 247.4^0$

The minimum value is $(247.4^0, -26)$

And maximum value occurs when

$\cos(\theta - 67.4^0) = 1$

⇨ Minimum value = 13(1) - 13 = 0

Now for $\cos(\theta - 67.4^0) = 1$

$\theta - 67.4^0 = 0^0$

$\theta = 67.4^0$

The maximum value is $(67.4^0, 0)$

Example 33

(a) Given that $p = 2\cos\theta + 3\cos2\theta$ and $q = 2\sin\theta + 3\sin2\theta$, show that $1 \le p^2 + q^2 \le 25$
If $p^2 + q^2 = 19$ and θ is acute, find θ and show that $pq = \frac{-5\sqrt{3}}{4}$

Solution

$p^2 = 4\cos^2\theta + 12\cos\theta\cos2\theta + 9\cos^2 2\theta$ ….(i)

$q^2 = 4\sin^2\theta + 12\sin\theta\sin2\theta + 9\sin^2 2\theta$ …….(ii)

Eqn. (i) + eqn. (ii)

$p^2 + q^2 = 4 + 12(\cos\theta\cos2\theta + \sin\theta\sin2\theta) + 9$

$p^2 + q^2 = 13 + 12\cos\theta$ [$\cos(-\theta) = \cos\theta$]

But $-1 \le \cos\theta \le 1$

Multiplying through by 12

$-12 \le 12\cos\theta \le 12$

Adding 13 throughout

$1 \le 12\cos\theta + 12 \le 25$

$\therefore 1 \le p^2 + q^2 \le 25$ as required

If $p^2 + q^2 = 19$, $\Rightarrow 13 + 12\cos\theta = 19$

$\cos\theta = \frac{1}{2}$; $\theta = 60^0$ [θ is acute]

⇨ $p = 2\cos60^0 + 3\cos120^0 = 1 - \frac{3}{2} = -\frac{1}{2}$

$q = 2\sin60^0 + 3\sin120^0 = \sqrt{3} + 3\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$\therefore pq = \left(-\frac{1}{2}\right)\left(\frac{5\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{4}$

(b) Express $f(x) = 5\sin^2\theta - 3\sin\theta\cos\theta + \cos^2\theta$ in the form $p + q\cos(2\theta - \alpha)$
Hence show that $\frac{1}{2} \le f(x) \le 5\frac{1}{2}$
Solution

Using $\sin^2\theta = \frac{1}{2}(1 - cos2\theta)$ and

$\cos^2\theta = \frac{1}{2}(1 + cos2\theta)$

$f(x) = \frac{5}{2}(1-\cos2\theta) - 3\sin\theta\cos\theta + \frac{1}{2}(1 + \cos2\theta)$

$= \frac{5}{2} - \frac{5}{2}\cos2\theta - 3.\frac{2}{2}\sin\theta\cos\theta + \frac{1}{2} + \frac{1}{2}\cos2\theta$

$= 3 - 2\cos2\theta - \frac{3}{2}\sin2\theta$

$= 3 - [2\cos\theta + \frac{3}{2}\sin2\theta]$

Now:

$3 - [2\cos\theta + \frac{3}{2}\sin2\theta] \equiv p + q\cos(2\theta - \alpha)$

$= 3 + [q\cos2\theta\cos\alpha + q\sin2\theta\sin\alpha]$

By comparing: $p = 3$, $q\sin\alpha = \frac{3}{2}$ and

$q\cos\alpha = 2$

⇨ $\tan\alpha = \frac{3}{4}$; $\alpha = 36.9^0$

And $q = \sqrt{\left\{\left(\frac{3}{2}\right)^2 + (2)^2\right\}} = \frac{5}{2}$

⇨ $3 - [2\cos\theta + \frac{3}{2}\sin2\theta] = 3 - \frac{5}{2}\cos(2\theta - 36.9^0)$

But $-1 \le \cos(2\theta - 36.9^0) \le 1$

Multiplying through by $-\frac{5}{2}$

$\frac{5}{2} \ge -\frac{5}{2}\cos(2\theta - 36.9^0) \ge -\frac{5}{2}$

Adding 3 throughout

$3 - \frac{5}{2} \le 3 - \frac{5}{2}\cos(2\theta - 36.9^0) \le 3 + \frac{5}{2}$

$\therefore \frac{1}{2} \le f(x) \le 5\frac{1}{2}$

(c) Find the maximum and minimum points of the function; $f(x) = 3\cos\theta - 4\sin\theta + 20$ for $0^0 \le \theta \le 360^0$
Solution
Let $3\cos\theta - 4\sin\theta = R\cos(\theta + \alpha)$

$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficient of cosθ and sinθ

$R\cos\alpha = 3$ ……. (i)

$R\sin\alpha = 4$ …… (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = \frac{4}{3}$; $\alpha = 53.1^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 3\cos\theta - 4\sin\theta = 5\cos(\theta - 53.1^0)$

⇨ $3\cos\theta - 4\sin\theta + 20 = 5\cos(\theta - 53.1^0) + 20$

The minimum value occurs when

$\cos(\theta - 53.1^0) = -1$

⇨ Minimum value = 5(-1) +20 = 15

Now for $\cos(\theta - 53.1) = -1$

$\theta - 53.1^0 = 180^0$

$\theta = 126.8^0$

The minimum value is $(126.8^0, 15)$

And maximum value occurs when

$\cos(\theta - 53.1^0) = 1$

⇨ Minimum value = 5(1) 20 = 25

Now for $\cos(\theta - 53.1^0) = 1$

$\theta + 53.1^0 = 0^0, 360^0$

$\theta = -53.1^0, 306.8^0$

The maximum value is $(306.8^0, 25)$

Example 34

Find the maximum and minimum points of the following

(a) $f(\theta) = \frac{1}{3+sin\theta-2cos\theta}$

Solution

Let $\sin\theta - 2\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 1$ ……. (i)

$R\sin\alpha = 2$ …… (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 2$; $\alpha = 63.4^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2 = 5$

$R^2[\cos^2\alpha + \sin^2\alpha] = 5$

$R^2 = 5$

$R = \sqrt{5}$

$\therefore \sin\theta - 2\cos\theta = \sqrt{5}\sin(\theta - 63.4^0)$

⇨ $3 + sin\theta - 2cos\theta = \sqrt{5}\sin(\theta - 63.4^0) + 3$

⇨ $f(\theta) = \frac{1}{3+\sqrt{5}\sin(\theta-63.4^0)}$

Note: for a fractional function, a maximum point is obtained when the denominator is minimum and the vice versa for the maximum point

The minimum value occurs when

$\sin(\theta - 63.4^0) = 1$

⇨ Minimum value = $\frac{1}{3+\sqrt{5}} = 0.31$

Now for $\sin(\theta - 63.4) = 1$

$\theta - 63.4^0 = 90^0$

$\theta = 153.4^0$

The minimum value is $(153.4^0. 0.31)$

And maximum value occurs when

$\sin(\theta - 63.4^0) = -1$

⇨ Maximum value = $\frac{1}{3+\sqrt{5(-1)}} = 1.31$

Now for $\sin(\theta - 63.4^0) = -1$

$\theta - 63.4^0 = 270^0$

$\theta = 333.4^0$

The maximum value is $(333.4^0, 1.31)$

(b) $f(\theta) = \frac{1}{4sin\theta-3cos\theta+6}$

Solution

Let $4\sin\theta - 3\cos\theta = R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

Comparing coefficient of $\cos\theta$ and $\sin\theta$

$R\cos\alpha = 4$ ……. (i)

$R\sin\alpha = 3$ …… (ii)

Eqn (ii) ÷ eqn (i)

$\tan\alpha = 0.75$; $\alpha = 36.9^0$

Eqn. $(i)^2$ + eqn. $(ii)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2 = 25$

$R^2[\cos^2\alpha + \sin^2\alpha] = 25$

$R^2 = 25$

$R = 5$

$\therefore 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.9^0)$

⇨ $4\sin\theta - 3\cos\theta + 6 = 5\sin(\theta - 36.9^0) + 6$

⇨ $f(\theta) = \frac{1}{5\sin(\theta-36.9^0)+6}$

The minimum value occurs when

$\sin(\theta - 36.9^0) = 1$

⇨ Minimum value = $\frac{1}{5(1)+6} = \frac{1}{11}$

Now for sin(θ – 63.4) = 1

$\theta - 36.9^0 = 90^0$

$\theta = 126.9^0$

The minimum value is $(126.9^0. \frac{1}{11})$

And maximum value occurs when

$\sin(\theta - 36.9^0) = -1$

⇨ Maximum value = $\frac{1}{5(-1)+6} = 1$

Now for $\sin(\theta - 36.9^0) = -1$

$\theta - 36.9^0 = 270^0$

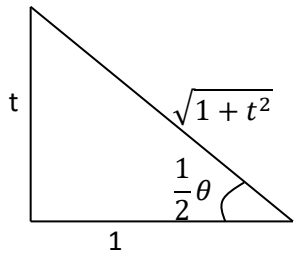
$\theta = 306.9^0$

The maximum value is $(306.9^0, 1)$

The t-formula

Although this form has been tackled indirectly, it is formally stated here

Suppose that $t = \tan\frac{\theta}{2}$, we have



From the triangle above

$cos\frac{1}{2}\theta = \frac{1}{\sqrt{1+t^2}}$ and $sin\frac{1}{2}\theta = \frac{t}{\sqrt{1+t^2}}$

But $cos\theta = cos^2\frac{1}{2}\theta - sin^2\frac{1}{2}\theta$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore cos\theta = \frac{1-t^2}{1+t^2}$

And $sin\theta = 2sin\frac{1}{2}\theta cos\frac{1}{2}\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore sin\theta = \frac{2t}{1+t^2}$

The t- formula is used widely in solving equations and proving trigonometric identities. These can be extended as follows

(i) For t= tanθ, $\sin2\theta = \frac{2t}{1+t^2}$ and $cos2\theta = \frac{1-t^2}{1+t^2}$

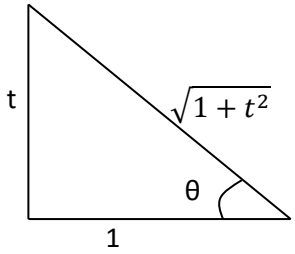
(ii) For $t = \tan\left(\frac{5x}{4}\right)$, $\sin\left(\frac{5x}{2}\right) = \frac{2t}{1+t^2}$ and

$cos\left(\frac{5x}{2}\right) = \frac{1-t^2}{1+t^2}$

Example 35

Show that if t = tanθ, then $\sin2\theta = \frac{2t}{1+t^2}$ and $2\theta = \frac{1-t^2}{1+t^2}$. Hence solve the equation $\sqrt{3}\cos2\theta + \sin 2\theta = 1$ for $0^0 \le \theta \le 360^0$.

Solution



From the triangle above

$cos\theta = \frac{1}{\sqrt{1+t^2}}$ and $sin\theta = \frac{t}{\sqrt{1+t^2}}$

But $cos2\theta = cos^2\theta - sin^2\theta$

$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$

$\therefore cos2\theta = \frac{1-t^2}{1+t^2}$

And $sin2\theta = 2sin\theta cos\theta$

$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$

$\therefore sin2\theta = \frac{2t}{1+t^2}$

Hence $\sqrt{3}\cos2\theta + \sin 2\theta = 1$

⇨ $\sqrt{3\left(\frac{1-t^2}{1+t^2}\right)} + \left(\frac{2t}{1+t^2}\right) = 1$

$\sqrt{3} - \sqrt{3t^2} + 2t = 1 + t^2$

$(1+\sqrt{3})t^1 - 2t + 1 - \sqrt{3} = 0$

$t = \frac{2\pm\sqrt{2^2-4(1+\sqrt{3})(1-\sqrt{3})}}{2(1+\sqrt{3})} = \frac{2\pm\sqrt{12}}{2(1+\sqrt{3})} = \frac{1\pm\sqrt{3}}{1+\sqrt{3}}$

t = $\frac{1+\sqrt{3}}{1+\sqrt{3}}$ = 1 or

t=$\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$ = -2 + $\sqrt{3}$

If tanθ = 1; θ = 450, 2250

If tanθ = -2 + $\sqrt{3}$; θ = 165^0, 345^0

∴ θ: θ = 45^0, 165^0, 225^0, 345^0

Example 36

Find all the solutions of the equation 5cosθ – 4sinθ = 6 for $-180^0 \leq \theta \leq 180^0$

Solution

Let t = tan$\frac{\theta}{2}$ then

$cos\theta = \frac{1-t^2}{1+t^2}$

$sin\theta = \frac{2t}{1+t^2}$

⇨ $5\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right) = 6$

$5(1-t^2) - 8t = 6(1+t^2)$

$5 - 5t^2 - 8t = 6 + 6t^2$

$11t^2 + 8t + 1 = 0$

$t = \frac{-8\pm\sqrt{8^2-4\ x\ 11\ x\ 1}}{2\ x\ 11} = \frac{-8\pm4.4721}{22}$

t = $\frac{-8+4.4721}{22} = -0.1604$ or

t = $\frac{-8-4.4721}{22}$ = -0.5669

Taking t = -0.1604

$tan\frac{\theta}{2} = -0.1604;\ \theta = -18.2^0$

Taking t = -0.5669

$tan\frac{\theta}{2} = -0.5669;\ \theta = -59.1^0$

∴θ = -59.1^0, -18.2^0

Example 37

Solve the equation

$3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$

Let t = tan θ

$3t^2 - 2(1 + t^2) = 2(5 - 3t)$

$5t^2 + 6t - 8 = 0$

t = $\frac{-6\pm\sqrt{6^2-4(5)(-8)}}{2(5)} = \frac{-6\pm14}{10}$ = -2 or $\frac{4}{5}$

Taking t = -2; θ = $\tan^{-1}(-2) = 116.57^0$

Taking t = $\frac{4}{5}$; θ = $\tan^{-1}(\frac{4}{5}) = 38.66^0$

Hence θ = 38.66^0, 116.57^0

Example 38

Show that tan4θ = $\frac{4t(1-t^2)}{t^4-6t^2+1}$, where t = tanθ.

Solution

tan4θ = $\frac{2\tan 2\theta}{1-tan^2\ 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-tan^2\theta}$

$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2}$

$= \frac{4t(1-t^2)}{t^4-6t^2+1}$

Example 39

Solve the equation cosθ + sinθ + 1 = 0 for $0^0 \leq \theta \leq 360^0$

Solution

cosθ + sinθ + 1

Let t = tan$\frac{\theta}{2}$

$\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} = 1$

$1 - t^2 + 2t = 1(1 + t^2)$

$2t + 2 = 0;\ t = -1$

∴ tan$\frac{\theta}{2}$ = -1

$\frac{\theta}{2} = 135^0, 315^0$

$\theta = 270^0, 630^0$

Hence θ = 270^0

Revision exercise 5

1. Solve equation 3cosθ + 4sinθ = 2 for $0^0 \leq \leq 360^0$ [119.6^0, 346.7^0]
2. (a) Show that cos4x = $\frac{tan^4x - 6tan^2x + 1}{tan^4x + 2tan^2x + 1}$
   (b) Show that if q = cos2x + sin2x, then $(1+q)\tan^2x - 2\tan x + q - 1 = 0$.

Deduce that if the roots of the above equation are $\tan x_1$ and $\tan x_2$, the $\tan(x_1 + x_2) = 1$

3. Find the values of R and tanα in each of the following equations
   (a) $2\cos\theta + 5\sin\theta = R\sin(\theta + \alpha)$ $\left[\sqrt{29}, \frac{2}{5}\right]$
   (b) $2\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$ $\left[\sqrt{29}, \frac{5}{2}\right]$
   (c) $\sqrt{3}\cos\theta + \sin\theta = R\cos(\theta - \alpha)$ $\left[2, \frac{1}{\sqrt{3}}\right]$
   (d) $5\sin\theta - 12\cos\theta = R\sin(\theta - \alpha)$ $\left[13, \frac{12}{5}\right]$
   (e) $\cos\theta - \sin\theta = R\cos(\theta + \alpha)$ $\left[\sqrt{2}, 1\right]$

4. Find the greatest and least values and state the smallest non-negative value of x for which each occurs
   (i) $12\sin x + 5\cos x$ [13, 67.4^0; -13, 247.4^0]
   (ii) $2\cos x + \sin x$
   $[\sqrt{5}, 26.6^0;\ -\sqrt{5}, 206.6^0]$
   (iii) $7 + 3\sin x - 4\cos x$
   [12, 143.1^0; 2, 323.1^0]
   (iv) $10 - 2\sin x + \cos x$
   $[10 + \sqrt{5}, 296.6^0; 10 - \sqrt{5}, 116.6^0]$
   (v) $\frac{1}{2+\sin x+\cos x}\left[\frac{2+\sqrt{2}}{2}, 225^0;\ \frac{2-\sqrt{2}}{2}, 45^0\right]$
   (vi) $\frac{1}{7-2\cos x+\sqrt{5}\sin x}\left[\frac{1}{4}, 311.8^0;\ \frac{1}{10}, 131.8^0\right]$
   (vii) $\frac{3}{5\cos x-12\sin x+16}$ [1, 112.6^0; $\frac{3}{29}$, 292.6^0]

5. Solve each of the following equations for $0^0 \le x \le 360^0$
   (a) $\sin x + \sqrt{3}\cos x = 1$ [90^0, 330^0]
   (b) $4\sin x - 3\cos x = 2$ [60.4^0, 193.3^0]
   (c) $\sin x + \cos x = \frac{1}{\sqrt{2}}$ [105^0, 345^0]
   (d) $5\sin x + 12\cos x = 7$ [80.0^0, 325.2^0]
   (e) $7\sin x - 4\cos x = 3$ [51.6^0, 187.9^0]
   (f) $\cos x - 3\sin x = 2$ [237.7^0, 339.2^0]
   (g) $5\cos x + 2\sin x = 4$[63.8^0, 339.8^0]
   (h) $9\cos 2x - 4\sin 2x = 6$ [14.2^0, 141.8^0, 194.2^0, 321.8^0]
   (i) $7\cos x + 6\sin x = 2$ [118.1^0, 323.1^0]
   (j) $9\cos x - 8\sin x = 12$ [313.6^0, 323.1^0]

The factor formulae

The following identities were developed from compound angles

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ …….(i)

$\cos(A - B) = \cos A\cos B + \sin A\sin B$ …… (ii)

$\sin(A + B) = \sin A\cos B + \sin B\cos A$ ……(iii)

$\sin(A - B) = \sin A\cos B - \sin B\cos A$ ……(iv)

eqn. (i) + eqn (ii)

$\cos(A + B) + \cos(A - B) = 2\cos A\cos B$

eqn. (i) – eqn (ii)

$\cos(A + B) - \cos(A - B) = -2\cos A\cos B$

eqn. (iii) + eqn (iv)

$\sin(A + B) + \sin(A - B) = 2\sin A\cos B$

eqn. (iii) - eqn (iv)

$\sin(A + B) - \sin(A - B) = -2\sin B\cos A$

For simplification, $A + B = \alpha$ and $A - B = \beta$

Add: $2A = \alpha + \beta$ i.e. $A = \left(\frac{\alpha+\beta}{2}\right)$

Subtract $2B = \alpha - \beta$ i.e. $A = \left(\frac{\alpha-\beta}{2}\right)$

Substituting for A and B in the above equation

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Example 40

Show that if X, Y and Z are angles of a triangle, then

(a) $\cos X + \cos Y + \cos Z - 1 = 4\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$

solution

LHS $\cos X + \cos Y + \cos Z - 1$

$= 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2} + 1 - 2\sin^2\frac{Z}{2} - 1$

(to eliminate -1)

$= 2\cos\frac{180^0 - Z}{2}\cos\frac{X-Y}{2} - 2\sin^2\frac{Z}{2}$

(since $X + Y = 180^0 - Z$)

= 2sin $\frac{Z}{2}$ cos $\frac{X-Y}{2}$ –2 sin$^2\frac{Z}{2}$

(Since cos(90^0 – A) = sinA)

= 2sin $\frac{Z}{2}\left[\cos\frac{X-Y}{2} - 2sin^2\left\{\frac{180^0-(X+Y)}{2}\right\}\right]$

= 2sin $\frac{Z}{2}\left[\cos\frac{X-Y}{2} - cos\left\{\frac{X+Y)}{2}\right\}\right]$

(Since sin (90^0 – A) = cos A)

= 2sin$\frac{Z}{2}\left[-2sin\frac{X}{2}sin\frac{-y}{2}\right]$

=2sin$\frac{Z}{2}\left[2sin\frac{X}{2}sin\frac{Y}{2}\right]$

(Since sin(-A) = -sinA)

$4sin\frac{X}{2}sin\frac{Y}{2}\sin\frac{Z}{2}$ as required

(b) Sin3X + sin3Y + sin3Z =

- 4cos$\frac{3X}{2}cos\frac{3Y}{2}cos\frac{3Z}{2}$

Solution

LHS: Sin3X + sin3Y + sin3Z

= 2sin$\frac{3(X+Y)}{2}$cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

=2sin$\frac{3(180^0-Z)}{2}$ cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

= -2cos$\frac{3Z}{2}$ cos$\frac{3(X-Y)}{2}$ + 2sin$\frac{3Z}{2}$cos$\frac{3Z}{2}$

Since sin(270^0 – A) = -cosA}

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - sin\frac{3Z}{2}\right]$

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - sin\frac{3\{180^0-(X+Y)\}}{2}\right]$

=-2cos$\frac{3Z}{2}\left[\cos\frac{3(X-Y)}{2} - cos\frac{3(X+Y)}{2}\right]$

=-2cos$\frac{3Z}{2}\left[2cos\frac{3X}{2} + cos\frac{-3Y}{2}\right]$

Since cos(-A) = cosA

=- 4cos$\frac{3X}{2}cos\frac{3Y}{2}cos\frac{3Z}{2}$

(c) cos4X + cos4Y + cos4Z + 1

= 4cos2Xcos2Ycos2Z

Solution

LHS: cos4X + cos4Y + cos4Z + 1

= 2cos2(X + Y)cos2(X – Y) + 2cos^{2}2Z -1 + 1

= 2cos2(180^0 – Z)cos2(X – Y) + 2cos^{2}2Z

= 2cos2Z[cos2(X – Y) + cos2{180^0 – (X + Y)}]

= 2cos2Z[cos2(X – Y) + cos2 (X + Y)]

= 2cos2Z[2cos2X cos-2Y]

Since cos(-A) = cosA

=4cos2Z2cos2X cos2Y

(d) sin^{2}Y + sin^{2}Z = 1 + cos(Y – Z)cosX

LHS: sin^{2}Y + sin^{2}Z

= $\frac{1}{2}(1 - cos2Y) + \frac{1}{2}(1 - cos2Z)$

=$\frac{1}{2}(2 - cos2Y - cos2Z)$

=$1 - \frac{1}{2}(cos2Y + cos2Z)$

= 1 – cos(180^0 –X)cos(Y – Z)

=1 + cos(Y – Z)cosX

Example 41

(a) Factorize cosθcos3θ – cos7θ + cos9θ and express it in the form Acospθsinqθsinrθ wher A, p, q and r are constants

Solution

f(θ)= cos9θ + cosθ – (cos7θ +cos3θ)

= 2cos5θcos4θ – 2cos5θcos2θ

=2cos5θ(cos4θ – cos2θ)

= -4cos5θ(-sin3θsinθ)

=-4cos5θsin3θsinθ

⇨ A = -4, p = 5, q = 3, r = 1

(b) Given that

p = sinα + sinβ

q = cosα + cosβ. Show that

$\frac{2pq}{p^2+q^2}$ = sin(α + β)

Solution

$\frac{2pq}{p^2+q^2}$

=$\frac{2(sin\alpha+sin\beta)(cos\alpha+cos\beta)}{sin^2\alpha+2sin\alpha sin\beta+sin^2\beta+cos^2\alpha+2cos\alpha cos\beta+cos^2\beta}$

$=\frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]\left[2cos\frac{\alpha+\beta}{2}cos\frac{\alpha-\beta}{2}\right]}{2+2(cos\alpha cos\beta+sin\alpha sin\beta}$

$=\frac{2\left[2sin\frac{\alpha+\beta}{2}cos\frac{\alpha+\beta}{2}\right]\left[2cos^2\frac{\alpha-\beta}{2}\right]}{2+2cos(\alpha-\beta)}$

$=\frac{2[\sin(\alpha+\beta)][1+\cos(\alpha-\beta)}{2[1+\cos(\alpha-\beta)]}$

= sin(α +β)

Example 42

Solve $5\cos^2 3\theta = 3(1 + \sin 3\theta)$ for $0^0 \leq \theta \leq 90^0$.

Solution

$5\cos^2 3\theta = 3(1 + \sin 3\theta)$

$5(1 - \sin^2 3\theta) = 3(1 + \sin 3\theta)$

$5 - 5\sin^2 3\theta = 3 + 3\sin 3\theta$

$5\sin^2 3\theta + 3\sin 3\theta - 0 = 0$

$(\sin 3\theta + 1)(5\sin 3\theta - 2) = 0$

$\sin 3\theta + 1 = 0$

$3\theta = \sin^{-1}(-1) = -90^0, 270^0$

Example 43

(a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$4\cos^2 x - 1 = 0$

$(2\cos x + 1)(2\cos x - 1) = 0$

Either

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}(-\frac{1}{2}) = 120^0$

Or

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \cos^{-1}(\frac{1}{2}) = 60^0$

$\therefore x(60^0, 120^0)$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$= \frac{4}{2}(1 + \cos 2x) - \frac{2}{2}(1 - cos2x)$

$= 2 + 2\cos 2x - 1 + \cos 2x$

$2\cos 2x + 1 = 0$

$\cos 2x = -\frac{1}{2}$

$2x = \cos_{-1}(-\frac{1}{2}) = 120^0, 240^0$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$\cos^2 x - \sin^2 x = 4\cos^2 x - 2\sin^2 x$

$3\cos^2 x - \sin^2 x = 0$

$\sin^2 x = 3\cos^2 x$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Either

$\tan x = \sqrt{3}$

$x = \tan^{-1}\sqrt{3} = 60^0$

Or

$\tan x = -\sqrt{3}$

$x = \tan^{-1} -\sqrt{3} = 120^0$

Hence $x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4(1 - \sin^2 x) - 2\sin^2 x$

$1 = 4 - 4\sin^2 x$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$x = 60^0, 120^0$

Alternatively

$\cos 2x = 4\cos^2 x - 2\sin^2 x$

$1 - 2\sin^2 x = 4\cos^2 x - 2\sin^2 x$

$4\cos^2 x = 1$

$\cos x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$x = 60^0, 120^0$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$.

Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$.

sinxcosα + coxsinα = p(sinxcosα - coxsinα)

cosxsinα (p + 1) = sinxcosα (p - 1)

$\text{cosxsin}\alpha\left(\frac{p+1}{P-1}\right) = \text{sinxcos}\alpha$

$\frac{cosxsin\alpha}{\text{sinxcos}\alpha}\left(\frac{p+1}{p-1}\right) = \frac{sinxcos\alpha}{\text{sinxcos}\alpha}$

$tanx = \left(\frac{p+1}{P-1}\right)\tan\alpha$

For $\sin(x + 20^0) = 2\sin(x - 20^0)$

$\tan x = \frac{2+1}{2-1}tan20^0 = 3\tan20^0$

$x = \tan^{-1}(3tan20^0) = 47.52^0$

Example 44

Prove that in any triangle ABC,

$\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$

Solution

$\frac{a^2-b^2}{c^2} = \frac{(2Rsin\,A)^2-(2Rsin\,B)^2}{(2Rsin\,C)^2}$

$= \frac{4R^2(sin^2A-sin^2B}{4R^2sin^2C}$

$= \frac{(\sin A+\sin B)(\sin A-\sin B)}{sin^2[180^0-(A+B]}$

$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{sin^2(A+B)}$

$= \frac{\sin(A+B)\sin(A-B)}{sin^2(A+B)}$

$= \frac{\sin(A-B)}{sin(A+B)}$

Inverse trigonometric functions

Note that

(a) If $\theta = \cos^{-1}\left(\frac{1}{2}\right)$ then $\cos\theta = \frac{1}{2}$

(b) $\tan^{-1}(tan\alpha) = \tan(\tan^{-1}\alpha) = \alpha$

(c) $\cos^{-1}[\cos(x+y)]$

$= \cos[\cos^{-1}(x+y) = x+y$

(d) $sin(\sin^{-1}\theta) = \sin^{-1}(\sin^{-1}\theta)$

To avoid errors test the values

Example 45

Show that

(a) $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{4}$

Solution

$A = \tan^{-1}\frac{1}{3}$ and $B = \sin^{-1}\frac{1}{\sqrt{5}}$

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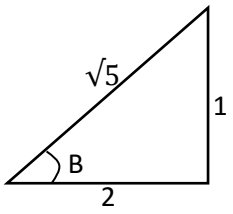
⇨ $\tan A = \frac{1}{3}$ and $B = \frac{1}{\sqrt{5}}$



⇨ $\tan B = \frac{1}{2}$

LHS = $\tan^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{\sqrt{5}} = A + B$

$= \tan^{-1}[\tan(A+B)]$

$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}\right)$

$= \tan^{-1}\frac{3+3}{6-1}$

$= \tan^{-1}\frac{5}{5} = \tan^{-1}1 = \frac{\pi}{4}$

(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

Solution

Let $A = \tan^{-1}\frac{1}{3}$ and $B = \tan^{-1}\frac{1}{7}$

⇨ $\tan A = \frac{1}{3}$ and $\tan B = \frac{1}{7}$

LHS: $\tan^{-1}\tan(2A+B)$ but $\tan 2A = \frac{2\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

$\therefore \tan^{-1}\tan(2A+B) = \frac{\frac{3}{4}+\frac{1}{7}}{1-\left(\frac{3}{4}\right)\left(\frac{1}{7}\right)}$

$= \tan^{-1}\left(\frac{21+4}{28-3}\right)$

$= \tan^{-1}1$

$= \frac{\pi}{4}$

(c) $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Solution

Let $\theta = \cos^{-1}x$; => $x = \cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$

$\sin x = \frac{\pi}{2} - \theta$

$\therefore \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

Example 46

Solve the equations

(a) $\tan^{-1}(2\theta + 1) + \tan^{-1}(2\theta - 1) = \tan^{-1}(2)$

Solution

Let A = $\tan^{-1}(2\theta + 1)$ and B =$\tan^{-1}(2\theta - 1)$
⇨ tanA = $2\theta + 1$ and tanB =$2\theta - 1$

∴ A + B = $\tan^{-1} 2$ or tan(A +B) =2

$\frac{2\theta+1+2\theta-1}{1-(2\theta+1)(2\theta-1)} = 2$

$4\theta = 2(1 - 4\theta^2 - 1)$

$2\theta^2 + \theta - 1 = 0$

$(2\theta - 1)(\theta + 1) = 0$

$\theta = \frac{1}{2}$ or $\theta = -1$

(b) $\tan^{-1}(1+ \theta) + \tan^{-1}(1 - \theta) = 32$
Let A = $\tan^{-1}(1 + \theta)$ and B =$\tan^{-1}(1 - \theta)$
⇨ tanA = $1 + \theta$ and tanB =$1 - \theta$

∴ A + B = 32 or tan(A +B) = tan32

Introducing tangents

$\frac{1+\theta+1-\theta}{1-(1+\theta)(1-\theta)} = tan32$

$\theta^2 \tan 32 = 2$

$\theta = \sqrt{2cot32} = \pm 1.789$

Example 47

If $x = \tan^{-1}\alpha$ and $y = \tan^{-1}\beta$;

Show that $x + y = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Solution

$\tan x = \alpha$; $\tan y = \beta$

$(x + y) = \tan[\tan^{-1}(x + y)]$

$= \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Example 48

Solve the equation

$\tan^{-1}\left(\frac{1}{x-1}\right) + \tan^{-1}(x + 1) = \tan(-2)$

Solution

Let A = $\tan^{-1}\left(\frac{1}{x-1}\right)$ and B = $\tan^{-1}(x + 1)$

⇨ A + B = $\tan^{-1}(-2)$

$\frac{\frac{1}{x-1}+(x+y)}{1-\left(\frac{1}{x-1}\right)(x+y)} = -2$

$\frac{1+x^2-1}{x-1-x-1} = -2$

$\therefore x^2 = 4; x = \pm 2$

Example 50

Without using tables or calculators determine the values of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

Solution

$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$.

$=\frac{\frac{1}{2}+\frac{1}{5}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)} + \tan^{-1}\frac{1}{8}$

$=\tan^{-1}\frac{7}{9} + \tan^{-1}\frac{1}{8}$

$=\frac{\frac{7}{9}+\frac{1}{8}}{1-\left(\frac{7}{9}\right)\left(\frac{1}{8}\right)} = \tan^{-1}\left(\frac{65}{65}\right) = \frac{\pi}{4}$

Example 51

Solve equations

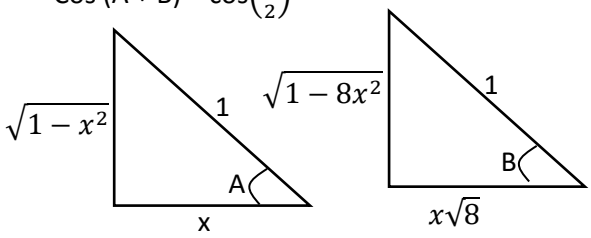
(a) $\cos^{-1} x + \cos^{-1} x\sqrt{8} = \frac{\pi}{2}$

Solution

Let A = $\cos^{-1} x$ and B =$\cos^{-1} x\sqrt{8}$

$A + B = \frac{\pi}{2}$

$\cos(A + B) = \cos\left(\frac{\pi}{2}\right)$



$$x\left(x\sqrt{8}\right) - \left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right) = 0$$

$$x\left(x\sqrt{8}\right) = \left(\sqrt{1-x^2}\right)\left(\sqrt{1-8x^2}\right)$$

$8x^4 = (1 - x^2)(1 - 8x^2)$

$1- 9x^2 = 0$

$(1 - 3x)(1 + 3x) = 0$

Either x = $\frac{1}{3}$ or x = $-\frac{1}{3}$

We discard the negative value, so the root is x = $\frac{1}{3}$

(b) $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}\left(x\sqrt{2}\right) = \frac{\pi}{2}$

Solution

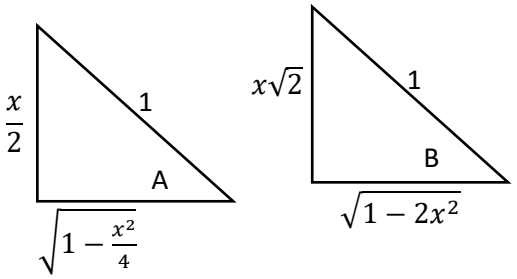
Let A = $\sin^{-1}\left(\frac{x}{2}\right)$ and B = $\sin^{-1}\left(x\sqrt{2}\right)$

2A + B = $\frac{\pi}{2}$

2A = $\frac{\pi}{2} - B$

Sin(2A) = sin $\left(\frac{\pi}{2} - B\right)$

2SinAcosA = cosB



$2\left(\frac{x}{2}\right).\sqrt{1-\frac{x^2}{4}} = \sqrt{1-2x^2}$

$x.\sqrt{\frac{4-x^2}{4}} = \sqrt{1-2x^2}$

$\frac{x}{2}.\sqrt{4-x^2} = \sqrt{1-2x^2}$

$\frac{x^2}{4}.(4-x^2) = (1-2x^2)$

$X^4 - 12x^2 + 4 = 0$

$x^2 = \frac{12 \pm \sqrt{144-4(4\,x\,1)}}{2\,x1}$

$x^2 = \frac{12 \pm \sqrt{128}}{2} = 6 \pm 4\sqrt{2}$

$x = \sqrt{6 \pm 4\sqrt{2}}$

After testing for $x = \sqrt{6+4\sqrt{2}}$ and for $x = \sqrt{6-4\sqrt{2}}$, the value that satisfies the equation is $x = \sqrt{6-4\sqrt{2}}$ = 0.5858

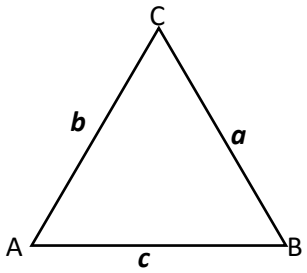
Hence the value of x = 0.5858

Revision exercise 6

1. If p = sinα + sin β and q = cosα + cosβ show that $\frac{p}{q} = tan\frac{\alpha+\beta}{2}$
2. (a) Prove that:
   (i) (sin2x – sinx)(1+ 2cosx) = sin3x
   (ii) cos4θ = $\frac{tan^4\theta - 6tan^2\theta + 1}{tan^4\theta + 2tan^2\theta + 1}$
   (iii) $\frac{sinx+2sin2x+sin3x}{sinx+2sinx+sin3x} = tan^2\frac{x}{2}$
3. Solve the equation for $0^0 \le x \le 180^0$:
   (a) sinx + sn3x + sin5x + sin7x = 0
   [x: x = 0^0, 45^0, 90^0, 135^0, 180^0]
   (b) sin7x + sinx + sin5x + sin3x = 0
   [x: x = 60^0, 180^0]
   (c) sinx + sin4x = 0
   [x: x= 0^0, 60^0, 72^0, 144^0, 180^0]
   (d) cos (x + 10^0) — -cos(x + 30^0) = 0
   [70^0]
   (e) cos5x – sin2x = cosx
   [x: x = 0^0, 70^0, 90^0, 110^0,180^0]
   (f) sin2x +sin10x + cos4x = 0
   [x: x = 22.5^0, 35^0, 55^0, 67.5^0, 95^0, 112.5^0, 115^0, 157.5^0, 175^0]
4. Show that
   (a) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
   (b) $2\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
   (c) the positive value that satisfies the equation $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$ is $\frac{1}{6}$
   (d) $\tan^{-1}(-x) = -\tan^{-1} x$
   (e) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$
5. Prove that
   (a) $\frac{sinA-sinB}{sinA+sinB} = tan\left(\frac{A-B}{2}\right)cot\left(\frac{A+B}{2}\right)$
   (b) Sin3x + sinx = 4sinxcos^{2}x
   (c) $\frac{sinx+sin2x+sin3x}{cosx+cos2x+cos3x} = tan2xx$
   (d) sin(A + B) – sin(A – B) =2cosAsinB
   (e) $\frac{sin5x+sinx}{sin4x+sin2x} = 2cosx - secx$
   (f) Cos3x + cosx = 4cos^{2}x – 2cosx

Solution to triangles

In a triangle ABC



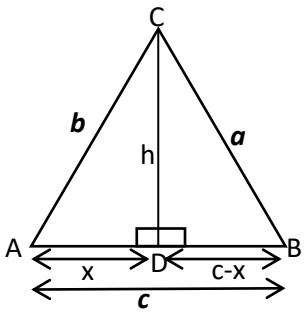
(a) Six elements are considered: three angles and three sides

Capital letters denote angles and ***small bold and italics letters*** sides

(b) The opposite side of angle A is a, of angle B is b and of angle C is c.
(c) The angle sum of a triangle is two right angles i.e. A + B + C = 180^0
(d) The sides are independent except that the sum of the two sides of the triangle should be equal to or greater than the third side

How to deal with triangles

1. The cosine rule
   (a) Given an acute angle A



From triangle
ACD: $x^2 + h^2 = \boldsymbol{b}^2$ .....(i)
BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ ..... (ii)
Substituting eqn. (i) into eqn. (ii)
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$
But
$x = \boldsymbol{b}\cos A$
⇨ $\boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A = a^2$
$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ (1)

Similarly;

$\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{ac}\cos B$ (2)
$\boldsymbol{c}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab}\cos C$ (3)

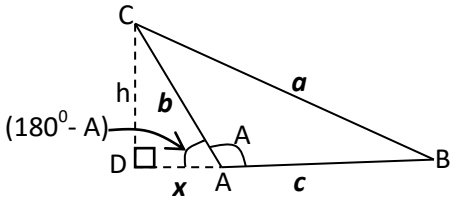
It follows that

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

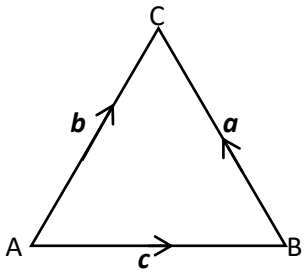
(b) Given an obtuse angle A



In triangle ABC, A is an obtuse angle and CD is the altitude.

From triangle
ACD: $x^2 + h^2 = \boldsymbol{b}^2$ .....(i)
BCD: $(\boldsymbol{c} - x)^2 + h^2 = \boldsymbol{a}^2$
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + x^2 + h^2 = \boldsymbol{a}^2$ ..... (ii)
Substituting eqn. (i) into eqn. (ii)
$\boldsymbol{c}^2 - 2\boldsymbol{c}x + \boldsymbol{b}^2 = \boldsymbol{a}^2$
But
$x = \boldsymbol{b}\cos(180^0 - A) = -\boldsymbol{bc}\cos A$
From triangle ACD
$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{bc}\cos A$ as before

The cosine rule can be derived using the vector approach.



Given a triangle above with BC = ***a***, AC = ***c*** and AB = ***b***

BC = BA + AC = AC – AB

$\boldsymbol{a} = \boldsymbol{b} - \boldsymbol{c}$

⇨ $\boldsymbol{a}.\boldsymbol{a} = (\boldsymbol{b}-\boldsymbol{c})(\boldsymbol{b}-\boldsymbol{c})$

$= \boldsymbol{b}.\boldsymbol{b} - 2\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{c}$

$= \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{c} - 2\boldsymbol{b}.\boldsymbol{c}$

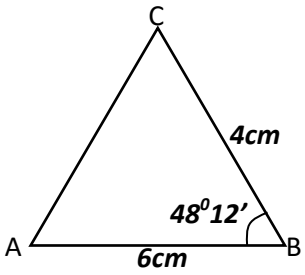
$\therefore \boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{b}\boldsymbol{c}\cos A$

since $\boldsymbol{b}.\boldsymbol{c} = |bc|\cos A$

Example 52

Solve the triangle in which AB = 6cm, BC = 4cm and angle ACB = $48^0 12'$

Solution



Using: $\boldsymbol{b}^2 = \boldsymbol{a}^2 + \boldsymbol{c}^2 - 2\boldsymbol{a}\boldsymbol{c}\cos B$

$= 6^2 + 4^2 - 2(6)(4)\cos 48.2^0$

1^0 (degree) = 60' (minutes)

b = 4.47cm

Using: $cosA = \frac{b^2+c^2-a^2}{2bc} = \frac{20.0+36-16}{2(4.47)(6)}$

A =41.8^0

But A + B + C = 180^0

$41.8^0 + 48.2^0 + C = 180^0$

C = 90^0

∴ AC = 4.47cm, angles BAC = 41.8^0 and ACB = 90^0

Example 53

In a triangle ABC, prove that

(a) $a^2 = (b-c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$ hence that

a = (b – c)secα where $\tan\alpha = \frac{\sqrt{bc}\sin\left(\frac{A}{2}\right)}{b-c}$

From cosA = 1- 2 $\sin^2\left(\frac{A}{2}\right)$

Substituting for cosA into the cosine formula $\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{b}\boldsymbol{c}\cos A$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{b}\boldsymbol{c}[1 - 2\sin^2\left(\frac{A}{2}\right)]$

$\boldsymbol{a}^2 = \boldsymbol{b}^2 + \boldsymbol{c}^2 - 2\boldsymbol{b}\boldsymbol{c} + 4\sin^2\left(\frac{A}{2}\right)$

$a^2 = (b-c)^2 + 4bc\sin^2\left(\frac{A}{2}\right)$

Hence, substituting for $\sin^2\left(\frac{A}{2}\right)$into tanα expression we get

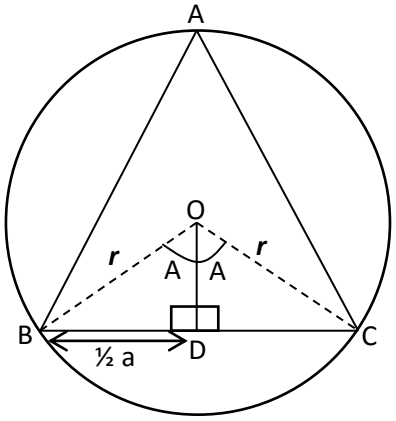
$a^2 = (b-c)^2 + (b-c)^2\tan^2\alpha$

$a^2 = (b-c)^2(1 + \tan^2\alpha)$

$a^2 = (b-c)^2\sec^2\alpha$

$a = (b-c)\sec\alpha$

2. The Sine Rule



The figure shows a circle with centre O and radius r circumscribing triangle ABC

Angle BOC = 2A [angle subtended by the same arc at the centre of the circle is twice the angle formed at any point on the circumference]

Triangle BOC is isosceles

OD bisects angle BOC and side BC

∴BD = ½a

From triangle BOD

$\sin A = \frac{a}{2r}$ i.e. $\frac{a}{sinA} = 2r$

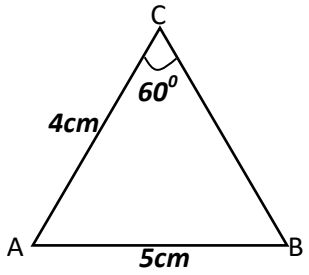
if instead we consider triangles AOC and AOB, we obtain $\frac{b}{sinB} = 2r$ and $\frac{c}{sinC} = 2r$

In general: $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

Example 54

Solve the triangle in which AB = 5cm, AC = 4cm and angle ACB = 600

Solution



Using sine rule

$\frac{b}{sinB} = \frac{c}{sinC}$ => B = $\sin^{-1}\left(\frac{bsinC}{c}\right)$

B = $\sin^{-1}\left(\frac{4}{5}sin60^0\right)$ = 43.9^0

From A + B + C+ = 180^0

A = $(180 - 60 - 43.9)^0 = 76.1^0$

Similarly a = $\frac{bsinA}{sinB} = \frac{4sin76.1^0}{sin43.9^0}$ = 5.6cm

$\therefore\ \overline{AB} = 5.6cm, B\hat{A}C = 76.1^0, A\hat{B}C = 43.9^0$

Example 55

Prove that in any triangle

$\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{a^2-b^2}{c^2} = \frac{(2rsinA)^2-(2rsinB)^2}{(2rsinC)^2} = \frac{sin^2A-sin^2B}{sin^2C}$

But A + B + C = 180^0

C = 180^0 – (A + B)

sinC = sin[180^0 – (A + B)] = sin (A + B)

By substitution

$\frac{a^2-b^2}{c^2} = \frac{sin^2A-sin^2B}{\sin(A+B)} = = \frac{(sinA+\sin B)(sinA-sinB}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B).2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{\sin(A+B)}$

$= \frac{2sin\frac{1}{2}(A-B)cos\frac{1}{2}(A-B)}{} = \frac{\sin(A-B)}{\sin(A+B)}$

Hence $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$

Example 56

Prove that in any triangle ABC,

$sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

Solution

From sine rule formula;
a = 2rsinA, b = 2rsinB, c = 2rsinC

By substitution

$\frac{b-c}{a} = \frac{2rsinB-2rsinC}{2rsinA} = \frac{SinB-sinC}{sinA}$

But A + B + C = 180^0

A = 180^0 – (B + C)

sinA = sin[180^0 – (B + C)] = sin(B + C)

By substitution

$\frac{b-c}{a} = \frac{SinB-sinC}{sinA} = \frac{SinB-sinC}{\sin(B+C)}$

$= \frac{2cos\frac{1}{2}(B+C)sin\frac{1}{2}(B+C)}{2cos\frac{1}{2}(B+C)cos\frac{1}{2}(B+C)}$

$= \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}(B+C)}$

From A + B + C = 180^0

B + C = 180^0 – A

$\frac{1}{2}(B+C) = \left(90^0 - \frac{1}{2}A\right)$

$sin\frac{1}{2}(B+C) = sin\left(90^0 - \frac{1}{2}A\right) = cos\frac{1}{2}A$

By substitution

$\frac{b-c}{a} = \frac{sin\frac{1}{2}(B+C)}{cos\frac{1}{2}A}$

$\therefore sin\frac{1}{2}(B-C) = \frac{b-c}{a}cos\frac{1}{2}A$

3. The Tangent Rule

It states that in a triangle ABC

$$tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$$

$$tan\frac{1}{2}(C-A) = \left(\frac{c-a}{c+a}\right)cot\frac{1}{2}B$$

$$tan\frac{1}{2}(b-c) = \left(\frac{b-c}{b+c}\right)cot\frac{1}{2}A$$

Proof

From $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

a = 2rsinA, b = 2rsinB, c = 2rsinC

$$\frac{a-b}{a+b} = \frac{2rsinA-2rsinB}{2rsinA+2rsinB} = \frac{sinA-sinB}{sinA+sinB}$$

$$= \frac{2cos\frac{1}{2}(A+B)sin\frac{1}{2}(A-B)}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)}$$

$$= \frac{2\cos(90-\frac{1}{2}C)sin\frac{1}{2}(A-B)}{2\sin(90-\frac{1}{2}C)cos\frac{1}{2}(A-B)}$$

$$= \frac{\cos(90-\frac{1}{2}C)tan\frac{1}{2}(A-B)}{\sin\frac{1}{2}(90-C)}$$

$$= \frac{\sin\frac{1}{2}Ctan\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$$

$$\frac{a-b}{a+b} = tan\frac{1}{2}Ctan\frac{1}{2}(A-B)$$

$$\therefore tan\frac{1}{2}(A-B) = \left(\frac{a-b}{a+b}\right)cot\frac{1}{2}C$$

Example 56

Show that in a triangle PQR

$$tan\frac{1}{2}(Q-C) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$$

Hence solve the triangle in which q =15.32, r = 28.6 and P = $39^0 52'$

Solution

From $\frac{p}{sinP} = \frac{q}{sinQ} = \frac{r}{sinR}$

p = 2rsinP, q = 2rsinQ, r = 2rsinR

$$\frac{q-r}{q+r} = \frac{2rsinQ-2rsinR}{2rsinQ+2rsinR} = \frac{sinQ-sinR}{sinQ+sinR}$$

$$= \frac{2cos\frac{1}{2}(Q+R)sin\frac{1}{2}(Q-R)}{2sin\frac{1}{2}(Q+R)cos\frac{1}{2}(Q-R)}$$

$$= \frac{2\cos(90-\frac{1}{2}P)sin\frac{1}{2}(Q-R)}{2\sin(90-\frac{1}{2}P)cos\frac{1}{2}(Q-R)}$$

$$= \frac{\cos(90-\frac{1}{2}P)tan\frac{1}{2}(Q-R)}{\sin\frac{1}{2}(90-P)}$$

$$= \frac{\sin\frac{1}{2}Ptan\frac{1}{2}(Q-R)}{\cos\frac{1}{2}P}$$

$$\frac{q-r}{q+r} = tan\frac{1}{2}Ptan\frac{1}{2}(Q-R)$$

$$\therefore tan\frac{1}{2}(Q-R) = \left(\frac{q-r}{q+r}\right)cot\frac{1}{2}P$$

Hence

$$tan\frac{1}{2}(Q-R) = \frac{15.32-28.6}{15.32+29.6}cot39^0 52'$$

= -0.3621

$\frac{1}{2}(Q-R) = -19.9^0$ i.e. Q – R = -39.9^0

But P + Q + R = 180

Q + R = 180 - 39.9 = 140.1^0

Solving Q = 50.15^0 and R = 89.95^0

Now p = $\frac{qsinP}{sinQ} = \frac{15.32\sin\left[39+\frac{52}{60}\right]^0}{\sin 50.15}$ = 12.79

∴p = 12.79, Q =50.15^0, R = 89.95^0

Example 57

Show that $\frac{a+b-c}{a+b+c} = tan\frac{1}{2}Atan\frac{1}{2}B$

Solution

LHS = $\frac{a+b-c}{a+b+c}$

$$= \frac{2rsinA+2rsinB-2rsinC}{2rsinA+2rsinB+2rsinC}$$

$$= \frac{sinA+sinB-sinC}{sinA+sinB+sinC}$$

$$= \frac{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2sin\left(90-\frac{1}{2}C\right)cos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{2cos\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}{2os\frac{1}{2}Ccos\frac{1}{2}(A-B)-2sin\frac{1}{2}Ccos\frac{1}{2}C}$$

$$= \frac{cos\frac{1}{2}(A-B)-sin\frac{1}{2}C}{cos\frac{1}{2}(A-B)+sin\frac{1}{2}C}$$

$$=\frac{cos\frac{1}{2}(A-B)-sin\left(90-\frac{1}{2}(A+B)\right)}{cos\frac{1}{2}(A-B)+sin\left(90-\frac{1}{2}(A+B)\right)}$$

$$=\frac{cos\frac{1}{2}(A-B)-cos\frac{1}{2}(A+B)}{cos\frac{1}{2}(A-B)+cos\frac{1}{2}(A+B)}$$

$$=\frac{-2sin\frac{1}{2}Asin\left(-\frac{1}{2}B\right)}{cos\frac{1}{2}A+cos\frac{1}{2}B}$$

$$=\tan\frac{1}{2}A\tan\frac{1}{2}B$$

Expressions for sinA, $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ in terms of the sides of the triangle

(a) sinA

From the identity

$\sin^2A = 1 - \cos^2A = (1 - \cos A)(1 + \cos A)$

$$=\left(1-\frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\left(\frac{2bc-b^2+c^2-a^2}{2bc}\right)\left(\frac{2bc+^2+c^2-a^2}{2bc}\right)$$

$$=\frac{[a^2-(b-c)^2][(b-c)^2-a^2]}{4b^2c^2}$$

$\therefore \sin^2A = \frac{(a+c-b)(a+b-c)(b+c-a)(b+c+a)}{4b^2c^2}$

Let $s = \frac{1}{2}[perimeter\ of\ triangle]$

$$=\frac{1}{2}[a+b+c]$$

$2s = [a + b + c]$

$a + b = 2s - c$; i.e. $a + b - c = 2s - c - c = 2(s - c)$

$a + c = 2s - b$; i.e. $a + c - b = 2s - b - b = 2(s - b)$

$b + c = 2s - a$; i.e. $b + c - a = 2s - a - a = 2(s - a)$

$\therefore \sin^2A = \frac{2(s-b).2(s-c).2(s-a).2s}{4b^2c^2}$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly, $\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

(b) $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$

From $\sin^2\frac{1}{2}A = \frac{1}{2}(1 - cosA)$

$$=\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\left(\frac{2bc-b^2+c^2-a^2}{4bc}\right)$$

$$=\left(\frac{a^2-(b-c)^2}{4bc}\right)$$

$$=\left(\frac{(a+c-b)(a+b-c)}{4bc}\right)$$

$$=\left(\frac{2(s-b).2(s-c)}{4bc}\right)$$

$$=\left(\frac{(s-b)(s-c)}{bc}\right)$$

$$\therefore \sin\frac{1}{2}A = \sqrt{\left(\frac{(s-b)(s-c)}{bc}\right)}$$

Similarly;

$$\sin\frac{1}{2}B = \sqrt{\left(\frac{(s-b)(s-c)}{ac}\right)}$$

$$\sin\frac{1}{2}C = \sqrt{\left(\frac{(s-b)(s-c)}{ab}\right)}$$

Also;

$\cos^2\frac{1}{2}A = \frac{1}{2}(1 + cosA)$

$$=\left(\frac{2bc+b^2+c^2-a^2}{4bc}\right)$$

$$=\left(\frac{(b+c)^2-a^2}{4bc}\right)$$

$$=\left(\frac{(b+c-a)(a+b+c)}{4bc}\right)$$

$$=\left(\frac{2(s-a).2s}{4bc}\right)$$

$$=\left(\frac{s(s-a)}{bc}\right)$$

$$\therefore \cos\frac{1}{2}A = \sqrt{\left(\frac{(s-a)}{bc}\right)}$$

Similarly;

$$\cos\frac{1}{2}B = \sqrt{\left(\frac{(s-b)}{ac}\right)}$$

$$\cos\frac{1}{2}C = \sqrt{\left(\frac{(s-c)}{ab}\right)}$$

The expression for $\boldsymbol{tan\frac{1}{2}A}$ can be deduced as follows

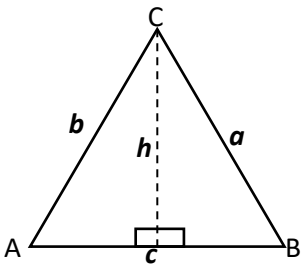
$\tan\frac{1}{2}A = \frac{sin\frac{1}{2}A}{cos\frac{1}{2}A} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly;

$\tan\frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\tan\frac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Area of a triangle



Area, $\Delta = \frac{1}{2}(base)(perpendicular\ height)$

$= \frac{1}{2}ch$

$= \frac{1}{2}cbsinA$

Substituting for

$sinA = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \frac{1}{2}bcx\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

This a convenient form given the three sides of a triangle. The formula is called Hero's formula from the first mathematician who suggested it.

Example 58

The area of a triangle is 336m^2. The sum of the three sides is 84m and one side is 28m. Calculate the length of the remaining two sides

Solution

Given Δ = 336, a + b + c =84 and a = 28

$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(84) = 42$

28 + b + c = 84

b + c = 56, or c = 56 – b

But $\Delta^2 = s(s-a)(s-b)(s-c)$

$336^2 = 42(42–28)(42– b)(42 – 56 + b)$

$b^2 – 56b + 780 = 0$

$b = \frac{56 \pm \sqrt{56^2 - 4x\ 1\ x\ 780}}{2\ x\ 1}$

b = 30 or 26

substituting for c = 56 – b

c = 26 or 30

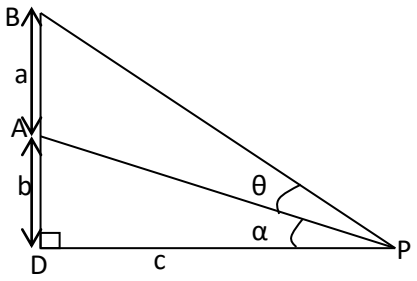
∴the remaining sides are 30m and 26m

Applications of trigonometry in finding distances and bearings

Example 59

A vertical pole BAD stands with its base D on a horizontal plane where BA = a and AD = b. A point P is situated on the horizontal plane at a distance C from D and the angle APB = θ.
Prove that $\theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Solution



Let angle APD = α

For triangle APD: $\tan\alpha = \frac{b}{c}$

For triangle DPB: $\tan(\theta+\alpha) = \frac{a+b}{c}$

⇨ $\frac{tan\theta+tan\alpha}{1-tan\theta tan\alpha} = \frac{a+b}{c}$

Substituting for tanα

⇨ $\frac{tan\theta+\frac{b}{c}}{1-\left(\frac{b}{c}\right)tan\theta} = \frac{a+b}{c}$

$c^2tan\theta + bc = ac + bc - abtan\theta - b^2tan\theta$

$(b^2 + ab + c^2)tan\theta = ac$

$tan\theta = \frac{ac}{b^2+ab+c^2}$

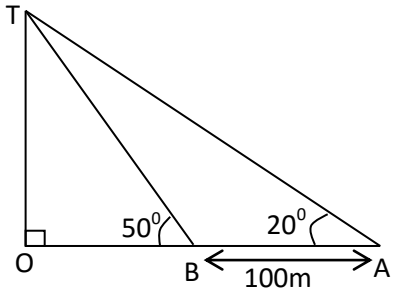
$\therefore\theta = \tan^{-1}\left(\frac{ac}{b^2+ab+c^2}\right)$

Example 60

The angle of the top of a vertical tower from a point A is 20^0 and from another point B is 50^0. Given that A and B lie on the same horizontal plane in the same direction where AB = 100m. Find the height of the tower

Solution

Let OT be the height of the tower



$A\hat{T}B = 50 - 30 = 30^0$

Using sine rule

$\frac{TB}{\sin 20^0} = \frac{100}{\sin 30^0}$

$TB = \frac{100\sin 20^0}{\sin 30^0}$

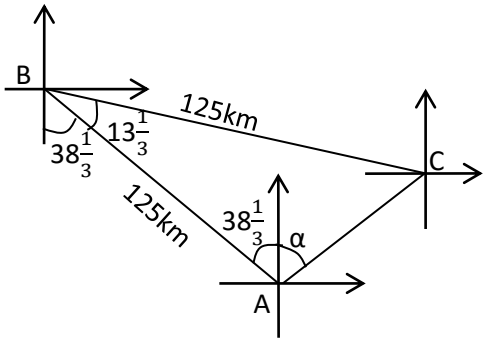
But $OT = TB\sin 50^0$

$OT = \frac{100\sin 20^0 \sin 50^0}{\sin 30^0} = 26.2m$

Example 61

From a point A, a pilot flies in the direction $N38^020'W$ to point B 125km from A. He then flies in the direction $S50^040'E$ for 125km. He wishes to return to A from this point. How far and in what direction must he fly.

Solution



From the digram

Let $B\hat{A}C = B\hat{C}A = \theta$

⇨ $2\theta + 13\frac{1^0}{3} = 180^0$

$\theta = 83\frac{1^0}{3}$

But $38\frac{1}{3} + \alpha = \theta$

$38\frac{1}{3} + \alpha = 83\frac{1^0}{3}$

$\alpha = 45^0$

From the sine rule

$\frac{AC}{sin13\frac{1}{3}} = \frac{125}{sin83\frac{1}{3}}$

AC = 29km

∴ he has to fly 29km in the direction $S45^0W$

Example 62

(a) Prove that $\tan(A - B) = \frac{\tan A - tanB}{1+ tanAtanB}$

$\tan(A - B) = \frac{\sin(A-B)}{\cos(A-B)}$

$= \frac{sinA\ cosB - cosA\ sinB}{\cos A sinB + sinAsin\ B}$

Diving numerator and denominator on the R.H.S by cosAcosB

$\tan(A - B) = \frac{\frac{sinA\ cosB}{\cos A cosB} - \frac{cosA\ sinB}{\cos A cosB}}{\frac{\cos A cosB}{\cos A cosB} + \frac{sinAsin\ B}{\cos A cosB}}$

$= \frac{\tan A - tanB}{1+ tanAtanB}$

Hence show that $\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{1}{\sqrt{3}}$

$$\frac{1-\tan 15^0}{1+\tan 15^0} = \frac{\tan 45^0 - tan15^0}{1+\tan 45^0 tan15^0}$$

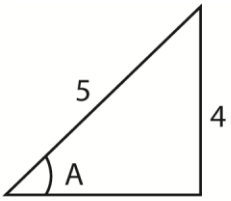
$= \tan(45^0 - 15^0)\tan 30^0 = \frac{1}{\sqrt{3}}$

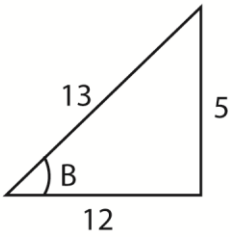
(b) Given that $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$ where A and B are acute, find the values of

(i) tan (A + B)

(ii) cosec (A + B)

Solution





$\cos A = \frac{3}{5}$ $\cos B = \frac{12}{13}$

$\sin A = \frac{4}{5}$ $\sin B = \frac{5}{13}$

$\tan A = \frac{4}{3}$ $\tan B = \frac{5}{12}$

(i) $\tan(A+B) = \frac{sinA\ cosB + cosA\ sinB}{\cos A sinB - sinA sin\ B}$

$= \frac{\frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13}}{\frac{3}{5}.\frac{5}{13} - \frac{4}{5}.\frac{5}{13}} = 3.9375$

(ii) $\text{cosec}(A+B) = \frac{1}{\sin(A+B}$

$= \frac{1}{sinAcosB + cosAsinB}$

$= \frac{1}{\frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13}}$

$= 1.0317$

Example 63

Express $\cos(\theta + 30)^0 - \cos(\theta + 48)^0$ in the form RsinPsinQ, where R is constant.

Hence solve th3 equation

$\cos(\theta + 30)^0 - \cos(\theta + 48)^0 = 0.2$

Solution

$\cos(\theta + 30)^0 - \cos(\theta + 48)^0$

$= -2\sin\left(\frac{\theta + 30^0 + \theta + 48^0}{2}\right)\sin\left(\frac{\theta + 30^0 - \theta - 48^0}{2}\right)$

$= -2\sin(\theta + 39^0)\sin(-9^0)$

$\cos(\theta + 30)^0 - \cos(\theta + 48)^0 = 0.$

⇨ $-2\sin(\theta + 39^0)\sin(-9^0) = 0.2$

$\sin(\theta + 39^0) = 0.63925$

$\theta + 39^0 = 39.74^0$

$\theta = 0.74^0$

Example 64

Express $7\cos 2\theta + 6\sin 2\theta$ in form $R\cos(2\theta - \alpha)$, where R is a constant and α is an acute angle.

$7\cos 2\theta + 6\sin 2\theta \equiv R\cos(2\theta - \alpha)$

$7\cos 2\theta + 6\sin 2\theta \equiv R\cos 2\theta \cos\alpha + R\sin 2\theta \sin\alpha$

Comparing both sides

$R\cos\alpha = 7$ …… (i)

$R\sin\alpha = 6$ …… (ii)

(i)2 + (ii)2 gives

$R = \sqrt{7^2 + 6^2} = \sqrt{85}$

From equation (i)

$\sqrt{85}\cos\alpha = 7$

$\alpha = cos^{-1}\left(\frac{7}{\sqrt{85}}\right) = 40.6^0$

Hence solve $7\cos 2\theta + 6\sin 2\theta = 5$ for $0^0 \leq \theta \leq 180^0$. (07marks)

$\therefore 7\cos 2\theta + 6\sin 2\theta = \sqrt{85}\cos(2\theta - 40.6^0) = 5$

$2\theta - 40.6 = cos^{-1}\left(\frac{5}{\sqrt{85}}\right) = 57.16^0, 302.84^0$

$\theta = 48.88^0, 171.72^0$

Revision exercise 7

1. Solve the triangles

(a) a = 17m, b = 21.42m, B = $51^0 34'$
[A = 38.44^0, C = 90^0, c = 27.34m]

(b) b = 107.2m, c= 76.69m, B = $102^0 25'$
[A = 33.26^0, C = 44.32^0, a = 60.21m]

(c) a = 7m, b = 3.59m, C= 47^0
[A= $103^0 2'$, B = $29^0 52'$, c= 5.25m]

(d) A = 60^0, b = 8m, C = 15
[a = 13, B= 32.2^0, C = 87.8^0]

2. Show that for all values of x

$cosx + cos\left(x + \frac{2\pi}{3}\right) + cos\left(x + \frac{3\pi}{3}\right) = 0$

3. (a) Simplify $\frac{sin3\theta}{sin\alpha} - \frac{cos3\theta}{cos\alpha}\left[\frac{2\sin(3\theta-\alpha)}{sin2\alpha}\right]$

   (b) Express 5sinθ + 12cosθ in the form rsin(θ + α) where r and α are constant. Hence determine the minimum value of 5sinθ + 12cosθ + 7.
   [r =13, α = 67.4^0, -6]

   (c) Given that tanθ = $\frac{3}{4}$, where θ is acute, find values of tan2θ and tan$\frac{\theta}{2}$
   [tan2θ =$\frac{24}{7}$ and tan$\frac{\theta}{2}$ = $\frac{1}{3}$]

4. (a) Show that $2\tan^{-1}\left(\frac{1}{3}\right) + tan\left(\frac{1}{7}\right) = \frac{\pi}{4}$

   (b) Find x given that
   $\tan^{-1}(1+x) + \tan^{-1}(1-x) = 32$
   [x = ± 1.789]

   (c) Given that sinα+ sinβ = p and cosα + cos β = q
   Show that sin(α + β) = $\frac{2pq}{p^2+q^2}$

5. (a) By expressing 2sinθsin(θ + α) as a difference of cosines of two angles or otherwise, where αis constant, find the least value [minimum value = cosα – 1. It occurs when θ = $\frac{-a}{2}$]

   (b) Solve for x in the equation cosx – cos(x + 60^0) = 0.4 for $0^0 \leq x \leq 360^0$ [x: x = 126.4^0, 353.6^0]

6. (a) Prove that in any triangle ABC
   $\frac{b^2-c^2}{a^2} = \frac{\sin(B-C)}{\sin(B+C)}$

   (b) Show that for any isosceles triangle ABC with AB = c the base, is given by Δ =$\frac{1}{2}c\sqrt{s(s-c)}$ where s is the perimeter of the triangle
   Given that Δ= $\sqrt{3}$ and s = 4, determine the sides of the triangle [1, 3.5, 3.5]

7. Given that $\tan^{-1}\alpha = x$ and $\tan^{-1}\beta = y$, by expressing α and β as tangents ratio of x and y and manipulating the ratios show that x + y = $\tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$
   Hence or otherwise
   (i) Solve for x in
   $\tan^{-1}\left(\frac{1}{x-1}\right) + \tan(x+1) = \tan(-2)$
   [x = ± 2]

   (ii) Without using tables of calculators determine the value of
   $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\left[\frac{\pi}{4}\right]$

8. (a) Prove that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$ where ABC has all angles acute and R is the radius of the circumcircle.

   (b) From the top of a vertical cliff 10m high, the angle of depression of ship A is 10^0 and ship B is 15^0. The Bearings of A and B from the cliff are 162^0 and 202.5^0 respectively. Find the bearing of B from A [301.5^0]

9. (a) Prove that
   (sin2x –sinx)(1 + 2cosx) = sin3x

   (b) A vertical pole BAO stands with its base O on a horizontal plane, where BA = c and AO = b, a point P is situated on horizontal plane at a distance x from O and angle APB = θ
   Prove that tanθ =$\frac{cx}{x^2+b^2+bc}$
   As P takes different positions on the horizontal plane, find the value of x for which θ is greatest.
   [$18^0$26', when x = b = c]

10. (a) Prove that sin3x = 3sinx – $4\sin^2 x$.

    (b) Find all the solutions to $2\sin^2 x = 1$ for $00 \leq x \leq 360^0$. [x =10^0, 50^0, 130^0, 170^0, 250^0, 290^0]

11. Solve cosx +$\sqrt{3}$sinx =2 for $0^0 \leq x \leq 360^0$
    [x = 60^0]

12. From the top of a tower 12.6m high, the angles of depression of ship A and B are 12^0 and 18^0 respectively. the bearing of ship A and ship B from the tower are 148^0 and 209.5^0 respectively
    Calculate
    (i) How far the ships are from each other [53.14m]
    (ii) The bearing of ship A from ship B[108.1^0]

13. (a) Solve sin3x + $\frac{1}{2}$ = $2\cos^2 x$ for $0^0 \leq x \leq 360^0$
    [x = 30^0, 60^0, 120^0, 150^0, 240^0, 300^0]

    (b) Given that in any triangle ABC, $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}cot\left(\frac{A}{a}\right)$ solve the triangle with two sides 5 and 7 and the included angle 45^0.
    [A = 45^0, B = 89.4^0, C= 45.6^0]

14. (a) Solve $\cot^2 x = 5(\text{cosec}x + 1)$ for $0^0 \le x \le 360^0$ [9.6^0, 170.4^0, 270^0]
    (b) Use $\tan\frac{\theta}{2} = t$ to solve $5\sec\theta - 2\sin\theta = 2$ for $0^0 \le x \le 360^0$ [46.4^0, 270^0]
15. Given that $\sin 2x = \cos 3x$, fins the values of $\sin\theta$, $0 \le x \le \pi$ [0.309 3dp]
16. (a) Show that $\tan\left(\frac{A+B}{2}\right) - \tan\left(\frac{A-B}{2}\right) = \frac{2sinB}{cosA+cosB}$
    (b) Find in radians the solution of the equation $\cos\theta + \sin 2\theta = \cos 3\theta$ for $0 \le \theta \le \pi$ $\left[0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right]$
17. (a) Show that $\cot A + \tan 2A = \cot A \sec 2A$
    (b) Show that $tan3x = \frac{3t-t^3}{1-3t^2}$, where $t = \tan x$. Hence or otherwise show that $\tan^{-1}\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$
18. (a) Find all the values θ, $00 \le \theta \le 3600$, which satisfies the equation $\sin^2\theta - \sin 2\theta - 3\cos^2\theta = 0$ [$\theta = 135^0$, 315^0]
    (b) Show that $\frac{cosx}{1+sinx} = cot\left(\frac{x}{2} + 45^0\right)$. Hence or otherwise solve $\frac{cosx}{1+sinx} = \frac{1}{2}$; $0^0 \le x \le 360^0$ [$x = 36.8^0$]
19. (a) Given that X, Y and Z are angles of a triangle XYZ. Prove that $tan\left(\frac{X-Y}{2}\right) = \frac{x-y}{x+y}cot\frac{Z}{2}$.
    Hence solve the triangle if x = 9cm, y = 5.7cm and $Z = 57^0$. [z = 7.6cm, $X = 84.4^0$]
    (b) Use the substitution $t = \tan\left(\frac{\theta}{2}\right)$ to solve the equation $3\cos\theta - 5\sin\theta = -1$ for $0^0 \le \theta \le 360^0$ [40.84^0, 201.1^0]
20. Prove that $tan\left(\frac{\pi}{4} + \theta\right) - tan\left(\frac{\pi}{4} - \theta\right) = 2tan2\theta$
21. (a) Solve the equation $3\cos x + 4\sin x = 2$ for $0^0 \le x \le 360^0$ [$x = 119.5^0$, 346.7^0]
    (b) If A, B, C are angles of a triangle. Show that

    $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B$
22. (a) Solve $2\sin 2\theta = 3$ for $-180^0 \le x \le 180^0$ [-90^0, 48.6^0,90^0, 131.4^0]
    (b) Solve $\sin x - \sin 4x = \sin 2x - \sin 3x$ for $-\pi \le x \le \pi$

    $\left[-\frac{\pi}{5}, -\frac{\pi}{2}, -\frac{3\pi}{5}, 0, \frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}\right]$
23. Without using tables or calculator, show that $\tan 150 = 2-\sqrt{3}$
24. (a) Solve the equation $\cos x + \cos 2x = 1$ for $0^0 \le x \le 360^0$ [$x = 38.67^0$, 321.33^0]
    (b) (i) Prove that $\frac{cosA+cosB}{sinA+sinB} = cot\frac{A+B}{2}$
    (ii) $\frac{cosA+cosB}{sinA+sinB} = tan\frac{C}{2}$ where A, B and C are angles of a triangle
25. Given that $\sin(\theta - 45^0) = 3\cos(\theta + 45^0)$ show that $\tan\theta = 1$. Hence find θ if $0^0 \le \theta \le 360^0$ [45^0, 225^0]
26. (a) Use the factor formula to show that $\frac{\sin(A+2B)+sinA}{coa(A+2B)+cosA} = \tan(A+B)$
    (b) Express $y = 8\cos x + 6\sin x$ in the form $R\cos(x - \alpha)$ where R is positive and α is acute
    Hence find the maximum and minimum values of $\frac{1}{8cosx+6sinx+15}$ [0.2, 0.04]
27. Express $\sin x + \cos x$ in the form $R\cos(x - \alpha)$. Hence, find the greatest value of $\sin x + \cos x - 1$. [0.4142]
28. (a) Solve $\cos x + \cos 3x = \cos 2x$, $0 \le x \le 360^0$ [$x = 45^0$, 60^0, 135^0, 225^0, 300^0, 315^0]
    (b) Show that $tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1+sin\theta}{cos\theta}$
29. Show that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{7}{9}$
30. (a) Solve $3\sin x + 4\cos x = 2$ for $-180^0 \le x \le 180^0$. [-29.55^0, 103.29^0]
    (b) Show that in any triangle ABC $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$
31. (a) Prove that $\frac{2tanx}{1+tan^2x} = sin2x$
    (b) Solve $\sin 2x = \cos x$; $0^0 \le x \le 90^0$ [$x = 30^0$, 90^0]
32. (a) Solve the equation $8\cos^4 x - 10\cos^2 x + 3 = 0$; $0^0 \le x \le 180^0$ [30^0, 45^0, 135^0, 150^0]
    (b) Prove that $\cos 4A - \cos 4B - \cos 4C = 4\sin 2B\sin 2C\cos 2A - 1$ given that A, B and C are angles of a triangle
33. Given that $\cos 2A - \cos 2B = -p$ and $\sin 2A - \sin 2B = q$, prove that $\sec(A + B) = \frac{1}{q}\sqrt{p^2 + q^2}$
34. Solve
(a) $4\sin^2\theta - 12\sin 2\theta + 35\cos^2\theta = 0$; for $0^0 \le \theta \le 90^0$ [74.0^0]

(b) $3\cos\theta - 2\sin\theta = 2$, for $0^0 \leq \theta \leq 360^0$ $[\theta: \theta = 22.62^0, 270.00^0]$

35. Solve the equation $\sin 2\theta + \cos 2\theta \cos 4\theta = \cos 4\theta \cos 6\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$. $\left[\theta = 0, \frac{3\pi}{16}\right]$

36. (a) solve the equation $\cos 2x = 4\cos^2 x - 2\sin^2 x$ for $0 \leq \theta \leq 180^0$ $[\theta = 60^0, 120^0]$

(b) Show that if $\sin(x + \alpha) = p\sin(x - \alpha)$ then $\tan x = \left(\frac{p+1}{p-1}\right)\tan\alpha$. Hence solve the equation $\sin(x + \alpha) = p\sin(x - \alpha)$ for $p = 2$ and $\alpha = 20^0$. $[x = 47.52^0]$

37. Solve the equation $3\tan^2\theta + 2\sec^2\theta = 2(5 - 3\tan\theta)$ for $0^0 < \theta < 180^0$ $[\theta = 38.66^0, 116.57^0]$

38. (a) Show that $\tan 4\theta = \frac{4t(1-t^2)}{t^4-6t^2+1}$, where $t = \tan\theta$

(b) Solve the equation $\sin x + \sin 5x = \sin 2x + \sin 4x$ for $0^0 < x < 90^0$. $[x = 60^0]$

39. Solve $2\cos 2\theta - 5\cos\theta = 4$ for $0^0 \leq \theta \leq 360^0$. $[\theta = 138.59^0, 221.41^0]$

Thank you

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Permutations and combinations

Permutation

A permutation is an ordered arrangement of a number of objects

Consider digits 1, 2 and 3; find the possible arrangements of the digits

123, 132, 321
231, 321, 321 } The total of six

This problem may also be solved as follows:

Given the three digits above, the first position can take up three digits, the second position can take up two digits and the third position can take up 1 digit only

1st position	2nd position	3rd position
3	2	1

The total is thus 3 x 2 x 1 = 6

If the digits were four say 1, 2, 3, 4 the arrangement would be

1st	2nd	3rd	4th
4	3	2	1

The total is thus 4 x 3 x 2 x 1 = 24

In summary the number of ways of arranging n different items in a row is given by n(n – 1)(n – 2)(n -3) x …….2 x 1 and can be expressed as n!

If the total number of books is 6

The total number of arrangements = 6!

= 6 x 5 x 4 x 3 x 2 x 1 = 720 ways

Example 1

Find the values of the following expression

(a) 5!
Solution
5!= 5 x 4 x 3 x 2 x 1 = 120

(b) $\frac{8!}{5!}$
Solution
$\frac{8!}{5!} = \frac{8\ x\ 7\ x\ 6\ x\ 5!}{5!} = 336$

(c) $\frac{10!}{6!x\ 5!\ x\ 2!}$
Solution
$\frac{10!}{6!x\ 5!\ x\ 2!} = \frac{10\ x\ 9\ x\ 8\ x\ 7\ x\ 6!}{6!x\ 5\ x\ 4\ x\ 3\ x\ 2\ x\ 1\ x\ 2x\ 1} = 21$

(d) Four different pens and 5 different books are to be arranged on a row. Find
(i) The number of possible arrangements of items
Solution
Total number of items = 4 + 5 = 9
Total number of arrangements = 9!
= 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1
= 362,880 ways
(ii) The number of possible arrangements if three of books must be kept together
Solution
The pens are taken to be one since they are to be kept together. So we consider total number of items to six.
The number of arrangements of six items = 6! = 6 x 5 x 4 x 3 x 2 x 1
= 720 ways
The arrangement of 4 pens = 4!
= 4 x 3 x 2 x 1 =24
Total number arrangements of all the items = 720 x 24 = 17, 280

Multiplication principle of permutation

If one operation can be performed independently in a different ways and the second in b different ways, then either of the two events can be performed in (a + b) ways

Example 2

There are 6 roads joining P to Q and 3 roads joining Q to R. Find how many possible routes are from P to R

From P to Q = 6 ways

From Q to R = 3 ways

Number of routes from P to R = 6 x 3 = 18

Example 3

Peter can eat either matooke, rice or posh on any of the seven days of the week. In how many ways can he arrange his meals in a week

Solution

For each of the 7 days, there are 3 choices

Total number of arrangements

= 3 x 3 x 3 x 3 x 3 x 3 x 3 = 3^7 = 2187 ways

Example 4

There are four routes from Nairobi to Mombasa. In how many different ways can a taxi go from Nairobi to Mombasa and returning if for returning:

(a) any of the route is taken
= 4 x 4 = 16 ways
(b) the same route is taken
= 4 x 1 = 4 ways
(c) the same route is not taken
= 4 x 3 = 12 ways

Example 5

David can arrange a set of items in 5 ways and John can arrange the same set of items in 3 ways. In how many ways can either David or John arrange the items?

Solution

Number of ways in which David arranges = 5

Number of ways in which John arranges = 3

Number of ways in which either David or John arrange the items = 5 + 3 = 8 ways

The number of permutation of r objects taken from n unlike objects

The permutation of n unlike objects taking r at a time is denoted by ${}^nP_r$ which is defined as

${}^nP_r = \frac{n!}{(n-r)!}$, where r ≤ n.

In case r = n, we have ${}^nP_n$ which is interpreted as the number of arranging n chosen objects from n objects denoted by n!

${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! => 0! = 1$

Example 6

How many three letter words can be formed from the sample space {a, b, c, d, e, f}

Solution

Total number of letters = 6 and r = 3

Total number of worms = ${}^6P_3$

$= \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\ x\ 5\ x\ 4\ x\ 3!}{3!} = 120$ways

Example 7

Find the possible number of ways of arranging 3 letters from the word MANGOES

Solution

Total number of letter in the word = 7

and r = 3

Number of ways ${}^7P^3 = \frac{7!}{(7-3)!}$

$=\frac{7\ x\ 6\ x\ 5\ x\ 4\ x\ 3!}{3!} = 840$ways

Example 8

Find number of ways of arranging six boys from a group of 13

Solution

Number of arrangements = $^{13}P_6$

$=\frac{13!}{(13-6)!} = \frac{13!}{7!}$

$= \frac{13x12x11x10\ x9\ x\ 8\ x\ 7!}{7!}$

= 1235520 ways

The number of permutations of n objects of which r are alike

The number of permutations of n objects of which r are alike is given by $\frac{n!}{r!}$

Example 9

Find the number of arranging in a line the letters B, C, C, C, C, C, C

The number of ways of arranging the seven letters of which of which 6 are alike
$= \frac{7!}{6!} = \frac{7\ x\ 6!}{6!} = 7$ ways

The number of ways of permutations on n objects of which p of one type are alike, q of the second type are alike, r of the third type are alike, and so on.

The number of ways of permutations on n objects of which p of one type are alike, q of the second type are alike, r of the third type are alike given by $\frac{n!}{p!\ x\ q!\ x\ r!}$

Example 10

Find the possible number of ways of arranging the letter of the word MATHEMATICS in line

Solution

The word MATHEMATICS has 11 letters and contains 2 M, 2A and 2T repeated

The number of ways = $\frac{11!}{2!x\ 2!\ x\ 2!}$

$= \frac{11\ x\ 10\ x\ 9\ \ x\ 8\ x\ 7\ x\ 6\ x\ 5\ x\ 4x3\ x\ 2\ x\ 1}{2\ x\ 2\ x\ 2}$

= 4,989,600

Example 11

Find the possible number of ways of arranging the letter of the word 'MISSISSIPPI' in line

Solution

The word 'MISSISSIPPI' has 10 letters with 4I, 4S, and 2P

The number of ways = $\frac{11!}{4!x\ 4!\ x\ 2!} = 34650$

The number of permutations of the like and unlike objects with restrictions

One should be cautious when handling these problems

Example 12

Find the possible number of ways of arranging

The letters of the word MINIMUM if the arrangement begins with MMM?

Solution

There is only one way of arranging MMM

The remaining contain four letters with 2I can be arranged in

$\frac{1\ x\ 4!}{2!} = \frac{1\ x\ 4\ x\ 3\ x\ 2!}{2!} = 12$ ways

Example 13

(a) How many 4 digit number greater than 6000 can be formed from 4, 5, 6, 7, 8 and 9 if:

(i) Repetitions are allowed

Solution

The first digit can be chosen from 6, 7, 8 and 9, hence 4 possible ways, the 2nd, 3rd and 4th are chosen from any of the six digits since repetitions are allowed

position	1st	2nd	3rd	4th
selections	4	6	6	6

Number of ways

= 4 x 6 x 6 x 6 = 864 ways

(ii) Repetition are not allowed

The first can be chosen from 6, 7, 8 and 9, hence 4 possible ways, the 2nd from 5, 3rd from 4 and 4th from 3 since no repetitions are allowed

position	1st	2nd	3rd	4th
selections	4	5	4	3

Number of ways
= 4 x 5 x 4 x 3 = 240 ways

(b) Find how many four digit numbers can be formed from the six digits 2, 3, 5, 7, 8 and 9 without repeating any digit.
Find also how many of these numbers
(i) Are less than 7000
(ii) Are odd

Solution

position	1st	2nd	3rd	4th
selections	6	5	4	3

Total number of ways = 6 x 5 x 4 x 3 = 360

(i) The 1st number is selected from three (2, 3, 5), the 2nd number from 5, the 3rd from 4 and the 4th from 3 digits

position	1st	2nd	3rd	4th
selections	6	5	4	3

Total number of less than 7000

= 3 x 5 x 4 x 3 = 180

(ii) The last number is selected from four odd digits (3, 5, 7, and 9), the 1st number selected from five remaining, 2nd from 4 and 3rd from 3

position	1st	2nd	3rd	4th
selections	5	4	3	4

Total number of odd numbers formed

= 5 x 4 x 3 x 4 = 240

(c) How many different 6 digit number greater than 500000 can be formed by using the digits 1, 5, 7, 7, 7, 8
Solution
The 1st digit is selected from five (5, 7, 7, 7, 8), the 2nd from remaining five, 3rd from four, 4th from three, 5th from two and 6th from one

1st	2nd	3rd	4th	5th	6th
5	5	4	3	2	1

Total number = $\frac{5\ x\ 5\ x\ 4\ x\ 3\ x\ 2\ x\ 1}{3!}$ = 100
NB. The number is divided by 3! Because 7 appears three times

(d) How many odd numbers greater than 60000 can be formed from 0, 5, 6, 7, 8, 9, if no number contains any digit more than once
Solution
Considering six digits
Taking the first digit to be odd, the first digit is selected from 3 digits (5, 7, 9) and the last is selected from 2 digits

1st	2nd	3rd	4th	5th	6th
3	4	3	2	1	2

Number of ways = 3 x 4 x 3 x 2 x 1 x 2
= 144
Taking the first digit to be even, the first digit is selected from 2 digits (6, 8) since the number should be greater than 60000 and the last is selected from 2 odd digits

1st	2nd	3rd	4th	5th	6th
2	4	3	2	1	3

Number of ways = 3 x 4 x 3 x 2 x 1 x 2
= 144
Considering five digits
Taking the first digit to be odd, the digit greater than 6 are 7 and 9 so first digit is selected from 2 digits and the last is selected from 2 digits

1st	2nd	3rd	4th	5th
2	4	3	2	2

Number of ways = 2 x 4 x 3 x 2 x 2
= 96
Taking the first digit to be even, the first digit is selected from 2 digits (6, 8) since the number should be greater than 60000 and the last is selected from 3 odd digits (5, 7, 9)

1st	2nd	3rd	4th	5th
2	4	3	2	3

Number of ways = 2 x 4 x 3 x 2 x 3
= 144
The total number of selections
= 144 + 144 + 96 + 144 = 528

Example 14

The six letter of the word LONDON are each written on a card and the six cards are shuffled and placed in a line. Find the number of possible arrangements if

(a) The middle two cards both have the letter N on them
Solution
If the middle letter are NN, the we need to find the number of different arrangements of the letter LODO.
With the 2O's, the number of arrangements = $\frac{4!}{2!} = 12$

(b) The two cards with letter O are not adjacent and the two cards with letter N are also not adjacent
Solution
If the two cards are not adjacent, the number of arrangements = Total number of arrangements of the word LONDON – number of arrangements when the two letters are adjacent
= = $\frac{6!}{2!2!} - 24 = 156$

Example 15

In how many different ways can letters of the word MISCHIEVERS be arranged if the S's cannot be together

Solution

There are 11 letters in the word MISCHIEVERS with 2S's, 2I's and 2E's

Total number of arrangements

= $\frac{11!}{2!2!2!}$ = 4989600

If S's are together, we consider them as one, so the number of arrangements

= $\frac{10!}{2!2!}$ = 907200

∴ the number of possible arrangements of the word MISCHIEVERS when S's are not together

= 4989600 – 907200 = 4082400

The number of permutation of n different objects taken r at a time, if repetition are permitted

Example 16

How many four digit numbers can be formed from the sample space {1, 2, 3, 4, 5} if repetitions are permissible

Solution

The 1st position has five possibilities, the 2nd five, the 3rd five, the 4th five

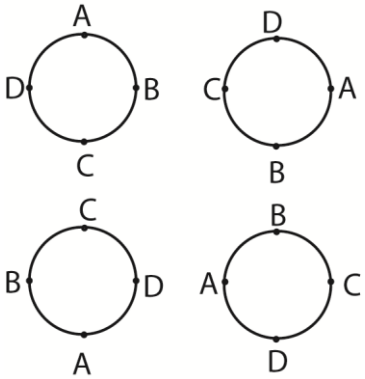
Number of permutations = 5 x 5 x 5 x 5 = 625

Circular permutations

Here objects are arrange in a circle

The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise are different.

Consider four people A, B, C and D seated at a round table. The possible arrangements are as shown below



With circular arrangements of this type, it is the relative positions of the objects being arranged which is important. The arrangements of the people above is the same. However, if the people were seated in a line the arrangements would not be the same, i.e. A, B, C, D is not the same as D, A, B, C. When finding the number of different arrangements, we fix one person say A and find the number of ways of arranging B, C and D.

Therefore, the number of different arrangements of four people around the table is 3!

Hence the number of different arrangements of n people seated around a table is (n – 1)!

Example 17

(a) Seven people are to be seated around a table, in how many ways can this be done
Solution
The number of ways = (7 – 1)! = 6!
= 720

(b) In how many ways can five people A, B, C, D and E be seated at a round table if
(i) A must be seated next to B

Solution

If A and B are seated together, they are taken as bound together. So four people are considered

The number of ways = (4 – 1)! = 3! = 6

The number of ways in which A and B can be arranged = 2

The total number of arrangements

= 6 x 2 = 12 ways

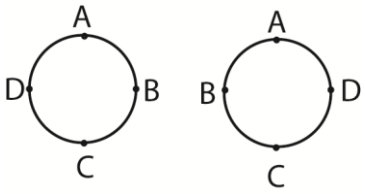
(ii) A must not seat next to B

If A and B are not seated together, then the number of arrangements = total number of arrangements – number of arrangements when A and B are seated together

= (5 – 1)! – 12 = 12 ways

The number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are the same

Consider the four people above, if the arrangement is as shown below



Then the above arrangements are the same since one is the other viewed from the opposite side

The number of arrangements = $\frac{3!}{2}$ = 3ways

Hence the number of ways of arranging n unlike objects in a ring when clockwise and anticlockwise arrangements are the same
= $\frac{(n-1)!}{2}$

Example 18

A white, a blue, a red and two yellow cards at arranged on a circle. Find the number of arrangements if red and white cards are next to each other.

Solution

If red and white cards are next to each other, they are considered as bound together. So we have four cards. Since anticlockwise and clockwise arrangements are the same and there are two yellow cards, the number of arrangements = $\frac{(4-1)!}{2\ x\ 2!} = \frac{3!}{2\ x\ 2!}$

The number of ways of arranging red and white cards = 2

Total number of ways of arrangements

=$\frac{3!}{2\ x\ 2!}\ x\ 2 = 3$

Revision exercise 1

1. In how many ways can the letters of the words below be arranged
   (a) Bbosa (5!]
   (b) Precious [8!]
2. How many different arrangements of the letters of the word PARALLELOGRAM can be made with A's separate [83160000]
3. How many different arrangements of the letters of the word CONTACT can be made with vowels separated? [900]
4. How many odd numbers greater than 6000 can be formed using digits 2, 3, 4, 5 and 6 if each digit is used only once in each number [12]

5. Three boys and five girls are to be seated on a bench such that the eldest girl and eldest boy sit next to each other. In how many ways can this be done [2 x 7!]
6. A round table conference is to be held between delegates of 12 countries. In how many ways can they be seated if two particular delegates wish to sit together [2 x 10!]
7. In how many ways can 4 boys and 4 girls be seated at a circular table such that no two boys are adjacent [144]
8. How many words beginning or ending with a consonant can be formed by using the letters of the word EQUATION? [4320]

Combinations

A combination is a selection of items from a group not basing on the order in which the items are selected

Consider the letters A, B, C, D

The possible arrangements of two letters chosen from the above letters are

AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC. AS seen earlier, the total number of arrangements of the above letters is expressed as $\frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$

However, when considering combinations, the grouping such as AB and BA are said to be the same groupings such as CA and AC, AD and DA, etc.

So the possible combinations are AB, AC, AD, BC, BD, CD which is six ways.

Thus the number of possible combinations of n items taken r at a time is expressed as ${}^{n}C_{r}$ or $\binom{n}{r}$ which is defined as ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$
where r ≤ n

Hence the number of combinations of the above letters taken two at a time is

$$\binom{4}{2} = \frac{4!}{2!2!} = 6$$

Example 19

A committee of four people is chosen at random from a set of seven men and three women

How many different groups can be chosen if there is at least one

(i) Woman on the committee
Solution
Possible combinations

7 men	3 women
3	1
2	2
1	3

The number of ways of choosing at least one woman

$$=\binom{7}{3} x \binom{3}{1} + \binom{7}{2} x \binom{3}{2} + \binom{7}{1} x \binom{3}{3}$$

$$=\frac{7!}{3!4!} x \frac{3!}{1!2!} + \frac{7!}{2!5!} x \frac{3!}{2!1!} + \frac{7!}{1!6!} x \frac{3!}{3!0!} = 175$$

(ii) Man on the committee

7 men	3 women
1	3
2	2
3	1
4	0

The number of ways of choosing at least one man

$$\binom{7}{1} x \binom{3}{3} + \binom{7}{2} x \binom{3}{2} + \binom{7}{3} x \binom{3}{1} + \binom{7}{4} x \binom{3}{0}$$

$$=\frac{7!}{1!3!} x \frac{3!}{0!3!} + \frac{7!}{5!2!} x \frac{3!}{1!2!} + \frac{7!}{4!3!} x \frac{3!}{2!1!} + \frac{7!}{3!4!} x \frac{3!}{3!0!}$$

=210

Example 20

A group of nine has to be selected from ten men and eight women. It can consist of either five men and four women or four men and five women. How many different groups can be chosen?

Solution

Possible combination

10 men	8 women
5	4
4	5

Number of groups = $\binom{10}{5} x \binom{8}{4} + \binom{10}{4} x \binom{8}{5}$

= $\frac{10!}{5!5!}$ x $\frac{8!}{4!4!} + \frac{10!}{6!4!}$ x $\frac{8!}{5!3!}$ = 29400

Example 21

A team of six is to be formed from 13 boys and 7 girls. In how many ways can the team be selected if it must consist of

(a) 4 boy and 2 girls

13 boys	7 girls
4	2

$\binom{13}{4} . \binom{7}{2}$= $\frac{13!}{9!4!} x \frac{7!}{5!2!}$ = 15015

(b) At least one member of each sex

Possible combinations

13 boys	7 girls
5	1
4	2
3	3
2	4
1	5

$=\binom{13}{5} . \binom{7}{1} + \binom{13}{4} . \binom{7}{2} + \binom{13}{3} . \binom{7}{3} + \binom{13}{2} . \binom{7}{4} + \binom{13}{1} . \binom{7}{5}$

$=\frac{13!}{8!5!} . \frac{7!}{6!1!} + \frac{13!}{9!4!} . \frac{7!}{5!2!} + \frac{13!}{10!3!} . \frac{7!}{4!3!} + \frac{13!}{11!2!} . \frac{7!}{3!4!} + \frac{13!}{12!1!} . \frac{7!}{2!5!}$

= 37037

Example 22

A team of 11 players is to be chosen from a group of 15 players. Two of the 11 are to be randomly elected a captain and vice-captain respectively. In how many ways can this be done?

Number of ways of choosing 11 players from 15 = $\binom{15}{11}$

A captain will be elected from 11 players and a vice-captain from 10 players

Total number of selection = $\binom{15}{11} \; x \; 11 \; x \; 10$

= $\frac{15!}{4!11!}$ x 11 x 10 = 150150

Example 23

(a) Find the number of different selections of 4 letters that can be made from the word UNDERMATCH.

Solution

There are 10 letters which are all different

Number of selections of 4 letters from 10 is given by $\binom{10}{4} = \frac{10!}{(10-4)!4!} = \frac{10!}{6!4!} = 210$

(b) How many selections do not contain a vowel?

Solution

Number of vowels in the word = 2

Number of letters not vowels = 8

Number of selections of 4 letters from 10 without containing a vowel = selecting 4 letters from 8 consonants =

$\binom{8}{4} = \frac{8!}{(8-4)!4!} = \frac{8!}{4!4!} = 70$

Example 24

In how many ways can three letters be selected at random from the word BIOLOGY is selection

(a) Does not contain the letter O

Solution

Number of selections without the letter O = number of ways of choosing three letters from B, I, L, G, y

$= \binom{5}{3} = \frac{5!}{2!3!} = 10$

(b) Contain only the letter O

Solution

Number of selections with one letter O = number of ways of choosing two letters from B, I, L, G, y

$= \binom{5}{2} = \frac{5!}{3!2!} = 10$

(c) Contains both of the letters O

Solution

Number of selections with two letter O = number of ways of choosing one letter from B, I, L, G, y

$= \binom{5}{1} = \frac{5!}{4!1!} = 5$

Example 25

In how many ways can four letters be selected at random from the word BREAKDOWN if the letters contain at least one vowel?

Solution

Vowels: E, A, O (3)

Consonants: B, R, K, D, W, N (6)

Consonants (6)	Vowels (3)
3	1
2	2
1	3

Number of selection of four letters with at least one vowel

$$= \binom{6}{3}.\binom{3}{1} + \binom{6}{2}.\binom{3}{2} + \binom{6}{1}.\binom{3}{3} = 111$$

Example 26

How many different selections can be made from the six digits 1, 2, 3, 4, 5, 6

Solution

Note: this an open questions because selections can consist of only one digit, two digits, three digits, four digits, five digits or six digits

Number of selection of 1 digit = ${}^6C_1 = 6$

Number of selection of 2 digits = ${}^6C_2 = 15$

Number of selection of 3 digits = ${}^6C_3 = 20$

Number of selection of 4 digits = ${}^6C_4 = 15$

Number of selection of 5 digits = ${}^6C_5 = 6$

Number of selection of 6 digits = ${}^6C_1 = 1$

Total number of selections

= 6 + 15 + 20 + 15 +6 + 1 = 63

This approach is tedious for a large group of objects.

The general formula for selection from n unlike objects is given by $2^n - 1$.

For the above problems, number of selections = $2^6 - 1 = 63$

Example 26

How many different selections can be made from 26 different letters of the alphabet?

Number of selection = $2^{26} - 1$

= 67,108,863

Cases involving repetitions

Suppose we need to find the number of possible selections of letters from a word containing repeated letters, we take the selections mutually exclusive

Example 27

How many different selections can be made from the letters of the word CANADIAN?

Solution

There are 3A's, 2N's and 3 other letters

The A's can be dealt with in 4 ways (either no A, 1A's, 2A's or 3A's)

The N's can be dealt in 3 ways (no N, 1N, or 2N's)

The C can be dealt with in 2 ways (no C, 1C)

The D can be dealt with in 2 ways (no D, 1D)

The I can be dealt with in 2 ways (no I, 1I)

The number of selections

= 4 x 3 x 2 x 2 x 2 – 1 = 95

Example 28

How many different selections can be made from the letters of the word POSSESS?

Solution

There are 4S's and 3 other letters

The S's can be dealt in 5 ways (no S, 1S, 2N's, 3S's, 4S's, or 5S's)

The P can be dealt with in 2 ways (no P, 1P)

The O can be dealt with in 2 ways (no O, 1O)

The E can be dealt with in 2 ways (no E, 1E)

Total number of selections = 5 x 2 x 2 x 2 – 1

= 39

Cases involving division into groups

The number of ways of dividing n unlike objects into say two groups of p and q where p + q = n is given by $\frac{n!}{p!q!}$

For three groups of p, q and r provided p + q + r = n

Number of ways of division = $\frac{n!}{p!q!r!}$

However, for the two groups above, if p = q then the number of ways of division = $\frac{n!}{p!p!2!}$

For three groups where p = q = r

then the number of ways of division = $\frac{n!}{p!p!p!3!}$

Example 29

The following letters a, b, c, d, e, f, g, h, I, j, k, l are to be divided into groups containing

(a) 3, 4, 5
(b) 5, 7
(c) 6, 6
(d) 4, 4, 4 letters. In how many ways can this be done?

Solution

(a) Number of ways = $\frac{12!}{3!4!5!}$ = 27720
(b) Number of ways = $\frac{12!}{5!7!}$ = 792
(c) Number of ways = $\frac{12!}{6!6!2!}$ = 462
(d) Number of ways = $\frac{12!}{4!4!4!3!}$ =5775

Example 30

Find the number of ways that 18 objects can be arranged into groups if there are to be

(a) Two groups of 9 objects each
(b) Three groups of 6 objects each
(c) 6 groups of 3 objects each
(d) Three groups of 5, 6 and 7 objects each

Solution

(a) Number of ways = $\frac{18!}{9!9!2!}$ = 24310
(b) Number of ways = $\frac{18!}{6!6!6!3!}$ = 2858856
(c) Number of ways = $\frac{18!}{3!3!3!3!3!6!}$ =190590400
(d) Number of ways = $\frac{18!}{5!6!7!}$ = 14702688

Example 31

(a) Find how many words can be formed using all letters in the word MINIMUM.

Solution

Number of ways of arranging the letters = 7!

There are 3M's and 2I's

Number of words formed = $\frac{7!}{3!2!} = 420$

(b) Compute the sum of four-digit numbers formed with the four digits 2, 5, 3, 8 if each digit is used only once in each arrangement

Solution

Number of ways of arranging a four digit number = 4!

Sum of any four digit number formed
= 2 + 5 + 3 + 8 = 18

Total sum of four digit numbers formed

= 18 x 4! = 432

(c) A committee consisting of 2 men and 3 women is to be formed from a group of 5 men and 7 women. Find the number of different committees that can be formed. If two of the women refuse to serve on the same committee, how many committees can be formed?

Solution

The committees formed = ${}^5C_2.{}^7C_3$

= 10 x 35 = 350

Suppose two women are to serve together, we take them as glued together, so the number of committees = ${}^5C_2.{}^6C_3$ = 200

Number of committees in which two women refuse to serve together = 350 – 200 = 150

Revision exercise 2

1. (a) Find the number of different selection of 3 letters that can be made from the word PHOTOGRAPH. [53]
   (b) How many of these selections contain no vowel [18]
   (c) How many of these selections contain at least one vowel? [35]
2. (a) find the number of different selections of 3 letters that can be made from the letters of the word SUCCESSFUL.[36]
   (c) How many of these selections contain only consonants [11]
   (d) How many of these selections contain at least one vowel [25]
3. (a) Find the value of n if ${}^{n}P_4 = 30{}^{n}C_5$ [8]
   (b) How many arrangement can be made from the letters of the name MISSISSIPPI
      (i) when all the letters are taken at a time [34650]

      (ii) If the two letters PP begin every word [630 ways]

   (c) Find the number of ways in which a one can chose one or more of the four girls to join a discussion group [15 ways]
4. Find in how many ways 11 people can be divided into three groups containing 3, 4, 4 people each. [5775]
5. A group of 5 boys and 8 girls. In how many ways can a team of four be chosen, if the team contains
   (a) No girl [5]
   (b) No more than one girl [85]
   (c) At least two boys [365]
6. Calculate the number of 7 – letter arrangements which can be made with the letters of the word MAXIMUM. In how many of these do all the 4 consonants appear next to each other? [840, 96]
7. In how many ways can a club of 5 be selected from 7 boys and 3 girls if it must contain
   (a) 3 boys and 3 girls [105]
   (b) 2 men and 3 girls [21]
   (c) At least one girl [231]
8. How many different 6 digit numbers greater than 400,000 can be formed form the following digits 1, 4, 6, 6, 6 7? [100]

Thank you

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Series

Introductions

Numbers arranged in a definite order a sequence. Each number in the sequence is derived from a particular rule.

The terms below are examples of sequences

(a) 1, 3, 5, 7, 9 ….. is a sequence of odd numbers
(b) 2, 3, 5, 7, 11 … is a sequence of prime number
(c) 4, 16, 64 …. is a sequence formed by multiplying the proceding number by 4 to give the next number

Series are categorized into two:

- Arithmetic progression (A.P)
- Geometric progression

Arithmetic progression (A.P)

This is a series in which each term is obtained from the preceding one by addition or subtraction of a constant quantity.

The series 1 + 3 + 5 + 7 + 9 ….. is an A.P

Note the following in an A.P

(i) The first term of an A.P is denoted a. the first letter of the English alphabet
(ii) There is a common difference d. in the progression, a = 1 and d = 2.
(iii) Given the first term, a and the common difference, d
1^{st} term = a
2^{nd} term = a + d
3^{rd} term = a + 2d
n^{th} term (U_n) = a + (n – 1)d

Example 1

Find the 30^{th} term of a series that has an n^{th} term given by $\frac{1}{2}(32 - n)$

Solution

$U_n = \frac{1}{2}(32 - n)$

$U_{30} = \frac{1}{2}(32 - 30) = 1$

Example 2

The first term of an arithmetic progression (A.P) is 73 and the 9^{th} term is 25. Determine the common difference

Solution

U_n = a + (n – 1)d

25 = 73 + (9 – 1)d

25 = 73 + 8d

d = -6

Example 3

The 3^{rd}, 5^{th} and 8^{th} terms of A.P are 3n + 8, n + 34, and n^3 + 15 respectively. Find the value of n and hence the common difference of the A.P

Solution

a + 2d = 3n + 8 …… (i)

a + 4d = n + 24 ……(ii)

a + 7d = n^3 + 15 ……..(iii)

Eqn. (ii) – eqn. (i)

$2d = -2n + 16$

$d = -n + 8$…… (iv)

eqn. (iii) – eqn. (i)

$5d = n^3 – 3n + 7$ …… (v)

Substituting eqn. (iv) into eqn. (v)

$5(-n + 8) = n^3 – 3n + 7$

$n^3 + 2n – 33 = 0$

By factorizing the equation

$(n- 3)(n^2 + 3n + 11) = 0$

Either $n – 3 = 0$ or $n^2 + 3n + 11 = 0$

$n = 3$ since $n^2 + 3n + 11 = 0$ has no real roots

Substituting for n in eqn. (iv)

$d = -n + 8$

$d = -3 + 8 = 5$

Hence n = 3 and the common difference is 5

Example 4

An A.P has the first term 3, common difference -2 and nth term – 15. Find n and the $(n -3)^{th}$ term

Solution

Given, $a = 3$, $d =-2$, $U_n = -15$

But $U_n = a + (n – 1)d$

$-15 = 3 + (n – 1) \times (– 2)$

$n = 10$

substitute n – 3 for n

$U_{(n – 3)} = 3 + [(10 – 3) – 1](-2)$

$= -9$

Example 5

The n^{th} term of a series is $Un = a3^n + bn + c$ given that $U_1 = 4$, $U_2 = 13$ and $U_3 = 46$, find the values of a, b and c.

Solution

By substituting for n

n = 1:

$3a + b + c = 4$ ……..(i)

n =2

$9a + 2b + c = 13$ ……(ii)

n = 3

$27a + 3b + 2 = 46$ ……(iii)

Eqn. (ii) – eqn. (i)

$6a + b = 9$ ……. (iv)

Eqn.(iii) – eqn (ii)

$18a + b = 33$ …… (v)

Eqn. (v) – eqn. (iv)

$12a = 24$

$a = 2$

Substitute for a in (v)

$36 + b = 33$

$b = -3$

Substituting for a and b in eqn. (i)

$3 \times 2 – 3 + c = 4$

$c = 1$

Hence a = 2, b = -3 and c = 1

The sum of the first n terms of an A.P

There are two formulas used for finding the sum of the first n terms of the A.P depending on the terms given

Formula A

If the first term, a and common difference are given, then sum (S_n) of the first n terms is given by

$S_n = \frac{n}{2}(2a + (n - 1)d)$

Formula B

If the first term is a and the last term is L, then sum of the first n terms (S_n)is given by

$S_n = \frac{n}{2}(a + L)$

Example 6

The first term of A.P is 73 and the common difference is -6, find the number of terms that must be added to give a sum of 96

Solution

$S_n = \frac{n}{2}(2a + (n-1)d$

$\frac{n}{2}(2x\ 73 - 6(n-1) = 93$

n(73 – 3(n – 1)) = 96

$73n - 3n^2 + 3n = 96$

$3n^2 - 76n + 96 = 0$

$n = \frac{76 \pm \sqrt{76^2 - 4x\ 3\ x\ 96}}{2\ x\ 3}$

$n = \frac{76 \pm 68}{6}$

n = 24

Hence the number of terms that must be added to give a sum of 96 are 24

Example 7

The tenth term of an arithmetic progression (A.P) is 29 and the fifteenth term is 44. Find the value of the first term and the common difference. Hence find the sum of the first 30 terms

Solution

U_n = a + (n – 1)d

U_{10} = a + (10 – 1)d

a + 9d = 29 …….(i)

U_{15} = a + (15 – 1)d

a + 14d = 44 …… (ii)

Eqn (ii) – eqn. (i)

5d = 15

d = 3

Substituting d in eqn. (i0

a + 9 x 2 = 29

a = 2

Sum of the first 30 terms

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_n = \frac{30}{2}(2\ x\ 2 + 3(30-1)) = 1365$

Example 8

The 5th term of an arithmetic progression (A.P) is 12 and the sum of the first 5 terms is 80. Determine the first term and common difference.

Solution

U_n = a + (n – 1)d [Un= nth term, a = first term and d = common difference}

U_5 = a + (5 – 1)d

a + 4d = 12 ……(i)

The sum of the first n terms,

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_5 = \frac{5}{2}(2a + (5-1)d)$

10a + 20d = 80 x 2 = 160 ……(ii)

Eqn. (ii) – 5eqn. (i)

5a = 100

a = 20

Substituting for a in eqn. (i)

20 + 4d = 12

d = -2

Hence the first term = 20 and the common difference = -2

Example 9

(a) Prove that $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

Solution

$S_n = 1 + 2 + 3 + ... + n$

$+ S_n = n + (n-1) + (n-2) + ... + 1$

$2S_n = (n+1) + (n+1) + (n+1) + ... + (n+1)$

$2S_n = n(n+1)$

$Sn = \frac{n}{2}(n+1)$

(b) Use your answer in (a) to deduce

(i) $\sum_{r=1}^{n}(3r-1) = \frac{n}{2}(3n+1)$

Note to deduce is to use the already existing result to work out other problems

$S_n = \frac{n}{2}[last\ term + first\ term]$

$\sum_{r=1}^{n}(3r-1) = \frac{n}{2}\big(3n-1+(3-1)\big)$

$= \frac{n}{2}(3n-1+2)$

$= \frac{n}{2}(3n+1)$

(ii) $\sum_{r=0}^{n}(r+5) = \frac{1}{2}(n+1)(n+10$

$= \sum_{r=1}^{n}(r+5) + \sum_{r=0}(r+5)$

$= \frac{n}{2}\big((n+1)+(1+5)\big) + 5$

$= \frac{1}{2}(n^2+11n) + 5$

$= \frac{1}{2}(n^2+11n+10)$

$= \frac{1}{2}(n+1)(n+10)$

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 10

(a) Insert two geometric means between 2 and 16

Solution

1st term a = 2

4th term, $ar^3 = 16$

$2(r^3) = 18$

$r = 2$

the second term, ar = 2 x 2 = 4

the third term, $ar^2 = 2 \times 2^2 = 8$

(b) Insert three geometric means between 1 and 81

a = 1

the 5th term $ar^4 = 81$

$1(r^4) = 81$

$r = 3$

the second term, ar = 1 x 3 = 3

the third term, $ar^2 = 1 \times 3^2 = 9$

the third term, $ar^3 = 1 \times 3^3 = 27$

Revision exercise 1

1. Find the 5th and 8th terms of a series that has an nth term given by $(-1)^n(2n+1)$ [-11, 17]
2. The first term of an arithmetic progression (A.P) is $\frac{1}{2}$. The sixth term of th A.P is four times the fourth term. Find the common difference of the A.P $\left[\frac{-3}{14}\right]$
3. The sum of p terms of an arithmetic progression is q and the sum of q terms is p; find the sum of p + q terms
4. (a) the first four terms of an A.P are 5, 11, 17 and 23. Find the 30th term and the sum of the first 30 terms [179, 2760]
   (b) the second term of an A.P is 7 and the 7th term is -8. Find the first term, common difference and the sum of the first 14 terms [10, -3, -133]
5. (a) An A.P has the first term of 2 and common difference 5. Given that the sum of the first n terms of the progression is 119, calculate n [7]
   (b) the sum of the first five terms of an A.P is $\frac{65}{2}$. Also , five times the 7th term is the same as six times the second term. Find the first term and the common difference $\left[a = 6, d = \frac{1}{4}\right]$

6. The sum of the first n terms of the a series is n(n + 2). Find the first three terms [3, 5, 7]
7. In an A.P, the 1st term is 13 and 15th term is 11. Find the common difference and sum of the first 20 terms [7, 1590]
8. (a) Show that $\ln 2^r$, r = 1, 2,3, is an arithmetic progression
   (b) find the sum of the first 10 terms of the progression [38.1231]
   (c) Determine the least value of m for which the sum of the first 2m terms exceeds 883.7 [25]
9. In an arithmetic progression $u_1 + u_2 + u_3 + u_4 = 15$ and $u_{16} = -3$. Find the greatest integer N such that $U_N \geq 0$. Determine the sum of the first N terms of the progression. [N = 14, $S_{14} = 136.5$]

Geometric progression (G.P)

It is a series in which each term is obtained from the preceding one by multiplication or division by a constant quantity.

Observations

- The first term of G.P is also denoted, a
- The common ratio is r
- Given a and r
  1st term = a
  2nd term = ar
  3rd term = ar^2
  4th term = ar^3
  nth term = ar^{n-1}

The sum (S_n) of the first n terms of G.P

The sum (S_n) of the first n terms of G.P is given by

$S_n = \frac{a(r^n-1)}{r-1}$ for r > 1 or

$S_n = \frac{a(1-r^n)}{1-r}$ for r< 1

The geometric mean (G.M)

Suppose three numbers a, b and c are consecutive terms of GP, then, the middle term is the geometric mean.

The common ratio, r = $\frac{b}{a} = \frac{c}{b}$

⇨ $b^2 = ac$
b = $\sqrt{ac}$, the geometric mean

Example 10

A GP has 3rd term 7 and 5th term 847. Find the possible values of the common ratio and the corresponding 4th terms

Solution

$U_3 = ar^2 = 7$

⇨ $ar^2 = 7$ .....(i)

$U_5 = ar^4 = 847$

⇨ $ar^4 = 847$ .....(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 121$

$r = \pm 11$

From eqn (i)

$a = \frac{7}{121}$

The 4th term, $U_4 = ar^3$

If r = 11, $U_4 = \frac{7}{121}(11)^3 = 77$

If r = -11, $U_4 = \frac{7}{121}(-11)^3 = -77$

Example 11

In a G.P the 2nd term is 15 and the 5th term is -405. Find the sum of the first 8 terms

Solution

$U_2 = ar = 15$

⇨ ar = 15 .....(i)

$U_5 = ar^4 = -405$

⇨ $ar^4 = -405$

Eqn. (ii) ÷ eqn. (i)

$r^3 = -27$

r = -3

From eqn. (i), a = -5

Since r < 1

$S_n = \frac{a(1-r^n)}{r-1} = \frac{-5(1-(-3)^8)}{1-(-3)} = \frac{32800}{4} = 8200 =$

Example 12

In the geometric series $u_1 + u_2 + u_3 + \ldots$

$u_1 + u_3 = 26$ and $u_3 + u_5 = 650$.

Find the possible values of u_4

Solution

$u_1 + u_3 = 26$

$a + ar^2 = 26$

$a(1 + r^2) = 26$ …… (i)

$u_3 + u_5 = 650$.

$ar^2 + ar^4 = 650$

$ar^2(1 + r^2) = 650$ ……(ii)

Eqn. (ii) ÷ eqn. (i)

$r^2 = 25$

$r = \pm 5$

From eqn. (i)

$a(1 + 25) = 26$

$a = 1$

$u_4 = ar^3$

If r = 5; $u_4 = a(5)^3 = 125$

If r = 5; $u_4 = a(-5)^3 = -125$

Example 13

In a Geometric Progression (G.P), the difference between the fifth and the second term is 156. The difference between the seventh and the fourth is 1404 Find the possible values of the common ratio.

$U_5 - U_2 = 156$
$ar^4 - ar = 156$
$ar(r^3 - 1) = 156$ ……. (i)
$U_7 - U_4 = 156$
$Ar^6 - ar^3 = 1404$
$ar^3(r^3 - 1) = 156$ ……. (ii)
Eqn. (ii) ÷ eqn. (i)
$\frac{ar^3(r^3-1)}{ar(r^3-1)} = \frac{1404}{156}$
$r^2 = 9$
$r = \pm 3$
∴ r = 3 and r = -3

Example 14

(a) The first three terms of a Geometric progression (G.P) are 4, 8 and 16. Determine the sum of the first ten terms of the G.P. (04marks)

solution

a = 4, ar = 6
4r = 8
r = 2
$S_n = \frac{a(1-r^n)}{r-1}$
$S_{10} = 4\left(\frac{2^{10}-1}{2-1}\right) = 4092$

(b) An Arithmetic Progression (A.P) has a common difference of 3. A Geometric Progression (G.P) has a common ration of 2. A sequence is formed by subtracting the term of the A.P from the corresponding terms of the G.P. The third term of the sequence is 4. The sixth term of the sequence is 79. Find the first term of the

(i) A.P (08 marks)

(ii) G.P (06 mars)

A.P
x, x+3, x +6, x + 9, x + 12, x+ 15, ……
G.P
y, 2y, 4y, 8y, 16y, 32y, ….
4y – (x + 6) = 4
4y – x = 10 ……..(i)
32y – (x + 15) = 79
32y – x = 94…….. (ii)
Eqn. (ii) – Eqn. (i)
28y = 84, => y = 3
Substituting for y into eqn. (i)
12 – x = 10

x =2

(i) A.P, $U_1 = 2$

(ii) G.P, $U_1 = 3$

Example 15

The sum of the first n terms of a geometric progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find the n^{th} term as an integral power of 2

Solution

$S_n = \frac{a(1-r^n)}{r-1}$

Comparing with $S_n = \frac{4}{3}(4^n - 1)$

a = 4

r – 1 = 3, r = 4

The n^{th} term, $U_n = ar^{n-1}$

$= 4 \times 4^{n-1} = 2^2 \times 2^{2(n-1)} = 2^{2+2n-2} = 2^{2n}$

Example 16

Find three numbers in geometrical progression such that their sum is 26 and their product is 216

Solution

Let the numbers be $\frac{a}{r}$, a and ar

Product = $\left(\frac{a}{r}\right)(a)(ar) = 216$

$a^3 = 216 = 6^3$

a = 6

∴ the terms are $\frac{6}{r}, 6 \; and \; 6r$

Sum of terms $\frac{6}{r} + 6 + 6r = 26$

⇨ $6r^2 – 26r + 6 = 0$

$3r^2 – 13r + 3 = 0$

$(r – 3)(3r – 1) = 0$

Either r – 3 = 0; r = 3

Or 3r - 1 = 0; r = $\frac{1}{3}$

When r = 3

the terms are $\frac{6}{3} = 2, 6 \; and \; 6x3 = 18$

when r = $\frac{1}{3}$

the terms are $6 \div \frac{1}{3} = 18, 6 \; and \; 6 \; x \; \frac{1}{3} = 2$

Hence the terms in their order are 2, 6, 18 or 18, 6, 2

Inserting geometric means

Like for A.Ps, the terms inserted between given two values of a G.P are known as geometric means.

If n terms are inserted, then the total number of terms will be n+ 2 with the two extreme values representing the first and last terms respectively

Example 16

(c) Insert two geometric means between 2 and 16

Solution

1^{st} term a = 2

4^{th} term, $ar^3 = 16$

$2(r^3) = 18$

$r = 2$

the second term, $ar = 2 \times 2 = 4$

the third term, $ar^2 = 2 \times 2^2 = 8$

(d) Insert three geometric means between 1 and 81

a = 1

the 5^{th} term $ar^4 = 81$

$1(r^4) = 81$

$r = 3$

the second term, $ar = 1 \times 3 = 3$

the third term, $ar^2 = 1 \times 3^2 = 9$

the third term, $ar^3 = 1 \times 3^3 = 27$

Mixed terms of A.P and G.P

These are problems involving both A.Ps and G.Ps. when handling we make use of their respective properties.

Example 17

A geometric progression (G.P) and an arithmetic progression (A.P) have the same first term. The sum of their first, second and

third terms are 6, 10.5 and 18 respectively. Calculate the sum of their 5th terms.

Solution

Terms	G.P	A.P	Sum
1st	a	a	2a = 6 ……(i)
2nd	a + d	ar	a + d + ar = 10.5 ….(ii)
3rd	a + 2d	ar^2	a + 2d + ar^2 = 18 ..(iii)

From eqn. (i): 2a = 6; a = 3

From eqn. (ii): 3 + d + 3r = 10.5

d + 3r = 7.5 …….. (iv)

From eqn. (iii) 3 + 2d + $3r^2$ = 18

2d + $3r^2$ = 15 ……..(v)

Eqn. (v) – 2eqn. (iv)

$3r^2 – 6r = 0$

3r(r -2) = 0

r -2 = 0

r = 2

Substitute for r into eqn. (iv)

d + 6 = 7.5

d = 1.5

Sum of their fifth terms

= (a + 4d) + ar^4

= (3 + 4 x 1.5) + 3 x 2^4 = 57

Example 18

The 1st, 4thand 8th terms of A.P form a G.P. if the first term is 9, find the

(i) Common difference of the A.P
(ii) Common ratio of the G.P
(iii) Difference in sums of the first 6 terms of the progressions.

Solution

Given that a, a + 3d, a + 7d form a G.P

Substituting for a = 9, the terms are

9, 9 + 3d, 9 + 7d

For a G.P, r = $\frac{9+3d}{9} = \frac{9+7d}{9+3d}$

⇨ (9 + 3d)2 = 9(9 + 7d)
$81 + 54d + 9d^2 = 81 + 63d$
$9d^2 – 9d = 0$
9d(d – 1) = 0
Either d – 1 = 0; d = 1
Or d = 0

When d = 0 all terms of A.P are equal

Hence the common difference d = 1

Example 19

(a) The sum of the first m terms of a progression is m(2m + 11)
   (i) Show that the progression is an A.P
   (ii) Determine the nth term of the progression

Solution

Given S_m = m(2m + 11)

First term = S_1 = 1(2 x 1 + 11) = 13

S_2 = 2(2 x 2 + 11) = 30

Second term = 30 – 13 = 17

S_3 = 3(2 x 3 + 11) = 51

Third term = 51 – 20 = 21

The progression is 13, 17, 21, ….. Hence A.P with the first term 13 and common difference, d= 4

(ii) $U_n = S_n – S_{n-1}$

= n(2n + 11) – (n – 1)(2(n -1) + 11)

= 9 + 4n

Example 20

(a) The first, second and last term of an A.P are a, b, c respectively. Prove that the sum of all terms is $\frac{(a+b)(b+c-2a)}{2(b-a)}$
   Solution
   If the 1st term is a and the second term is b; the common difference, d = (b – a)

Last term (nth terms c = a + (n – 1)d

c = a + (n – 1)(b – 1)

i.e. n – 1 = $\frac{c-a}{b-a}$ => n = $\frac{b+c-2a}{b-a}$

but $S_n = \frac{1}{2}n(a+L)$

$=\frac{1}{2}\left(\frac{b+c-2a}{b-a}\right)(a+c)$

$=\frac{(a+b)(b+c-2a)}{2(b-a)}$

(b) The first, second and last terms of a GP are a and b. show that the sum of the first n terms is $\frac{a^n-b^n}{a^{n-2}(a-b)}$

Solution

Common ratio $=\frac{b}{a}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a\left(1-\left(\frac{b}{a}\right)^n\right)}{1-\left(\frac{b}{a}\right)}$

$= \frac{a(a^n-b^n)a}{a^n(a-b)} = \frac{a^n-b^n}{a^{n-2}(a-b)}$

Sum to infinity of a G.P

We have seen that the sum of n terms of a G.P for r < 1 is $S_n = \frac{a(1-r^n)}{1-r}$

Now for -1 < n < 1 i.e. $|r| < 1$, as n →∞, r^n →0

Therefore $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$

Hence the sum of a GP to infinity for $|r| < 1$ converges to $S_\infty = \frac{a}{1-r}$ and diverges for r > 1 and r < 1

Example 21

(a) Calculate the sum to infinity of the following terms

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

a = 1 and r = $\frac{1}{2}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

Solution

a = $\frac{1}{5}$ and r = $\frac{1}{5}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$

(b) Work out the following

(i) $\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r$

Solution

$\sum_{r=0}^{\infty}\left(\frac{1}{3}\right)^r = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$

a = 1 and r = $\frac{1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

(ii) $\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r$

Solution

$\sum_{r=2}^{\infty}\left(-\frac{1}{8}\right)^r = \frac{1}{64} - \frac{1}{512} + \frac{1}{4096} - \cdots$

A = $\frac{1}{64}$ and r = $-\frac{1}{8}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{64}}{1+\frac{1}{8}} = \frac{1}{72}$

(iii) $\sum_{r=0}^{\infty} a^r$

Solution

a = 1 and r = a

$S_\infty = \frac{a}{1-r} = \frac{1}{1-a}$

(iv) $\sum_{r=1}^{\infty}(3x)^{r+1}$

Solution

$\sum_{r=1}^{\infty}(3x)^{r+1} = 9x^2 + 27x^3 + 81x^4 + \ldots$

a = $9x^2$ and r = 3x

$S_\infty = \frac{a}{1-r} = \frac{9x^2}{1-3x}$

Example 22

(a) Express the following as fractions using approach of sum of a G.P to infinity

(i) $0.\dot{4}$

Solution

$0.\dot{4} = \frac{4}{10} + \frac{4}{100} + \frac{1}{1000} + \cdots$

$=\frac{4}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$

$=\frac{4}{10}\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{4}{10}x\frac{10}{9} = \frac{4}{9}$

(ii) $3.1\dot{2}\dot{7}$

Solution

$3.1\dot{2}\dot{7} = 3 + \frac{1}{10} + \frac{27}{1000} + \frac{27}{10000} + \cdots$

$= 3 + \frac{1}{10} + \frac{27}{1000}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right)$

$= \frac{31}{10} + \frac{27}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{31}{10} + \frac{27}{1000}\left(\frac{100}{99}\right)$

$= \frac{31}{10} + \frac{3}{110} = \frac{344}{110} = \frac{172}{55}$

Hence $3.1\dot{2}\dot{7} = \frac{172}{55}$

(b) The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Find the common ratio and the first term of the GP with positive common ratio

Solution

$S_\infty = 7$

$=> \frac{a}{1-r} = 7$

$a = 7(1 - r)$ .....(i)

But $S_2 = \frac{48}{7}$

$a + ar = \frac{48}{7}$

$a(1 + r) = \frac{48}{7}$ .....(ii)

substituting eqn. (i) into eqn. (ii)

$7(1 - r)(1 + r) = \frac{48}{7}$

$49\ (1 - r^2) = 48$

$49r^2 = 1$

$r^2 = \frac{1}{49}$

$r = \pm\frac{1}{7}$

Considering appositive ratio

From eqn. (i)

$a = 7(1 - \frac{1}{7}) = 6$

Revision exercise 2

1. The common ration of a GP is -5 and the sum of the first seven terms of the progression is 449. Find the first three terms. $\left[\frac{1}{29}, \frac{-5}{29}, \frac{25}{29}\right]$
2. In the geometrical series $\sum_{r=1}^{n} u_r$, $u_5 - u_2 = 156$ and $u_7 - u_4 = 1404$. Find the possible values of the common ratio and corresponding values of u_1
   [r = 3, a = 156; r = -3, a = $\frac{13}{7}$]
3. The sum of the second and third terms of a G.P is 9. It the seventh term is eight times the fourth term, find the
   (a) The first term and the common ration [a = $^3/_2$ and r = 2]
   (b) The sum of the fourth and first term [36]
4. Find the sum of ten terms of geometrical series 2, -4, 8 ..... [-682]
5. The second and the third terms of a G.P progression are 24 and 12(b + 1) respectively. Find b if the sum of the first three terms of the progression is 76 $\left[\frac{1}{2}\ or\ 2\right]$
6. The sum of the 2nd and 3rd terms of a G.P is 12. The sum of the 3rd and 4th terms is -36. Find the 1st term and common ratio [a = 2, r = -3]
7. What is the smallest number of terms of GP 5, 10, 20 .... that can give a sum greater than 500, 000 [n= 17]
8. The first, fourth and eighth terms of Arithmetic progression (A.P) form a geometric progression. If the first term is 9, find
   (i) The common difference of A.P [1]
   (ii) The common ratio of the G.P $\left[\frac{4}{3}\right]$
   (iii) The difference in the sums of the first 6 terms of the progressions [55.7049]
9. The second, third and ninth terms of an A.P form a G.P. find the common ratio of the G.P [6]
10. (a) The sum of the first 10 terms of an AP is 120. The sum of the next 8 terms is 240. Find the sum of the next 6 terms [264]
    (b) the arithmetic mean of the a and b is three times their geometric mean. Show that $\frac{a}{b} = 7 \pm 12\sqrt{2}$
11. The first three terms of a geometric series are 1, p, and q. Given also that 10, q and p are the first three terms of an arithmetic series. Show that $2p^2 - p - 10 = 0$
    Hence find the possible values of p and q
    [p = -2 and q = 4 or p = $\frac{5}{2}$ and q = $\frac{25}{4}$]

Application of A.Ps and G.Ps to interest rates

If a sum of money P is invested at a simple interest rate of r% per annum, the amount received after n years is given by A = P + I where I = $\frac{P\ x\ r\ x\ n}{100}$

By substitution we have

A = P + $\frac{P\ x\ r\ x\ n}{100}$ = P$\left(1 + \frac{nr}{100}\right)$

The interest for one year is $\frac{Pr}{100}$, for 2 years is $\frac{2Pr}{100}$, for n year = $\frac{nPr}{100}$. Therefore the various amounts of interest after one, two, three, etc. years form an AP

On the other hand, if the principal P is invested at compound interest rate of r% per annum, the interest being added annually, the amount after one year is $\left(1 + \frac{r}{100}\right)$, after two years is $P\left(1 + \frac{r}{100}\right)^2$, after 3 years is

$P\left(1 + \frac{r}{100}\right)^3$ and after n years $P\left(1 + \frac{r}{100}\right)^n$

Hence the amounts after one, two, three, etc. years for a GP.

Note: if with compound interest is added half annually as much as when added yearly, but it is added twice as much. Hence amount

A = $P\left(1 + \frac{r}{100}\right)^{2n}$

Now suppose that instead of adding the interest annually, it is the principal, P which is added annually,

Amount after 1st year = $P\left(1 + \frac{r}{100}\right)^1$

Amount after 2nd year = $P\left(1 + \frac{r}{100}\right)^2$

Amount after 3rd year = $P\left(1 + \frac{r}{100}\right)^3$

Amount after nth year = $P\left(1 + \frac{r}{100}\right)^n$

Total amount after n years

= $P\left(1 + \frac{r}{100}\right)^1 + P\left(1 + \frac{r}{100}\right)^2 + \ldots + P\left(1 + \frac{r}{100}\right)^n$

= P$\left[\left(1 + \frac{r}{100}\right)^1 + \left(1 + \frac{r}{100}\right)^2 + \cdots + \left(1 + \frac{r}{100}\right)^n\right]$

This is a G.P with:

first term = $\frac{100+r}{100}$ = (100 + r)%

And common ratio $\frac{100+r}{100}$ = (100 + r)%

$s_n = P(100 + \text{r})\%\left[\frac{(100 + \text{r})\% - 1}{(100 + \text{r})\% - 1}\right]$

Example 23

(a) Find the amount at the end of ten years when 500000 shillings is invested at 5% compound interest

(i) the interest being added annually

Solution

A = $P\left(1 + \frac{r}{100}\right)^n$

$= 500000\left(1 + \frac{5}{100}\right)^{10}$

A = 814,447.3134

(ii) the interest being added twice a year

Solution

A = $P\left(1 + \frac{r}{100}\right)^{2n}$

$= 500000\left(1 + \frac{5}{100}\right)^{20}$

A = 1,326,648.853

(b) Find the amount at the end of ten years when 500000 shillings is invested at 5% simple interest

A = $\frac{nPr}{100} = \frac{10\ x\ 500000\ x\ 5}{100}$ =750,000

Proof by induction

This is a mathematical technique that uses the reasoning that if a statement is true for a particular value say n = 1, then it must be true for n = 2, 3, 4, …. This involves the proof that the series on the LHS must be equal to the terms on the RHS

Example 24

Prove by induction that

1 + 2 + 3 + …+ n = $\frac{1}{2}n(n+1)$

Solution

Here we need to show that the above series agrees for all values of n = 1, 2, 3 .., q and q +1

Suppose n = 1

LHS = 1 [taking only the 1st number]

RHs = $\frac{1}{2}(1)((1)+1)$= 1[substituting for n = 1]

∴LHS = RHS => the series hold for n = 1

Suppose n = 2

LHS = 1+ 2 = 3 [taking first 2 numbers]

RHs = $\frac{1}{2}(2)((2)+1)$= 3[substituting for n = 2]

∴LHS = RHS => the series hold for n = 2

Suppose n = q

1 + 2 + 3 + …+ q = $\frac{1}{2}q(q+1)$

For n = q +1(i.e. adding k + 1 on both sides)

1 + 2 + 3 + …+ q + (q+1) = $\frac{1}{2}q(q+1)$+ (q + 1)

$=(q+1)(\frac{1}{2}q+1)$

$=\frac{1}{2}(q+1)(q+2)$

The result is true for n = q + 1, hence true for all positive values of n

Example 25

Prove by induction

(a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Solution

For n = 1

LHS = 1^2 = 1;

RHS = $\frac{1}{6}(1)((1)+1)(2(1)+1)$ = 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $1^2 + 2^2$ = 5;

RHS = $\frac{1}{6}(2)((2)+1)(2(2)+1)$ = 5

∴ LHS = RHS => the series holds for n = 2

For n = k

$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$

For n = k + 1

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$

$=[k+1]\left[\frac{1}{6}k(2k+1)+(k+1)\right]$

$=[k+1]\left[\frac{1}{6}(2k^2+7+6)\right]$

$=\frac{1}{6}(k+1)(k+2)(2k+3)$

∴ LHS = RHS => the series holds for n = k+1 hence for all positive values of n

(b) $\sum_{r=1}^{n=r} r^3 = \frac{1}{4}n^2(n+1)^2$

Solution

For n = 1

LHS = $\sum_{r=1}^{r=1} r^3 = 1^3 = 1$

RHS = $\frac{1}{4}(1)^2(1+1)^2$= 1

∴ LHS = RHS => the series holds for n = 1

For n = 2

LHS = $\sum_{r=1}^{r=2} r^3 = 1^3 + 2^3 = 1$

RHS = $\frac{1}{4}(2)^2(2+1)^2$= 9

∴ LHS = RHS => the series holds for n = 2

For n = k

$\sum_{r=1}^{n=k} r^3 = \frac{1}{4}k^2(k+1)^2$

For n = k+1

$\sum_{r=1}^{n=k+1} r^3 = \sum_{r=1}^{n=k} r^3 + (k+1)^3$

$=\frac{1}{4}k^2(k+1)^2+(k+1)^3$

$=(k+1)^2\left[\frac{1}{4}k^2+k+1\right]$

$=\frac{1}{4}(k+1)^2[k^2+4k+4]$

$=\frac{1}{4}(k+1)^2(k+2)^2$

(c) $p + pq + pq^2 + \ldots + pq^{n-1} = p\left(\frac{1-q^n}{1-q}\right)$

For n = 1,

LHS = p

RHS = $p\left(\frac{1-q^1}{1-q}\right)$ = p

∴the identity is true for n = 1

For n = 1,

LHS = p + pq =p(1+q)

RHS =$p\left(\frac{1-q^2}{1-q}\right)$ = $p\left(\frac{(1+q)(1-q)}{1-q}\right)$=p(1+q)

∴the identity is true for n = 2

For n = k

$p + pq + pq^2 + \ldots + pq^{k-1} = p\left(\frac{1-q^k}{1-q}\right)$

For n = k+1

$p + pq + pq^2 + \ldots + pq^{k-1} + pq^k = p\left(\frac{1-q^k}{1-q}\right) + pq^k$

$= p\left(\frac{1-q^k+q^k+p^{k+1}}{1-q}\right)$

$= p\left(\frac{1-q^{k+1}}{1-q}\right)$

∴the identity is true for n = k+1, hence true for all positive values of n

Revision exercise 3

1. Five millions shillings is invested each year at a rate of 15% compound interest by a certain bank.
   (a) Find how much he will receive at the end of ten years [116.7464m]
   (b) How many years will it take to accumulate to more than 50m [6]
2. John opened an account in the bank and deposited200,000 shillings every month for ten months without withdrawing. Find how much money he accumulated after 10 months if the bank offered 10% compound interest per month. [3,506,233.412]
3. Peter deposited sh. 100,000 at the beginning of every year for 5 years; find how much he got at the end of the fifth year at the compound interest rate of 2% per annum. [530,812.1]
4. Prove by induction
   (i) $\sum_{r=1}^{n} 3^{r-1} = \frac{3^n-1}{2}$
   (ii) $1^3 + 2^3 + 3^3 + \ldots + n^3$
   $= \frac{1}{4}n^2(3n+1)(5n+1)$
   (iii) $\sum_{r-1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$
   (iv) $\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$
   (v) $\sum_{r=1}^{n} \frac{1}{r(1+1)} = \frac{n}{n+1}$
   (vi) $\sum_{r=1}^{n} ap^{r-1} = \frac{a(p^n-1)}{p-1}$
5. The sum of the first n terms of a Geometric Progression (G.P) is $\frac{4}{3}(4^n - 1)$. Find its nth term as an integral power of 2 [2^{2n}]
6. Prove by mathematical induction the $3^{2n} - 1$ is a multiple of 8 for all positive integers n
7. Use the method of induction to prove that 6n – 1is divisible by 5 for all positive integral values of n
8. Prove by induction that
   $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$
   hence evaluate $\sum_{r=1}^{20} r(r+1)$

Thank you

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Linear equations in one unknown

Linear equation is an equation of a straight line. It is an equation whose unknown variable has the highest power one. E.g. y = x + 5, 4y = 3x + 4, etc.

Linear equation in one unknown is the one which has got only one unknown variable, such as x + 2 = 5, $\frac{2x-3}{x+2} = 4$, ...

Solving for unknown of the equation is the same as making that variable the subject of the equation

Example 1

Solve the following equations

(i) 4x + 4 = 10

Solution

4x + 6 = 10

4x = 10 – 6 = 4

x= $\frac{4}{4} = 1$

(ii) 3x + 2 = x + 8

Solution

3x + 2 = x + 8

3x – x = 8 – 2

2x = 6

x = 3

(iii) $\frac{5x-3}{4} = \frac{4x-3}{3}$

Solution

$\frac{5x-3}{4} = \frac{4x-3}{3}$

3(5x – 3)= 4(4x – 3)

15x – 9 = 16x – 12

-x = -3

x = 3

(iv) $\frac{1}{5}(2x-1) - \frac{1}{4}(3x-4) = 0$

$solution$

$\frac{1}{5}(2x-1) - \frac{1}{4}(3x-4) = 0$

Multiplying through by 20

$4(2x-1) - 5(3x-4) = 0$

8x – 4 – 15x + 20 = 0

-7x = - 16

x = $\frac{16}{7}$

Revision exercise

1. Solve the following equation
   (a) 2x + 4 = 0 [x=3]
   (b) 5x-6 = 24 [x =2]
   (c) 3x + 2 = x + 8 [x =3]
   (d) 2x + 5 = 29 -10x [x= 2]
   (e) 3(x – 8) + 2(4x – 1) = 3 $\left[x = \frac{29}{11}\right]$
   (f) 3(2x – 5) – 4(x – 2) = 5(x – 8) [x = 11]
2. Solve the following equations
   (a) $\frac{5x-3}{4} = \frac{4x-3}{3}$ [x = 3]
   (b) $\frac{7}{1-x} = \frac{3}{x+2}$ $\left[x = -\frac{17}{4}\right]$
   (c) 2x + 3 + $\frac{5x-1}{4} = \frac{3x-2}{8}$ $\left[x = -\frac{24}{23}\right]$
   (d) $\frac{2x-3}{4} + \frac{6x-4}{3} = \frac{2x+5}{6}$ $\left[\frac{35}{26}\right]$
   (e) $\frac{1}{5}(x-2) + \frac{1}{3}(5x-4) = \frac{1}{2}x$ $\left[x = \frac{52}{41}\right]$
   (f) $\frac{3x+1}{2x-3} = \frac{6x+1}{4x-5}$ $\left[x = \frac{2}{5}\right]$
   (g) $\frac{4-x}{2x+3} = \frac{x-1}{3-2x}$ $\left[x = \frac{5}{4}\right]$

Thank you

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Centre of gravity

This is the point where the resultant force due to attraction acts

General formula for the centre of gravity

Consider a system of particles of weight, w_1, w_2 …. w_n located at points with coordinates (x_1, y_1), (x_2, y_2) ….. (x_n, y_n) in the x-y plane

The resultant of weight $w_1 + w_2 + \ldots. + w_n$ have a C.O.G at point $(\bar{x}, \bar{y})$

Taking moments along the y-axis

$(w_1 + w_2 + \ldots. + w_n)\bar{x} = w_1x_1 + w_2x_2 + \ldots. + w_nx_n$

$$\bar{x} = \frac{\sum w_1x_i}{\sum w_i}$$

Similarly taking moments along the x-axis

$$\bar{y} = \frac{\sum w_1x_i}{\sum w_i}$$

Alternatively:

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[w_1\begin{pmatrix}x_1\\ y_1\end{pmatrix}+w_2\begin{pmatrix}x_2\\ y_2\end{pmatrix}+\cdots.+w_n\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right]}{(w_1+ w_2+\cdots+w_n)}$$

Or

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[m_1g\begin{pmatrix}x_1\\ y_1\end{pmatrix}+m_2g\begin{pmatrix}x_2\\ y_2\end{pmatrix}+\cdots.+m_ng\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right]}{(m_1g+ m_2g+\cdots+w_ng)}$$

Example 1

Find the position of the centre of gravity of three particles of masses 1kg, 5kg, and 2g which lie on the y-axis at points (0, 2), (0, 4) and (0,5) respectively

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[m_1g\begin{pmatrix}x_1\\ y_1\end{pmatrix}+m_2g\begin{pmatrix}x_2\\ y_2\end{pmatrix}+\cdots.+m_ng\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right]}{(m_1g+ m_2g+\cdots+w_ng)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[1g\begin{pmatrix}0\\ 2\end{pmatrix}+5g\begin{pmatrix}0\\ 4\end{pmatrix}+2g\begin{pmatrix}0\\ 5\end{pmatrix}\right]}{(1g+ 5g+2g)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\begin{pmatrix}0\\ 32\end{pmatrix}}{8} = \begin{pmatrix}0\\ 4\end{pmatrix}$$

Example 2

Find the coordinates of the centre of gravity of four particles of masses 5kg, 2kg, 2kg and 3kg which are situated at (3, 1), (4, 3), (5, 2) and (-3, 1) respectively

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[m_1 g\begin{pmatrix}x_1\\ y_1\end{pmatrix} + m_2 g\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots + m_n g\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right]}{(m_1 g + m_2 g + \cdots + w_n g)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[5g\begin{pmatrix}3\\ 1\end{pmatrix} + 2g\begin{pmatrix}4\\ 3\end{pmatrix} + 2g\begin{pmatrix}5\\ 2\end{pmatrix} + 3g\begin{pmatrix}-3\\ 1\end{pmatrix}\right]}{(5g + 2g + 2g + 3g)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\begin{pmatrix}24\\ 18\end{pmatrix}}{12} = \begin{pmatrix}2\\ 1.5\end{pmatrix}$$

Example 3

Three particles of masses 2kg, 1kg and 3kg are situated at (4, 3), (1, 0) and (a, b) respectively. If the centre of gravity of the system lies at (0,2), find the values of a and b.

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[m_1 g\begin{pmatrix}x_1\\ y_1\end{pmatrix} + m_2 g\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots + m_n g\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right]}{(m_1 g + m_2 g + \cdots + w_n g)}$$

$$\begin{pmatrix}0\\ 2\end{pmatrix} = \frac{\left[2g\begin{pmatrix}4\\ 3\end{pmatrix} + 1g\begin{pmatrix}1\\ 0\end{pmatrix} + 3g\begin{pmatrix}a\\ b\end{pmatrix}\right]}{(2g + 1g + 3g)}$$

$$\begin{pmatrix}0\\ 2\end{pmatrix} = \frac{\begin{pmatrix}9+3a\\ 6+3b\end{pmatrix}}{6}$$

$0 = \frac{9+3a}{6}$, a = -3

$2 = \frac{6+3b}{6}$,

b = 2

Example 4

Find the coordinates of the centre of gravity of four particles of weight, 6N, 5N, 4N and 7N which are situated at (i +2j), (2i), (3j) and (4i + 2j) respectively.

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[w_1\begin{pmatrix}x_1\\ y_1\end{pmatrix} + w_2\begin{pmatrix}x_2\\ y_2\end{pmatrix} + \cdots + w_n\begin{pmatrix}x_n\\ y_n\end{pmatrix}\right]}{(w_1 + w_2 + \cdots + w_n)}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[6\begin{pmatrix}1\\ 2\end{pmatrix} + 5\begin{pmatrix}2\\ 0\end{pmatrix} + 4\begin{pmatrix}0\\ 3\end{pmatrix} + 7\begin{pmatrix}4\\ 2\end{pmatrix}\right]}{(6 + 5 + 4 + 7)} = \frac{\begin{pmatrix}44\\ 38\end{pmatrix}}{22}$$

$(\bar{x}, \bar{y})$ = (2, 1.73)

Example 5

The rectangle EFGH has EF= 3m and EH = 2m, particles of masses 20g, 30g, 60g and 10g are placed at the mid-point of the sides EF, FG, GH and EH respectively. Find the distance of the centre of gravity of the system from each of the line EF and EH

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{\left[0.02\begin{pmatrix}1.5\\ 0\end{pmatrix} + 0.03\begin{pmatrix}3\\ 1\end{pmatrix} + 0.06\begin{pmatrix}1.5\\ 2\end{pmatrix} + 0.01\begin{pmatrix}0\\ 1\end{pmatrix}\right]}{(0.02 + 0.03 + 0.06 + 0.01)} = \frac{\begin{pmatrix}0.21\\ 0.16\end{pmatrix}}{0.12}$$

$(\bar{x}, \bar{y})$ = (1.75, 1.33)

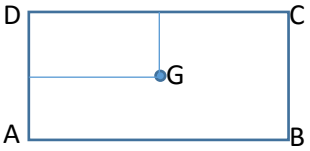
Revision exercise 1

1. Find the coordinates of the centre of gravity of four particles of masses 60g, 30g, 70g and 40g which are situated at (4i + 3j), (6i + 5j), (-6i + 5j) and (-5i – 2j) respectively. [-1, 3]
2. The rectangle ABCD has AB = 4cm and AD = 2cm, particles og masses 3kg, 5kg, 1kg and 7kg are placed at point points A, B, C and D respectively. Find the distance of the centre of gravity of the system from each line AB and AD. [1cm, 1.5cm]
3. Find the position of the centre of gravity of four particles of masses 5kg, 6kg, 2kg and 2kg which are situated at (5i – 7j), (-3i + 2j), (3i – 5j) and (i -6j) respectively. [(i - 3j)]
4. Particles of weight 1N, 2N, 3N and 4N are situated at (6i), (i – 5j), (3i + 2j) and (ai + bi) respectively. If the centre of gravity of the system lies at the points with position vector (2.5i – 2j), find the values of a and b. [2, -4]

5. Find the position of the centre of gravity of four particles of weight 2N, 1N, 5N and 2N which are situated at (4, -5), (1, 2), (3, -6) and (0, 3) respectively. [2.4, -3.2]
6. Particles of mass 1kg, 2kg and mkg are situated at (5, 2), (1, 5) and (1, -2) respectively. If the centre of gravity of this system lies at (2, $\bar{y}$), find the value of m and $\bar{y}$. [m = 1kg, $\bar{y} = 2.5$]
7. Particles of masses 2kg, 1kg and 3kg are situated on y-axis at (0, 7), (0, 4), and (0, -2) respectively. Where must a 6kg mass be placed to ensure that the centre of gravity of this system lies at the origin. [(0, -2)]
8. Particles of weight 5N, 4N and 3N are situated at (-5, 0), (4, 0.5), (-4, -3) respectively. Where must a 7N particles be placed to ensure that the centre of gravity of this system lies at the origin (3,1)

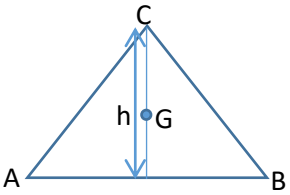
Centre of gravity of lamina

1. C.O. G of a rectangle



G = $\left(\frac{AB}{2}, \frac{AD}{2}\right)$

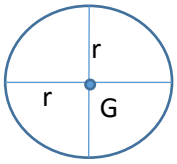
2. C.O. G of an isosceles triangle



G = $\left(\frac{AB}{2}, \frac{h}{3}\right)$

C.O.G lies along the line of symmetry a distance $\frac{h}{3}$

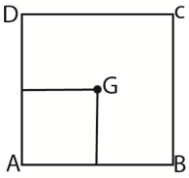
3. C.O. G of a circle



C.O.G lies along the centre

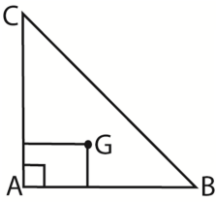
G = (r, r)

4. C.O.G of a square



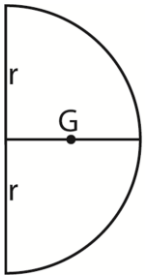
G = $\left(\frac{AB}{2}, \frac{AD}{2}\right)$

5. C.O.G right angled triangle



G = $\left(\frac{AB}{3}, \frac{AC}{3}\right)$

6. C.O.G of a semi-circle



7. C.O.G of **a sector of a circle** subtending an angle 2α at the centre line along the line of symmetry at a distance $\frac{2rsin\alpha}{3\alpha}$ from the centre.

8. C.O.G of a circular arc subtending an angle 2α at the centre line along the line of symmetry at a distance $\frac{rsin\alpha}{\alpha}$ from the centre.

C.O.G a semi-circular arc of radius r is at a distance $\frac{2r}{\pi}$ from the centre.

Example 6

Find the position of the centre of gravity of a uniform lamina in form of a triangle whose co-ordinates are

(i) (0, 0), (2, 6) and (4, 0)

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{3}(0+2+4,\ 0+6+0)$ = [(2,2)]

(ii) (0, 3), (3, 0) and (6,3) [3,2]

(iii) (0, 0), (0,6) and (6,0) [2, 2]

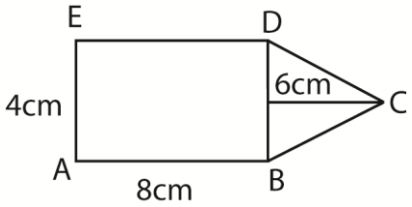
(iv) (0, 0), (0, 6) and (3,0) [1,2]

(v) (1, 0), (5, 0) and (0, 6) [2,2]

(vi) (3, 0), (6,0) and (0,6) [3, 2]

Example 7

The figure below shows a lamina formed by joining together a rectangular solid and triangular solid. Find the C.O.G of the composite lamina from AE and AB.



Lamina	Area (cm^2)	Weight	Distance of C.O.G from	
			AE	AB
ABDE	32	32w	4	2
BCD	12	12w	10	2
Composite	44	4w	$\bar{x}$	$\bar{y}$

C.O.G from AE

44w$\bar{x}$ = 32w x 4 + 12w x 10

$\bar{x}$ =5.64cm

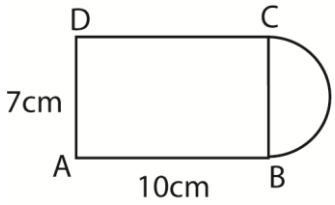
C.O.G from AB

44w$\bar{y}$ = 32w x 2 + 12w x 2

$\bar{y}$ =2cm

Example 8

The figure below shows a lamina formed b joining together a rectangular solid to a semi-circular solid. Find the C.O.G of the composite lamina



Lamina	Area (cm^2)	Weight	Distance of C.O.G from	
			AE	AB
ABCD	70	70w	5	3.5
Semi-circle	19.24	19.24w	10+ $\frac{4\ x\ 3.5}{3\pi}$ =11.49	3.5
Composite	89.24	89.24w	$\bar{x}$	$\bar{y}$

C.O.G from AD

89.24w$\bar{x}$ = 19.24w x 11.49 + 70w x 5

$\bar{x}$ =6.4cm

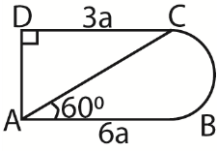
C.O.G from AB

89.24w$\bar{y}$ = 19.24w x 3.5 + 70w x 3.5

$\bar{y}$ =3.5cm

Example 9

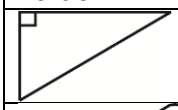
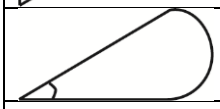
The figure below shows a uniform lamina consisting of a sector ABC of a circle centre A and of radius 6a and a triangle ADC where angle ADC = 90^0 and CAB = 60^0.



Show that the distance of the centre of gravity of the composite body from Ad is $\frac{27a\sqrt{3}}{4\pi+3\sqrt{3}}$

Solution

AD = $\sqrt{(6a)^2-(3a)^2} = 3a\sqrt{3}$; let w = weght per unit area

Portion	Area	Weight	C.O.G from AD
	$\frac{1}{2}x3a\ x3a\sqrt{3}$ =4.5$\sqrt{3a^2}$	4.5$\sqrt{3a^2}$w	$\frac{3a}{3} = a$
	$\frac{60^0}{360^0}\pi(6a)^2$ =6πa^2	6$\pi a^2 w$	$\frac{2\ x6a\ x\sin 30^0}{3\ x\frac{30^0\pi}{180^0}}\cos 30^0$ =$\frac{6a\sqrt{3}}{\pi}$
Composite	(4.5$\sqrt{3a^2}$+6πa^2)	(4.5$\sqrt{3a^2}$+6πa^2)w	$\bar{x}$

$\bar{x}\left(\frac{9}{2}\sqrt{3}+6\pi\right)a^2w = 6\pi a^2 w\ x\ \frac{6a\sqrt{3}}{\pi} + \frac{9}{2}\sqrt{3a^2}w\ x\ a$

$\bar{x}\left(9\sqrt{3}+12\pi\right) = 81a\sqrt{3}$

$\bar{x} = \frac{81a\sqrt{3}}{12\pi+9\sqrt{3}} = \frac{27a\sqrt{3}}{4\pi+3\sqrt{3}}$

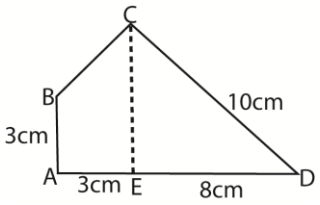
Example 10

Find the position of the centre of gravity of a uniform lamina in form of a triangle whose coordinates are (2, 2), (4, 6) and (0, 3)respectively

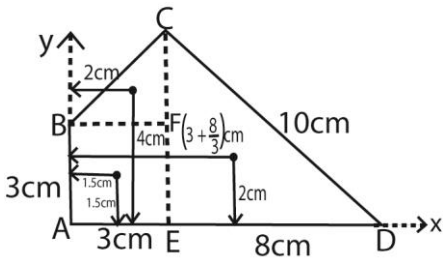
$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{3}(2+4+0, 2+6+3)$ = $\left(2,\ \frac{11}{3}\right)$

Example 11

Find the coordinates of the centre of the mass of the lamina shown below. Take A as the origin and AD, AB as the x – and y – axes respectively



Solution



Let m = mass per cm^2

Let $(\bar{x}, \bar{y})$= coordinates of C.O.G of composite

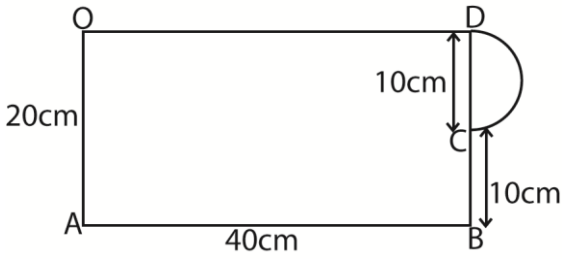
Portion	Area (cm^2)	Mass	C.O.G from AB (cm)	C.O.G from AD (cm)
ABFE	9	9m	1.5	1.5
BFC	4.5	4.5m	2	4
EDC	24	24m	$\left(3+\frac{8}{3}\right)=\frac{17}{3}$	2
ADCB	37.5	37.5m	$\bar{x}$	$\bar{y}$

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} 37.5m = 9m\begin{pmatrix}1.5\\ 1.5\end{pmatrix} + 4.5m\begin{pmatrix}2\\ 4\end{pmatrix} + 24m\begin{pmatrix}\frac{17}{3}\\ 2\end{pmatrix}$

$\bar{x} = 4.227cm$ and $\bar{y} = 2.12cm$

Example 12

The figure below is a lamina formed by welding together a rectangular metal sheet and a semi-circular meal sheet.



Find the position of the centre of gravity from the side OA and AB

Let w = weight per cm^2

Lamina	Weight	Distance of C.O.G from	
		OA	AB
ABDO	800w	20	10
Semi-circle	39.27w	$20+\frac{3\,x\,5}{8}$ =21.875	15
Composite	839.27w	$\bar{x}$	$\bar{y}$

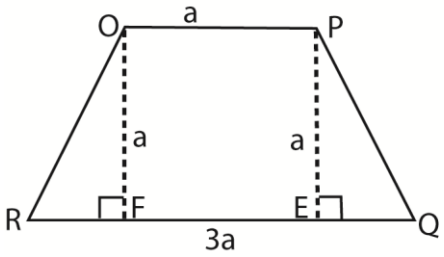
839.27w$\bar{x}$ = 800 w x 20 + 39.27w x 21.875; $\bar{x}$ = 20.088cm

839.27w$\bar{y}$ = 800 w x 10 + 39.27w x 15; $\bar{y} = 10.234$cm

∴ Position of C.O.G from OA and AB is 20.088 cm and 10 .234cm respectively

Example 13

The figure OPQR below shows a metal sheet of uniform material cut in a shape of a trapezium; OP= a, RQ = 3a and the vertical height of P from RQ = a.



Calculate the centre of gravity of the mass from R and RQ

Let w = weight per cm^2

Lamina	Weight	Distance of C.O.G from	
		R	RQ
ROF	$\frac{1}{2}a^2$w	$\frac{2a}{3}$	$\frac{a}{3}$
OPFE	a^2w	$\frac{2a}{3}+\frac{a}{2}=\frac{7a}{6}$	$\frac{a}{2}$
PEQ	$\frac{1}{2}a^2$w	$\frac{7a}{6}+\frac{a}{3}=\frac{9a}{6}$	$\frac{a}{3}$
Composite	$2a^2$w	$\bar{x}$	$\bar{y}$

$2a^2\text{w}\bar{x} = \frac{1}{2}a^2$ w x $\frac{2a}{3} + a^2$w x $\frac{7a}{6} + \frac{1}{2}a^2w$ x $\frac{9a}{6} = \left(\frac{2a^3}{6}+\frac{7a^3}{6}+\frac{9a^3}{6}\right)w = \frac{18a^3}{6}w$;

$\bar{x} = \frac{18a^3}{12a^2} = \frac{3a}{2}$

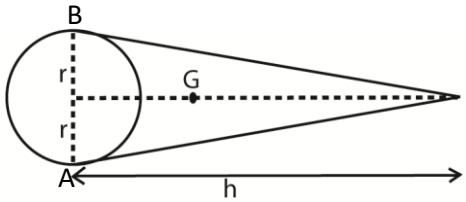
$2a^2\text{w}\bar{y} = \frac{1}{2}a^2$ w x $\frac{a}{3} + a^2$w x $\frac{a}{2} + \frac{1}{2}a^2w$ x $\frac{a}{3} = \left(\frac{a^3}{6}+\frac{a^3}{2}+\frac{a^3}{6}\right)w = \frac{5a^3}{6}w$

$\bar{x} = \frac{5a^3}{12a^2} = \frac{5a}{12}$

∴ Centre of gravity is $\left(\frac{3a}{2}, \frac{5a}{12}\right)$

Centre of gravity of solids

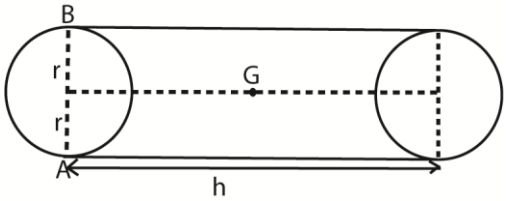
1. C.O.G of a solid cone, Pyramid and tetrahedron



C.O.G lies along the line of symmetry at a distance $\frac{h}{4}$ from AB

G = $\left(\frac{h}{4}, r\right)$

2. C.O.G of solid cylinder



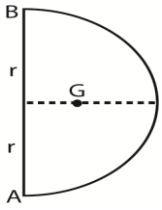
C.O.G lies along the line of symmetry at a distance $\frac{h}{2}$ from AB

G = $\left(\frac{h}{2}, r\right)$

3. C.O.G of solid hemisphere

C.O.G lies along the line of symmetry at a distance $\frac{3r}{8}$ from AB

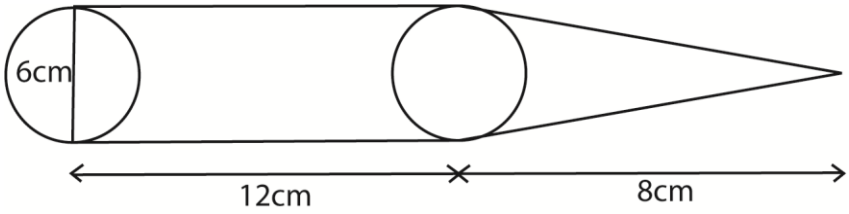
G = $\left(\frac{3r}{8}, r\right)$



4. C.O.G of *a hollow(thin) hemisphere* of radius r is at a distance $\frac{h}{3}$ from the base
5. C.O.G of a hollow (thin) cone, pyramid of tetrahedron of height h is at a distance $\frac{h}{3}$ from the base

Example 14

The figure below shows a lamina formed by joining together a cylindrical solid to a conical solid. Find the centre of gravity of the composite from AE and AB



Solution

Let w = weight per volume

Solid	Volume (cm^3)	Weight	Distance of C.O.G from	
			AE	AB
Cylinder	339.29	339.29w	6	3
Cone	75.4	75.4w	$12+\frac{8}{4}$ = 14	3
Composite	414.69	414.69w	$\bar{x}$	$\bar{y}$

C.O.G from AE = 414.69w$\bar{x}$ = 339.29 x 6 + 75.4 x 14

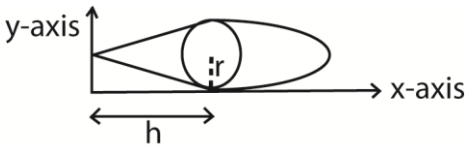
$\bar{x}$ = 7.45

C.O.G from AB = 414.69w$\bar{y}$ = 339.29 x 3 + 75.4 x 3

$\bar{y}$ =3cm

Example 15

A body consists of a solid hemisphere of radius r joined to a solid right angled circular cone of base radius r and perpendicular height h. The pane surface of the cone and hemisphere coincide and both solids are made of the same uniform material. Show that the C.O.G of the body lies on the axis of symmetry at a distance $\frac{3r^2-h^2}{4(h+2r)}$ from the base of the cone.



Let w = weight per volume

Solid	Weight	C.O.G from y-axis
Cone	$\frac{1}{3}\pi r^2 hw$	$\frac{3h}{4}$
Hemisphere	$\frac{2}{3}\pi r^3 w$	$h+\frac{3r}{8}$
Composite	$\frac{1}{3}\pi r^2(h+2r)w$	$\bar{x}$

$\frac{1}{3}\pi r^2(h+2r)w\bar{x}$

$=\frac{1}{3}\pi r^2 hw\ x\frac{3h}{4}+\frac{2}{3}\pi r^3 w\ x\left(h+\frac{3r}{8}\right)$

(h+ 2r) $\bar{x}=\frac{3h^2}{4}+2r\left(h+\frac{3r}{8}\right)$

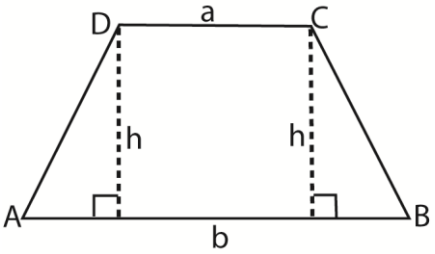
$\bar{x}=\frac{3h^2+8hr+3r^2}{4(h+2r)}$

C.O.G from the base = $\frac{3h^2+8hr+3r^2}{4(h+2r)}-h$

$=\frac{3r^2-h^2}{4(h+2r)}$

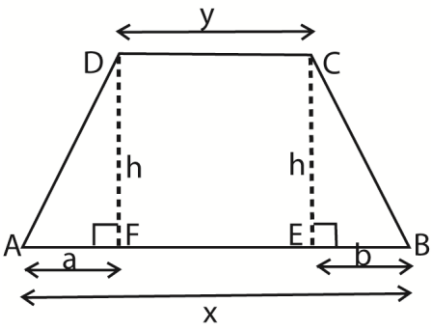
Revision exercise 2

1. ABCD is a trapezium in which AB and CD are parallel and length a and b respectively



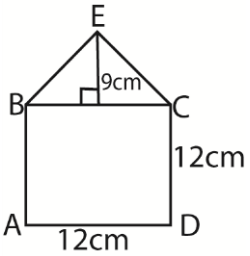
Prove that the distance of the centre of gravity of the centre of mass from AB is $\frac{1}{3}h\left(\frac{a+2b}{a+b}\right)$, where h is the distance between AB and CD

2. The figure ABCD below shows a metal sheet of uniform material cut in a shape of a trapezium: AB = x, CD = y, AF = a, EB = b and h is the vertical distance between AB and CD



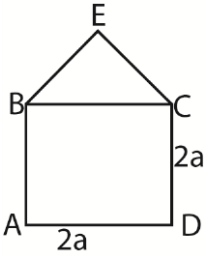
Prove that the distance of the centre of gravity of the centre of mass from AB is $\frac{1}{3}h\left(\frac{3y+a+b}{x+y}\right)$.

3. The figure below shows a sheet of metal in form of a square ABCD of side 12cm and an isosceles triangle BCE of height 9cm



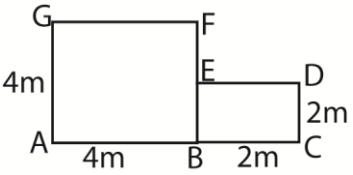
Find the distance of centre of gravity from line AB and AD [6cm, 8.45cm]

4. The figure below shows a sheet of a metal in form of a square ABCD of side 2a and an equilateral triangle BCE.



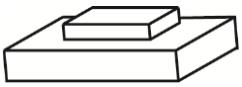
Find the distance of centre of gravity from line AB and AD [a, 1.896a]

5. The diagram shows two uniform squares ABFG and BCDE joined together. The mass per unit area of BCDE is twice that ABFG.



Find the distance of the centre of gravity of the composite from AB and AG. $\left[1\frac{2}{3}m, 3m\right]$

6. Two uniform square lamina, each of side 3m are joined together to form a rectangular lamina 6m x 3m. The weight per unit area of one square is twice the weight per unit mass of the other. Find the distance of the centre of gravity from the edge of the square [0.5m]
7. Two solid cubes one of side 4cm and the other of side 2cm are made of the same uniform material. The smaller cube is glued centrally to the one of the faces of the larger cube as shown below



Find how far the centre of gravity of the composite body is from the common surface of the cube $\left[1\frac{2}{3}cm\right]$

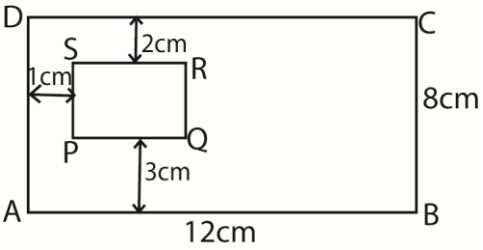
8. A uniform semi-circular lamina of radius 6cm is joined to another uniform semi-circular lamina of radius 3cm. The centre of the straight edges of each lamina coincides, but the laminas do not overlap. If the two laminas are made of the same material, find the centre of gravity of the composite lamina formed. [On the axis of symmetry, $\frac{28}{5\pi}$cm into the larger semicircle from the common diameter]
9. A uniform semi-circular lamina of radius 6cm is joined to another uniform semi-circular lamina of radius 3cm. The centre of the straight edges of each lamina coincides, but the laminas do not overlap. If the smaller semi-circle is made of a material having mass per unit area equal to twice that of the larger semi-circular lamina, find the position of the centre of gravity of the composite lamina formed. [On the axis of symmetry, $\frac{4}{\pi}$cm into the larger semicircle from the common diameter]

10. A solid right circular cylinder has a base of radius 3cm and height of 6cm. The cylinder's circular top form the base of a solid right circular cone of base radius 3cm and perpendicular height 4cm.The cylinder and the cone are made of the same uniform material. Find the position of the centre of gravity of the composite. [On the axis of symmetry, $3\frac{8}{11}$cm above the base of the cylinder]
11. A body consists of a solid hemisphere of radius 4cm joined to a solid circular cone of radius 4cm and perpendicular height 12cm. The plane surface of the cane and the hemisphere coincides and both solids are made of the same uniform material. Find the centre of gravity of the composite [On the axis of symmetry, 10.8cm from the tip of the cone]

Centre of gravity of a remainder.

Example 16

Find the centre of gravity of the remainder of the rectangle ABCD if a square PQRS is removed as shown below



Let w = weight per cm^2

Portions	Area (cm^2)	weight	Distance of C.O.G from	
			AD	AB
ABCD	96	96w	6	4
PQRS	9	9w	2.5	4.5
remainder	87	87w	$\bar{x}$	$\bar{y}$

C.O.G from AD

87w$\bar{x}$ = 96w x 6 – 9w x 2.5

$\bar{x}$ =6.362cm

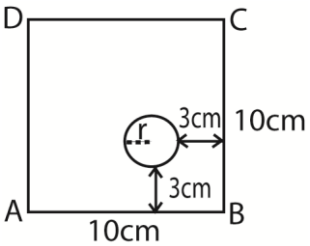
C.O.G from AB

87w$\bar{y}$ = 96w x 4 – 9w x 4.6

$\bar{y}$ = 3.95cm

Example 17

Find the centre of gravity of the remainder of the square ABCD offside 10cm if a circle of radius r = 3cm is removed as shown below



Let w = weight per cm^2

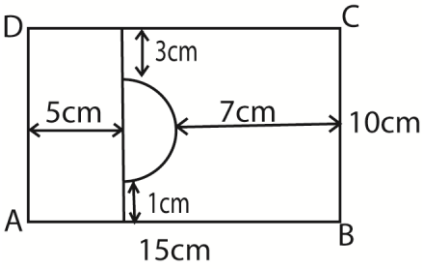
Portions	Area (cm^2)	weight	Distance of C.O.G from	
			AD	AB
ABCD	100	100w	5	5
PQRS	28.27	28.27w	4	6
remainder	71.73	71.73w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 71.73w$\bar{x}$ = 100w x 5 – 28.27w x 5; $\bar{x}$= 5.39cm

C.O.G from AB: 71.73w$\bar{y}$ = 100w x 5 – 28.27w x 6; $\bar{x}$= 4.61cm

Example 18

A rectangle ABCD is of AB = 15cm and AD = 10cm. If a semi-circle is removed as shown below. Find the centre of gravity of the remainder solid



Let w = weight per cm^2

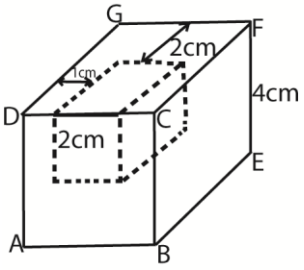
Portions	Area (cm^2)	weight	Distance of C.O.G from	
			AD	AB
ABCD	150	150w	7.5	5
Semi-circle	14.14	14.14w	$5+\frac{4\ x\ 3}{3\pi}$ =6.27	4
remainder	135.86	135.86w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 135.86w$\bar{x}$ = 150w x 7.5 – 14.14w x 6.27; $\bar{x}$= 7.63cm

C.O.G from AB: 135.86w$\bar{y}$ = 150w x 5 – 14.14w x 4; $\bar{x}$= 5.1cm

Example 19

A solid cube of 4cm is made from a uniform material. From this a smaller cube is removed as shown



Find the centre of gravity of the remaining body

Let w = weight per cm^3

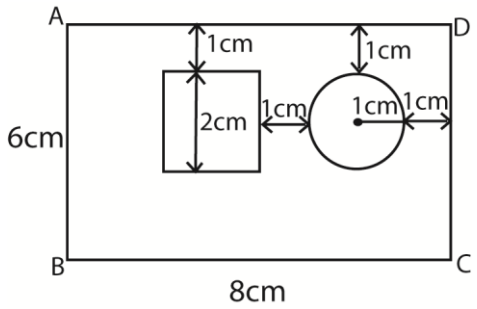
Portions	volume (cm^3)	weight	Distance of C.O.G from	
			AD	AB
Big cube	64	64w	2	2
Small cube	8	8w	3	3
remainder	56	56w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 56w$\bar{x}$ = 64w x 2 – 8w x 2; $\bar{x}$= 2cm

C.O.G from AB: 56w$\bar{y}$ = 64w x 2 – 8w x 3; $\bar{y}$= 1.86cm

Example 20

ABCD is a uniform rectangular sheet of cardboard of length 8cm and width 6cm. a square and a circular hole are cut off from the cardboard as shown below. Calculate the position of the centre of gravity of the remaining sheet



Let w = weight per cm^3

Portions	volume (cm^3)	weight	Distance of C.O.G from	
			AB	BC
ABCD	48	64w	4	3
square	4	4w	3	4
Circle	3.14	3.142w	6	4
remainder	40.86	40.86w	$\bar{x}$	$\bar{y}$

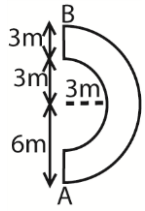
C.O.G from AB: 40.86w$\bar{x}$ = 48w x 4 – (4w x 3+ 3.14w x 6); $\bar{x}$= 3.944cm

C.O.G from BC: 40.86w$\bar{y}$ = 48w x 3 – (4w x 4+ 3.14w x 4); $\bar{y}$= 2.825cm

Revision exercise 3

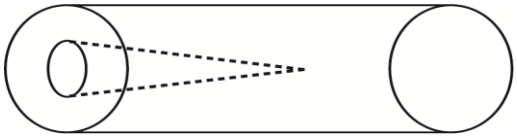
1. A circular lamina made of uniform material has its centre at the origin and radius 6cm. Two smaller circles are cut from this circle, one of radius 1cm and centre (-1, -3) and another of radius 3cm and centre (1, 2). Find the coordinates of the centre of gravity of the remaining shape. $\left[\frac{-4}{3}, \frac{-15}{26}\right]$

2. The diagram shows a uniform semi-circular lamina of radius 6m with a semi-circular portion of radius 3m missing.



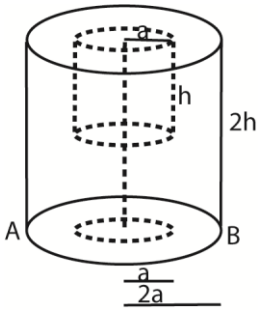
   Find the distance of the centre of gravity of the remaining shape from AB as shown above $\left[\frac{28}{\pi}\right]$

3. A conical hole is made in one end of a right circular cylinder, The axis of symmetry of the cone is the same as that of the cylinder



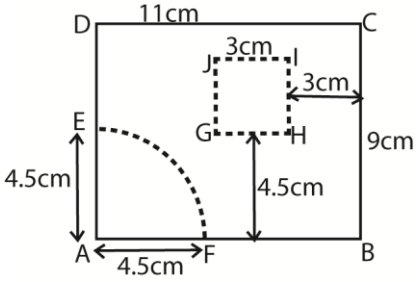
   The cylinder is of radius 2cm and length 6cm. The conical hole penetrates 4cm into the cylinder and the circular hole at the end of the cylinder is of radius 1.5cm. Find the position of the centre of gravity of the remaining body. [on axis of symmetry, $2\frac{5}{17}$cm from undrilled end]

4. The diagram below shows a uniform cylinder of radius 2a and height 2h with a cylindrical hole of radius a and height h drilled centrally at one plane end.



Show that the centre of gravity of the remaining solid is $\frac{13}{14}h$ from base AB

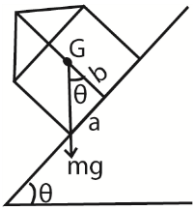
5. The figure below shows a uniform rectangular lamina ABCD with a square GHIJ of side 3cm and a quarter circular section AFE of radius 4.5cm cut off



Find the coordinates of the centre of gravity from sides AB and AD taken as the x and y axes respectively [6.149cm, 4.874cm]

Toppling

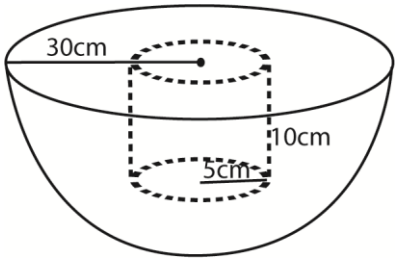
Consider a body which is resting on a slope which is rough to prevent slipping. When the angle of the slope is such that the weight acts through X the will be on the point of toppling.



Where, θ = $\tan^{-1}\left(\frac{a}{b}\right)$

Example 21

The figure shows a uniform hemispherical solid of radius 30cm with cylindrical hole of radius 5cm and height 10cm centrally drilled in it.



(a) Find the distance of centre of gravity from the flat surface

(b) If the figure is placed on an inclined plane with the flat surface in contact with the plane, calculate the angle that should be inclined, before toppling occurs assuming that sliding does not occur.

Solution

Let w = weight per cm^3

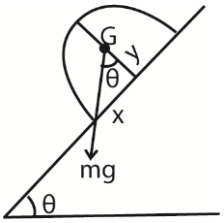
Portion	Volume (cm^3)	Weight	C.O.G from	
			Flat face (cm)	y-axis (cm)
Hemisphere	$\frac{2}{3}\pi r^3$ =5648.668	5648.668w	$\frac{3\ x\ 30}{8}$ = 11.25	30
Cylinder	$\pi r^2 h$ = 785.398	785.398w	5	30
remainder	4863.270	4863.270w	$\bar{y}$	$\bar{x}$

C.O.G from flat face

4863.270w$\bar{y}$ = 5648.668w x 11.25 - 785.398w x 5; $\bar{y}$ = 12.26

C.O.G from y-axis

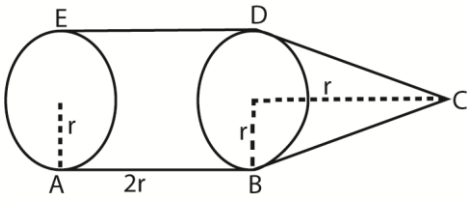
4863.270w$\bar{x}$ = 5648.668w x 30 - 785.398w x 30; $\bar{x}$ = 30



$\theta = \tan^{-1}\left(\frac{30}{12.26}\right) = 67.77^0$

Example 22

A body consists of a uniform solid cylinder of mass 6m, base radius r and height 2r, attached to a plane face of a uniform solid cone of mass 4m base radius r and height r.



(a) Find the position of the centre of gravity of the body

(b) The body is now placed with its plane face AE in contact with a horizontal table. The surface of the table is rough enough to prevent slipping as the table is slowly tilted. Find the angle through which the table has to be tilted when the body is on the point of toppling.

Let w = weight per cm^3

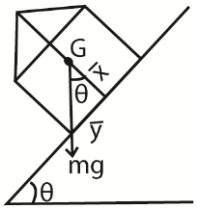
Portion	Weight	C.O.G from	
		AE(cm)	AB (cm)
Cylinder	6mg	r	r
cone	4mg	2r + $\frac{r}{4}$ = $\frac{9r}{4}$	r
composite	10mg	$\bar{x}$	$\bar{y}$

C.O.G from AE

10mg$\bar{x}$ = 4mg x $\frac{9r}{4}$ + 6mg x r; $\bar{x}$ = 1.5r

C.O.G from AB

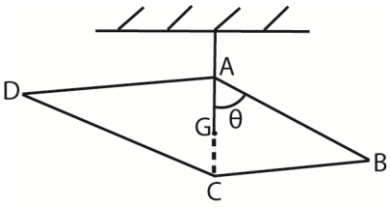
10mg$\bar{y}$ = 4mg x r + 6mg x r; $\bar{x}$ = r



$\theta = \tan^{-1}\left(\frac{r}{1.5r}\right) = 33.69^0$

Equilibrium of suspended lamina

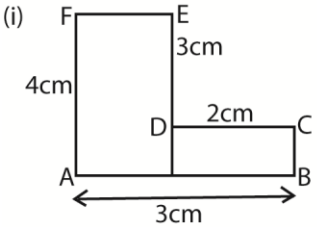
Consider a lamina freely suspended from point A. The centre of gravity passes through the point of suspension

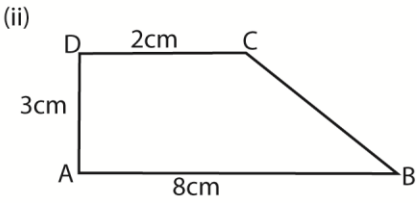


$\theta = \tan^{-1}\left(\frac{BC}{AB}\right)$

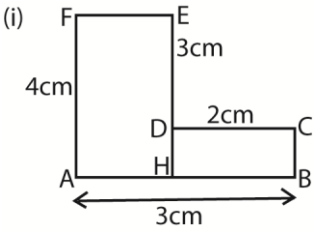
Example 23

The uniform laminas below freely suspended from point A. In each case find the angle that line AB makes with the vertical.



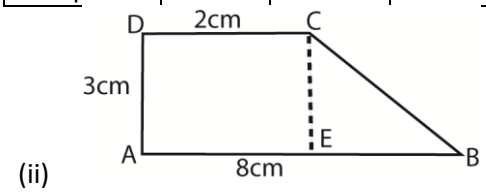


(i) Solution



Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from	
			AF	AB
AFEH	4	4w	1	2
HBCD	2	2w	1+1 =2	0.5
Composite	6	6w	$\bar{x}$	$\bar{y}$

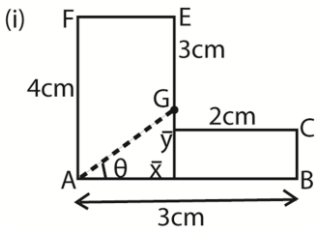


(ii)

C.O.G from AF: $6w\bar{x} = 4w \times 1 + 2w \times 2$; $\bar{x}$ = 1cm

C.O.G from AB: $6w\bar{y} = 4w \times 2 + 2w \times 0.5$;

$\bar{y}$ = 1.5cm



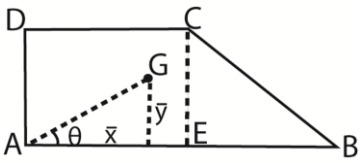
$\theta = \tan^{-1}\left(\frac{1.5}{1}\right) = 56.3^0$

Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from	
			AD	AB
ADCE	6	6w	1	1.5
EBC	9	9w	$2+\frac{6}{3}=4$	1
Composite	15	15w	$\bar{x}$	$\bar{y}$

C.O.G from AD: 15w$\bar{x}$= 6w x 1 + 9w x 4; $\bar{x}$ = 2.8cm

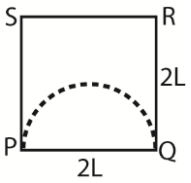
C.O.G from AB: 15w$\bar{y}$= 6w x 1.5 + 9w x 1; $\bar{y}$ = 1.2cm



$\theta = \tan^{-1}\left(\frac{1.2}{2.8}\right) = 23.2^0$

Example 24

The figure below shows a uniform lamina PQRS of side 2L with semi-circular lamina cut off as show below



(a) Show that the distance of the centre of gravity of the from PQ is $\frac{20L}{3(8-\pi)}$.

(b) The figure is freely suspended from point R. Find the angle RS makes with the vertical.

Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from PQ
PQRS	$4L^2$	$4L^2$w	L
Semi-circle	$\frac{1}{2}\pi L^2$	$\frac{1}{2}\pi L^2$ w	$\frac{4L}{3\pi}$
Composite	$\left(4-\frac{\pi}{2}\right)L^2$	$\left(4-\frac{\pi}{2}\right)L^2$ w	$\bar{y}$

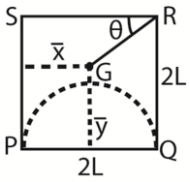
C.O.G from PQ:

$\left(4-\frac{\pi}{2}\right)L^2$ w$\bar{y}$= $4L^2$w x L + $\frac{1}{2}\pi L^2$ w x $\frac{4L}{3\pi}$;

$\left(\frac{8-\pi}{2}\right)\bar{y} = 4L - \frac{2L}{3}$

$\left(\frac{8-\pi}{2}\right)\bar{y} = \frac{10L}{3}$

$\bar{y} = \frac{20L}{3(8-\pi)}$



C.O.G from RS

$$2L - \frac{20L}{3(8-\pi)} = \frac{28L-6L\pi}{3(8-\pi)}$$

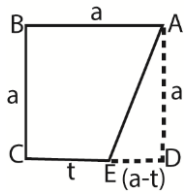
$$\theta = \tan^{-1}\left(\frac{\frac{28L-6L\pi}{3(8-\pi)}}{L}\right) = 32.12^0$$

Example 25

ABCD is a uniform square lamina of side a from which a triangle ADE is removed, E being a point of distance t from C

(i) Show that the centre of gravity of the remaining lamina is at a distance $\frac{a^2+at+t^2}{3(a+t)}$ from BC

(ii) If the lamina is placed in a vertical plane with CE resting on a horizontal table show equilibrium will not be possible if $t < \frac{a(\sqrt{3}-1)}{2}$



Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from BC
ABCD	a^2	a^2w	$\frac{a}{2}$
DAE	$\frac{1}{2}a(a-t)$	$\frac{1}{2}a(a-t)$ w	$a-\frac{1}{3}(a-t)=\frac{2a-t}{3}$
remainder	$\frac{1}{2}a(a+t)$	$\frac{1}{2}a(a+t)w$	$\bar{x}$

$\frac{1}{2}a(a+t)w\bar{x} = a^2w\ x\ \frac{a}{2} - \frac{1}{2}a(a-t)$ w x $\frac{2a-t}{3}$

$\bar{x}(a+t) = \frac{3a^2-(a-t)(2a+t)}{3}$

$\bar{x} = \frac{a^2+at+t^2}{3(a+t)}$

(ii) t < $\bar{x}$

$t < \frac{a^2+at+t^2}{3(a+t)}$

$3at + 3t^2 < a^2 + at + t^2$

$2t^2 + 2at - a^2 < 0$

$t < \frac{-2a\pm\sqrt{(2a)^2-4\ x\ 2\ x-a^2}}{2\ x\ 2}$

$t < \frac{-a\pm a\sqrt{3}}{2}$

$t < \frac{a(-1-\sqrt{3})}{2}$ or $t < \frac{a(\sqrt{3}-1)}{2}$

Since $\frac{a(-1-\sqrt{3})}{2}$ give a-t < 0 then

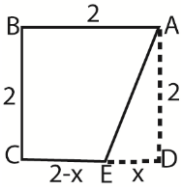
$t < \frac{a(\sqrt{3}-1)}{2}$

Example 26

ABCD is a uniform square lamina of side 2cm from which a triangle EDC is removed. E being a point on AD such that ED = x cm

(i) Show that the centre of gravity of the remaining lamina is at a distance $\frac{12-6x+x^2}{3(4-x)}$ from AB

(ii) If the lamina is placed in a vertical plane with AE resting on a rough horizontal table, show that it will topple if x> 3- $\sqrt{3}$



Let w= weight per cm^2

Lamina	Area (cm^2)	Weight	C.O.G from BC
ABCD	4	4w	1
DAE	$\frac{1}{2}(2x)$ =x	x w	$2-\frac{1}{3}x=\frac{6-x}{3}$
remainder	$(4-x)$	$(4-x)\ w$	$\bar{x}$

$\bar{x}(4-x)w = 4w\ x\ 1 - \frac{(6-x)wx}{3}$

$\bar{x}(4-x) = \frac{12-6x+x^2}{3}$

$\bar{x} = \frac{12-6x+x^2}{3(4-x)}$

(ii) 2-x > $\bar{x}$

$2 - x > \frac{12-6x+x^2}{3(4-x)}$

24-18x + 3x^2 > 12 – 6x + x^2

x^2-6x + 6 > 0

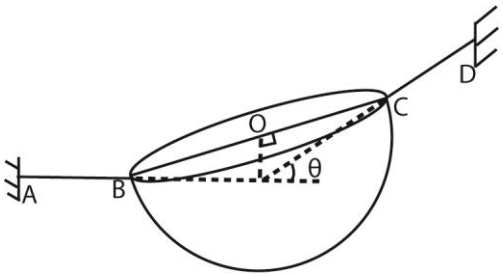
$x > \frac{6\pm\sqrt{6^2-4\ x\ 1\ x\ 6}}{2x1}$; $x > \frac{6\pm\sqrt{12}}{2}$

$x > 3 - \sqrt{3}$ or $x > 3 + \sqrt{3}$

Since $3 + \sqrt{3}$ then $x > 3 - \sqrt{3}$

Example 26

A uniform solid hemisphere is in equilibrium in space by two inelastic strings AB which is horizontal and CD which makes an angle θ with the horizontal. The strings and the diameter BC lie in the same vertical plane.

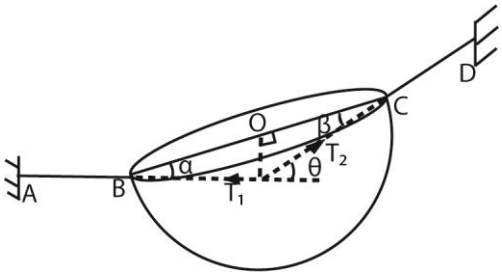


Given that the weight of the hemisphere is 24N, show that

(i) tanθ = $\frac{48}{55}$

(ii) Tension in strings AD and AB are 36.5N and 27.5N respectively

Solution



tanα = $\frac{3r}{8}\Big/_{r}$ = $\frac{3}{8}$ and tan β $\frac{3r}{8}\Big/_{r}$ = $\frac{3}{8}$

θ = α + β

tanθ = tan(α + β) = $\frac{tan\alpha + tan\beta}{1-tan\alpha tan\beta}$

tanθ = $\frac{\frac{3}{8}+\frac{3}{8}}{1-\frac{3}{8}x\frac{3}{8}} = \frac{48}{55}$

(→) T$_2$cosθ = T$_1$ …… (i)

(↑) T$_2$sinθ = 24 ……..(ii)

$\frac{T_2 sin\theta}{T_2 cos\theta} = \frac{24}{T_1}$

$\tan\theta = \frac{24}{T_1}$

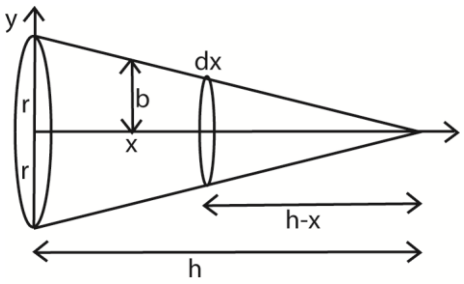
$T_1 = \frac{24}{\tan\left(\frac{48}{55}\right)}$ = 27.5N

$T_2 = \frac{24}{sin\theta} = \frac{24}{\sin\left(\tan^{-1}\frac{48}{55}\right)}$ 36.5N

Example 27

Prove that the centre of mass of a solid cone is $\frac{1}{4}$ of the vertical height from the base

By subdividing the cone into small discs



Let w= weight per unit volume

Volume of a disc = $\pi r^2 h = \pi b^2 dx$

By using similarity of figures

$\frac{b}{r} = \frac{h-x}{h}$; b = $\frac{(h-x)}{h} r$

By substitution

Volume of disc = $\pi \left(\frac{(h-x)}{h}\right)^2 r^2 dx$

Volume of a cone = $\frac{1}{3}\pi r^2 h$

Weight of a cone =$\frac{1}{3}\pi r^2 hw$

Moment of the whole cone about the y-axis

=$\frac{1}{3}\pi r^2 hwx$

Equating moments of the whole solid with the sum of moments of discs about y-axis

$\frac{1}{3}\pi r^2 hw\bar{x} = \pi \int_0^h \left(\frac{(h-x)}{h}\right)^2 r^2 wxdx$

$\frac{1}{3}\pi r^2 hw\bar{x} = \frac{\pi}{h^2}\int_0^h (h^2 - 2hx + x^2)\, r^2 wxdx$

$\frac{1}{3}h^3\bar{x} = \int_0^h (h^2 x - 2hx^2 + x^3)\, dx$

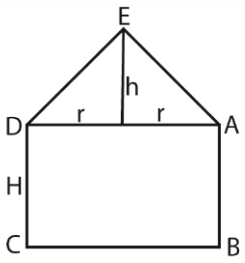
$\frac{h^3\bar{x}}{3} = \left[\frac{h^2x^2}{2} - \frac{2hx^3}{3} + \frac{x^4}{4}\right]_0^h$

$\frac{h^3\bar{x}}{3} = \frac{h^4}{12}$

$\bar{x} = \frac{h}{4}$

Example 28

The figure ABCDE below shows a solid cone of radius r, height, h joined to solid cylinder of the same material with the same radius and height, H.



(i) If the centre of gravity of the whole solid lies in the plane of the cone where the two solids are joined, find H.
Let w = weight per unit volume

Lamina	volume	Weight	C.O.G from BC
cone	$\frac{1}{3}\pi r^2 h$	$\frac{1}{3}\pi r^2 hw$	$H+\frac{h}{4}$
cylinder	$\pi r^2 H$	$\pi r^2 Hw$	$\frac{1}{2}H$
composite	$\pi r^2\left(H+\frac{h}{3}\right)$	$\pi r^2\left(H+\frac{h}{3}\right)w$	H

Equating moments

$\pi r^2\left(H+\frac{h}{3}\right)wH = \frac{1}{3}\pi r^2 hw\ x\ \left(H+\frac{h}{4}\right) + \pi r^2 H\ w\ x\ \frac{H}{2}$

$\left(H+\frac{h}{3}\right)H = \frac{1}{3}h\left(H+\frac{h}{4}\right)+\frac{H^2}{2}$

$H^2+\frac{Hh}{3} = \frac{Hh}{3}+\frac{h^2}{12}+\frac{H^2}{2}$

$\frac{H^2}{2} = \frac{h^2}{12}$

H =$\frac{h}{\sqrt{6}}$

(ii) If instead H= h and r = ½r, find the angle AB makes with the horizontal, if the body is hanged from A.

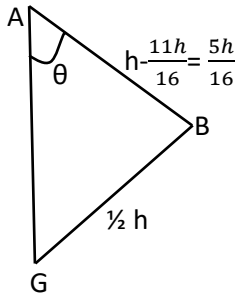
Let distance from BC to the centre of gravity be $\bar{y}$ and substituting H and r

$\pi\left(\frac{r}{2}\right)^2\left(h+\frac{h}{3}\right)w\bar{y} = \pi\left(\frac{r}{2}\right)^2 hw\ x\ \frac{1}{2}h + \frac{1}{3}\pi\left(\frac{r}{2}\right)^2 hw\ x\ \left(h+\frac{h}{4}\right)$

$\frac{4h}{3}\bar{y} = \frac{h^2}{2}+\frac{5h^2}{12} = \frac{11h^2}{12}$

$\bar{y} = \frac{11h}{16}$

If the body is hang from A



tanθ = $\left(\frac{1}{2}h\right)\div\left(\frac{5h}{16}\right)$

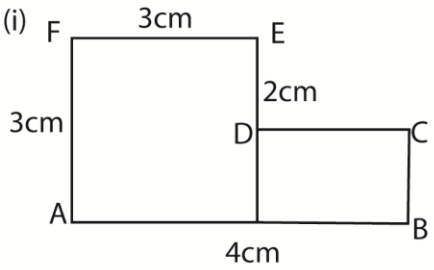
θ = 57.99^0

Therefore AB makes an angle 90 -57.99^0 = 32.01^0 with horizontal

Revision exercise 4

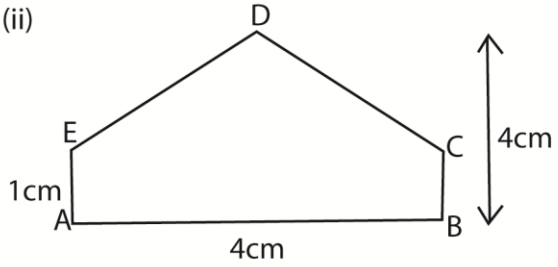
1. The uniform laminas below are freely suspended from point A. In each case find the angle that line AB makes with the vertical.

(i)

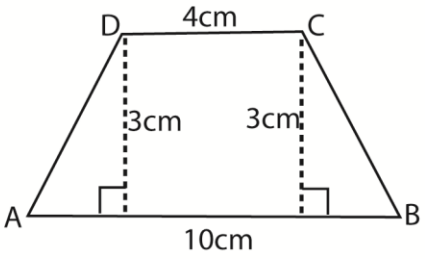


[39.5º]

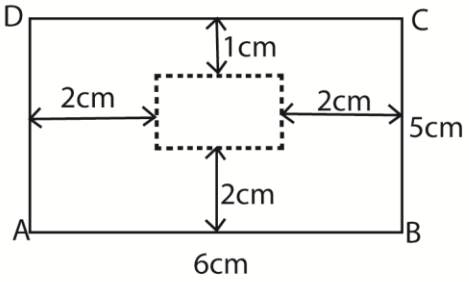
(ii)



[35º]

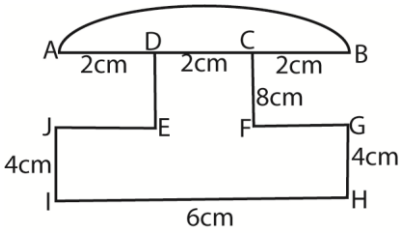


[14.4º]



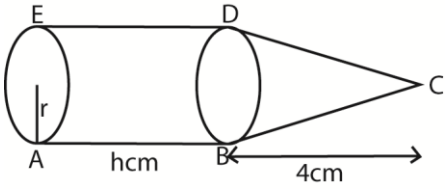
[38.9º]

2. The figure ABCDEFGHIJ shows a symmetrical composite lamina made up of a semi-circle of radius 3cm. a rectangle CDEF 2cm by 8cm and another rectangle GHIJ 6cm by 4cm.



Find the distance of the centre of gravity of this lamina from HI. If the lamina is suspended from H by means of a peg through a hole, calculate the angle of inclination of HG to the vertical [6.72cm, 24.1^0]

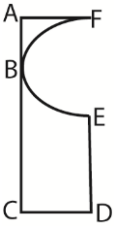
3. A, B, C and D are points (0, 0), (10, 0), (7, 4) and (3, 3) respectively. If AB, BC, CD and DA are made of a thin wire of uniform mass, find the coordinates of the centre of gravity.[5, 1.5]
4. A body consists of a uniform solid cylinder of base radius r and height h, attached to a plane face of a uniform solid cone of base radius r and height 4cm.



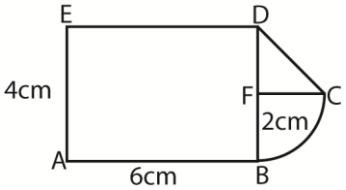
Show that the distance of the centre of gravity of the solid from AE is $\frac{3h^2+8h+8}{6h+8}$.

5. ABCD is a uniform square lamina of side 2a from which an isosceles triangle ABE is cutaway. In this triangle AB = BE and the distance of E from AB= d.
   (i) Show that G, the centre of gravity of the remaining body is a distance $\frac{12a^2-d^2}{3(4a-d)}$ from AB.
   (ii) If d = $\frac{a}{2}$ and the remaining body is suspended by the vertical string from A, find the angle which AD makes with the vertical. [41.8^0]

6. ABCD is a uniform rectangular lamina in which AB = p and BC = 3p. The point E is on AD such that ED = 3q.
   (i) Show that G the centre of gravity of the trapezium ABCE is a distance $\frac{3p^2-3pq+q^2}{(2p-q)}$ from AB and find its distance from BC.
   (ii) When the trapezium is suspended from E, the edge BC is horizontal, prove that q = $\frac{1}{2}p(3-\sqrt{3})$.
7. A child's toy consists of a solid uniform hemisphere of radius r and a solid right circular cone of base radius r and height h. If the density of the hemisphere is k time that of a cone;
   (i) Show that the distance from the vertex of the cone to the centre of gravity of the toy is $\frac{4r(3r+8h)+3h^2}{4(2kr+h)}$
   (ii) If the toy is suspended from a point on the rim of the common base and rests in equilibrium with the axis of the cone inclined at an angle θ to the downward vertical. Show that tanθ = $\frac{4r(2kr+h)}{h^2-3kr^2}$
8. A semi-circular lamina of radius r and bas AB is cut from a large semi-circular lamina of radius 2r with diameter AC.
   (i) Determine the centre of gravity of the remainder[2r, 0.99r]
   (ii) If the remainder is freely suspended from A, find the inclination of AB to the vertical[26.3^0]
9. The figure ABCDEF below shows a uniform lamina in form of a rectangle from which a hole in form of a semi-circle was made. The diameter of the semicircle is 6cm.



   (a) Find the centre of gravity from AC and CD if the semicircle is removed.[2.47cm, 4.06cm]
   (b) If the remaining lamina is suspended at D, find the angle CD makes with the vertical (82.56^0]
10. The figure below shows a uniform rectangular lamina ABCD with a triangle DFC and quarter circular lamina section BFC all of the same density. F is the centre of the circle from which the quadrant forms part.

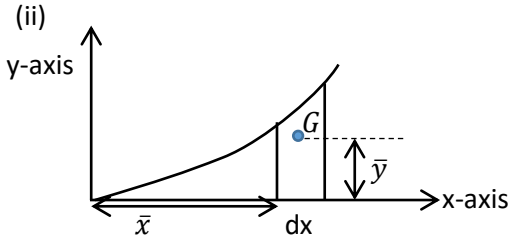


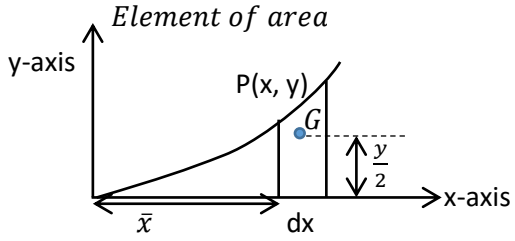
   (a)Find the coordinates of the centre of gravity from sides AB and AE taken as x and y-axes respectively
   (b) If the lamina is suspended about A, find the angle that AB makes with the horizontal.

Centre of gravity of the lamina whose area is bounded.

(i) C.O.G of the area bounded in the first quadrant

(ii)





C.O.G from y-axis: w$\bar{x}\int ydx = w\int xydx$

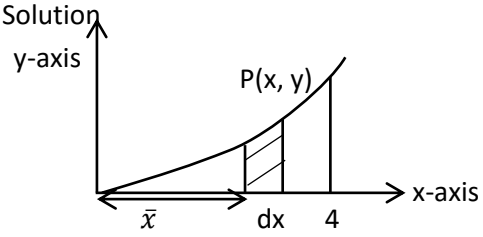
C.O.G from the x-axis: w$\bar{y}\int ydx = w\int \frac{y}{2}y\,dx$

Where w = weight per unit area

Example 28

Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve x^2, the x-axis and the line x = 4

Solution



C.O.G from y-axis: w$\bar{x}\int ydx = w\int xydx$

$\bar{x}\int_0^4 ydx = \int_0^4 xy\,dx$

$\bar{x}\int_0^4 x^2dx = \int_0^4 x(x^2)\,dx$

$\bar{x}\left[\frac{x^3}{3}\right]_0^4 = \left[\frac{x^4}{4}\right]_0^4$

$\bar{x} = 3$

w$\bar{y}\int ydx = w\int \frac{y^2}{2}dx$

$\bar{y}\int_0^4 x^2\,dx = \frac{1}{2}\int_0^4 x^4dx$

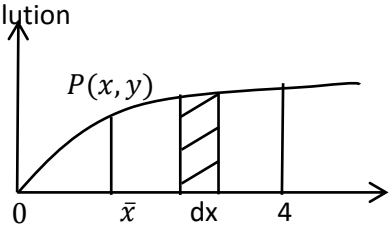
$\bar{y}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{2}\left[\frac{x^5}{5}\right]_0^4$

$\bar{y} = 4.8$

Example 29

Find the coordinates of the centre of gravity of uniform lamina enclosed by the curve $y^2 = 9x$, the x-axis and the line x =1 and x = 4 and lying in the first quadrant

Solution



$\bar{x}\int_1^4 ydx = \int_1^4 xy\,dx$

$\bar{x}\int_1^4 3x^{\frac{1}{2}}dx = \int_1^4 x\left(3x^{\frac{1}{2}}\right)dx$

$3\bar{x}\left[\frac{x^{\frac{3}{2}}}{2}\right]_1^4 = 5\left[\frac{x^{\frac{5}{2}}}{2}\right]_1^4$

$\bar{x} = 2.66$

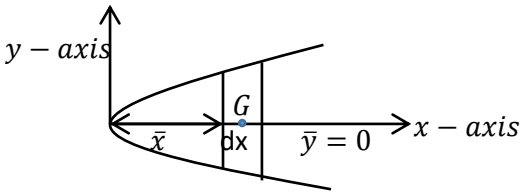
$w\bar{y}\int y dx = w\int \frac{y^2}{2}dx$

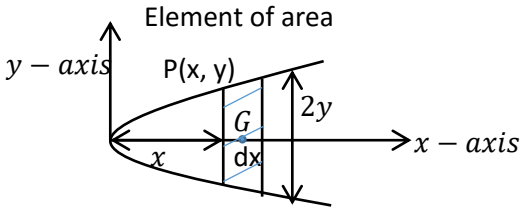
$\bar{y}\int_1^4 3x^{\frac{1}{2}}dx = \frac{1}{2}\int_1^4 9x dx$

$\bar{y}\left[\frac{x^{\frac{3}{2}}}{2}\right]_1^4 = \frac{3}{2}\left[\frac{x^2}{2}\right]_1^4$

$\bar{y} = 2.41$

(ii) C.O.G of the area bounded in the first and fourth quadrant



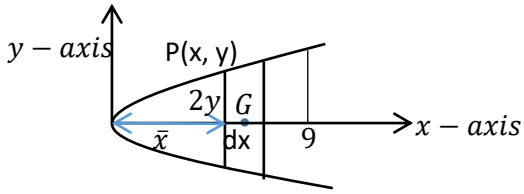


Taking moments about the y-axis: $w\bar{x}\int 2y dx = w\int x(2y)dx$

Where w- weight per unit area

Example 30

Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y^2 = 4x$ and the line x = 9



$w\bar{x}\int_0^9 2y dx = \int_0^9 x(2y)dx$

$\bar{x}\int_0^9 x^{1/2}\,dx = \int_0^9 x.x^{1/2}dx$

$3\bar{x}\left[\frac{x^{\frac{3}{2}}}{2}\right]_0^9 = 5\left[\frac{x^{\frac{5}{2}}}{2}\right]_0^9$

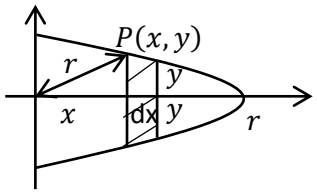
$\bar{x} = 5.4$

$(\bar{x}, \bar{y})$ = (5.4, 0)

Example 30

Show that the position of C.O.G of a uniform semi-circular lamina of radius r is $\frac{4r}{3\pi}$ from th straight edge.

Solution



Area of semi-circle = Area of element of semi-circle

$W\frac{1}{2}\pi r^2\bar{x} = w\int_0^r x(2y)dx$

But a semi-circle is part of a circle of radius r whose

Equation is $x^2 + y^2 = r^2$

y =$(r^2 - x^2)^{\frac{1}{2}}$

$\frac{1}{2}\pi r^2\bar{x} = 2\int_0^r x(r^2 - x^2)^{\frac{1}{2}}dx$

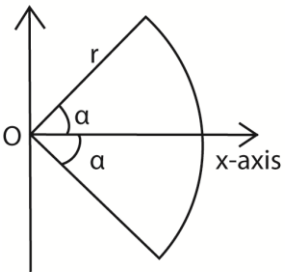
$\frac{1}{2}\pi r^2\bar{x} = 2\left[-\frac{(r^2-x^2)^{\frac{3}{2}}}{3}\right]_0^r$

$\frac{1}{2}\pi r^2\bar{x} = \frac{2r^3}{3}$

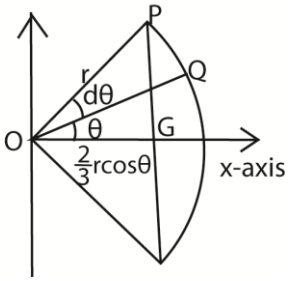
$\bar{x} = \frac{4r}{3\pi}$

Example 31

Show that the centre of gravity of a uniform lamina in shape of a sector of a circle of radius r and subtending an angle 2α at the centre O is given by $\frac{2rsin\alpha}{3\alpha}$ from O.



Element of area



The strip OPQ approximates a triangle and its C.OG is at a distance $\frac{2}{3}r$ from O. The distance of C.O.G from O is therefore $\frac{2}{3}r\cos\theta$

Area of sector = element of area of sector

$w\frac{1}{2}r^2(2\alpha)\bar{x} = w\int_{-\alpha}^{\alpha} xydx$

$\frac{1}{2}r^2(2\alpha)\bar{x} = \int_{-\alpha}^{\alpha}\frac{2}{3}rcos\theta\left(\frac{1}{2}r\right)d\theta$

$\alpha\bar{x} = \frac{1}{3}\int_{-\alpha}^{\alpha} rcos\theta d\theta$

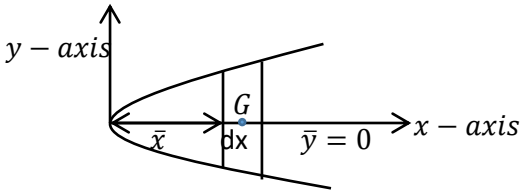
$\alpha\bar{x} = \frac{r}{3}[sin\theta]_{-\alpha}^{+\alpha}$

$\bar{x} = \frac{2rsin\alpha}{3\alpha}$

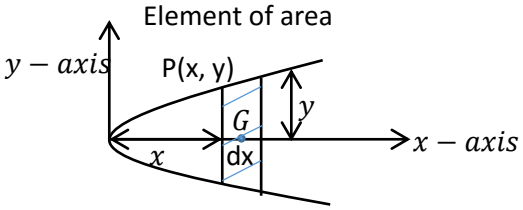
For a complete semi-circle, $\alpha = \frac{\pi}{2}$

$\therefore \bar{x} = \frac{4r}{3\pi}$

Centre of gravity of solids of revolution



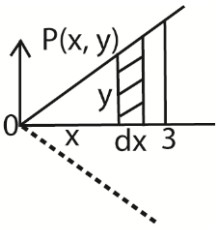
Element of area



Taking moments about y-axis: $\bar{x}w\pi\int y^2dx = w\pi\int xy^2dx$; where w = weight per unit volume

Example 32

Find the centre of gravity of a solid generated by rotating about the x-axis, the area under y = x from x =0 and x = 3.



$\bar{x}w\pi\int y^2dx = w\pi\int xy^2dx$

$\bar{x}\int_0^3 x^2dx = \int_0^3 x(x^2)dx$

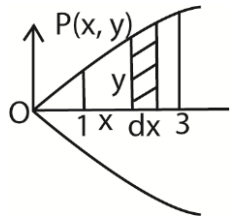
$\bar{x}\left[\frac{x^3}{3}\right]_0^3 = \left[\frac{x^4}{4}\right]_0^3$

$\bar{x} = 2.25$

$(\bar{x}, \bar{y}) = (2.25, 0)$

Example 33

Find the centre of the solid generated by rotating about the x-axis, the area bounded by $y^2 = 5x$ the x-axis, the lines x = 1 and x = 3 and lies in the first quadrant.



$\bar{x} w\pi \int y^2 dx = w\pi \int xy^2 dx$

$\bar{x} \int_1^3 5x dx = \int_1^3 x(5x) dx$

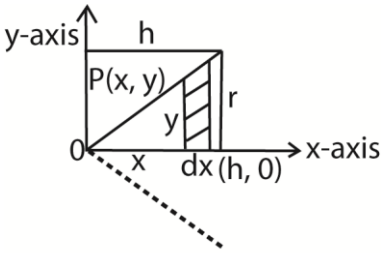
$\bar{x} \left[\frac{x^2}{2}\right]_1^3 = \left[\frac{x^3}{3}\right]_1^3$

$\bar{x} = 2.17$

$(\bar{x}, \bar{y}) = (2.17, 0)$

Example 34

Show that the position of centre of gravity of a uniform solid right circular cone of base radius r and height h is given by $\frac{h}{4}$ from the straight edge.



$\bar{x} w \frac{1}{3}\pi r^2 h = w\pi \int xy^2 dx$

$\bar{x} \frac{1}{3} r^2 h = \int xy^2 dx$

From similarity $\frac{y}{x} = \frac{r}{h}$

y = $\frac{r}{h}x$

$\bar{x} \frac{1}{3} r^2 h = \int x \left(\frac{r}{h}x\right)^2 dx$

$\bar{x} \frac{1}{3} r^2 h = \left(\frac{r^2}{h^2}\right)\left[\frac{x^4}{4}\right]_0^h$

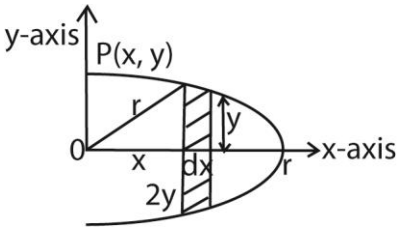
$\bar{x} = \frac{3h}{4}$

From the straight edge

$\bar{x} = h - \frac{3h}{4} = \frac{h}{4}$

Example 35

Show that the position of the centre f gravity of a <u>uniform solid hemisphere</u> of radius r is $\frac{3r}{8}$ from the straight edge.



$\frac{2}{3}\pi r^3 \bar{x} w = w\pi \int_0^r xy^2 dx$

But $x^2 + y^2 = r^2 => y^2 = r^2 - x^2$

$\frac{2}{3} r^3 \bar{x} = \int_0^r x(r^2 - x^2) dx$

$\frac{2}{3} r^3 \bar{x} = \left[r^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$

$\frac{2}{3} r^3 \bar{x} = \frac{r^4}{2} - \frac{r^4}{4}$

$\bar{x} = \frac{3r}{8}$

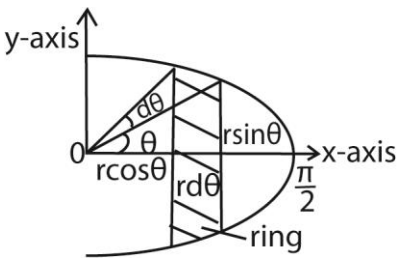
Revision exercise 5

1. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$, the x-axis and the line x= 2 [1.5, 1.2]
2. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = 2x – x^2$ and the x-axis. [1.0, 0.4]
3. Find the coordinates of the centre of gravity of a uniform lamina enclosed by the curve $y = x^2 + 2$, the x-axis and the lines x = 2 and x = 2 [1.56, 2.25]
4. Find the coordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y^2 = 8x$, the x-axis and the lines x = 2 and x = 8 [5.31, 3.21]
5. Find the coordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y^2 = x - 8$ and the axes [0.4, 0.375]
6. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^2$ and the line y = 3x. [1.5, 3.6]
7. Find the coordinates of the centre of gravity of the uniform lamina lying in the first quadrant and enclosed by the curve $y = 4 – x^2$, $y = 3x^2$ and the y-axis. [0.375, 2.2]
8. Find the coordinates of the centre of gravity of the uniform lamina enclosed by the curve $y = x^3$, the x-axis and the line x = 3 [2.4, 7.14]
9. The area enclosed by the curve $y^2 = x$, the x-axis, the line x =4 and lying in the first quadrant is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed. [2.67, 0]
10. The area enclosed by the curve $y^2 = x$, the x-axis, the line x = 2, x=4 and lying in the first quadrant is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed. [3.39, 0]
11. The area enclosed by the curve $y = x^2 + 3$, the x-axis, the y-axis and the line x = 2 is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed. [1.3, 0]
12. The area enclosed by the curve $y = x^3$, the x-axis and the line x = 3 is rotated about the x-axis through one revolution. Find the coordinates of the centre of gravity of the uniform solid formed.[2.625, 0]

Surface of revolution

Example 36

Show that the centre of gravity of a uniform thin hemispherical cup of radius r is at a distance $\frac{r}{2}$ from the base



Surface area of = element of surface area

a hemisphere

$w2\pi r^2\bar{x} = w2\pi \int_0^{\frac{\pi}{2}} xydx$

$2r^2\bar{x} = 2\int_0^{\frac{\pi}{2}}(rsin\theta\ x\ rcos\theta)rd\theta$

$2r^2\bar{x} = r^3\int_0^{\frac{\pi}{2}} 2sin\theta\ x\ rcos\theta d\theta$

$\bar{x} = \frac{r}{2}\int_0^{\frac{\pi}{2}} sin2\theta d\theta$

$\bar{x} = \frac{r}{2}\left[-\frac{1}{2}cos2\theta\right]_0^{\frac{\pi}{2}}$

$\bar{x} = -\frac{r}{4}\left[cos\left(\frac{\pi}{2}\right) - cos2(0)\right]$

$\bar{x} = -\frac{r}{4}(-1-1) = \frac{r}{2}$

Thank you

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Nurture your dreams

Circular motion

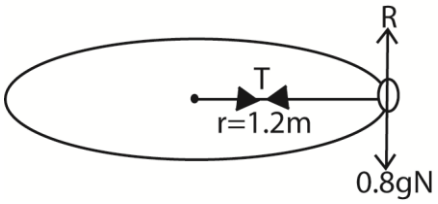
Circular motion on a smooth horizontal surface

Example 1

A particle of mass 0.8kg is attached to one end of a light inextensible string of length 1.2m. The other end is fixed to a point P on a smooth horizontal table. The particle is set moving in a circular path. If the speed of the particle is $16ms^{-1}$;

(i) Determine the tension in the string and the reaction on the table

(ii) If the string snaps when the tension in the string exceed 100N, find the greatest angular velocity at which the particle can travel.



(i) T = F= $\frac{mv^2}{r} = \frac{0.8\ x\ 16^2}{1.2}$ =170.667N

R = 0.8gN = 0.8 x 9.8 = 7.4N

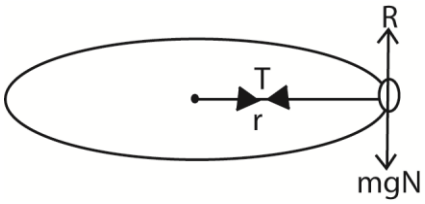
(ii) T = $m\omega^2 r$

100 = 0.8 x ω^2 x 1.2

ω = 10.21rads^{-1}

Example 2

A ball is tied on an elastic sting of natural length 30m to a fixed point on a smooth horizontal table upon which a ball is describing a circle around a point at a constant speed. If the modulus of elasticity of the string is 100times the weight of the ball and the number of revolution per minute is 20. Show that extension in the string is approximately 4.7m



T= F = $m\omega^2 r$

$\omega = 2\pi f = 2\pi\left(\frac{20}{60}\right) = \frac{2}{3}\pi$ and r = 30 + e

$T = m\left(\frac{2}{3}\pi\right)^2 (30 + e)$ .....(i)

T= $\frac{\lambda}{L}e = \frac{100mg}{30}e$ ..... (ii)

Eqn. (i) and (ii)

$\frac{100mg}{30}e = m\left(\frac{2}{3}\pi\right)^2 (30 + e)$

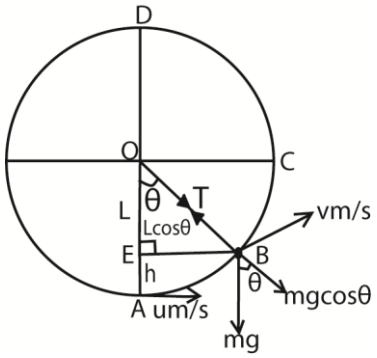
98 x 3e = $4\pi^2(30 + e)$

$e = \frac{4\pi^2 x\ 30}{294-4\pi^2}$ = 4.6532m $\approx$ 4.7m

Motion in a vertical plane

Particle in fourth quadrant

Consider a body of mass m attached to a string (light rod) of length L and whirled in a vertical circle with a constant speed v. if there is no air resistance to the motion, then the net force towards the centre is the centripetal force. Tension in the string acts in the same way as the reaction R



At equilibrium at B: T- mgcosθ = $\frac{mv^2}{L}$

$T = \frac{mv^2}{L} +$ mgcosθ ..... (i)

But $v^2 = u^2 + 2as$

a = -g, s = h = L –Lcosθ

$v^2 = u^2 – 2g(L –Lcosθ)$ ..... (ii)

when the particle comes momentarily to

rest at some point A, then v = 0

$o = u^2 – 2g(L –Lcosθ)$

$cos\theta = L - \frac{u^2}{2gL}$ ..... (iii)

If the particle is attached to <u>a rod</u>, it can it

can complete circle when v > 0 and θ= 180^0

$v^2 = u^2 – 2g(L –Lcosθ)$

$u^2 – 2g(L –Lcos180) > 0$

$u^2 > 2g(L + L)$

$u^2 > 4gL$ .....(iv)

Put (ii) into (i)

$T = \frac{m[u^2-2g(L-Lcos\theta)]}{L} + mgcos\theta$

$T = \frac{m[u^2-2g(L-Lcos\theta)]+mgLcos\theta}{L}$

$T = \frac{m}{L}(u^2 - 2gL + 3gLcos\theta)$ ..... (v)

If the particle attached to <u>a string</u>, it can complete circle when T > 0 and θ = 180^0

$T = \frac{m}{L}(u^2 - 2gL + 3gLcos\theta)$

$\frac{m}{L}(u^2 - 2gL + 3gLcos\theta) > 0$

$\frac{m}{L}(u^2 - 2gL + 3gLcos180) > 0$

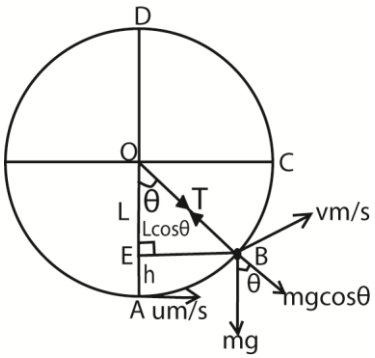
$(u^2 - 2gL - 3gL) > 0$

$u^2 > 5gL$ ..... (vi)

Example 3

A particle P of mass 5kg is suspended from a fixed point O by light inextensible string of length 1m. The particle is projected from its lowest position at A, with horizontal speed of $4ms^{-1}$. When the angle AOB = 60^0, find

(a) Speed at P (b) the tension in the string at P



$5\left(\frac{1}{2}x\ 4^2 + 9.8\ x\ 0\right) = 5\left(\frac{1}{2}x\ 4^2 + 9.8\ x\ h\right)$

$5\left(\frac{1}{2}x\ 4^2 + 9.8\ x\ 0\right) = 5\left(\frac{1}{2}x\ 4^2 + 9.8\ (1 - cos60\right)$

v = 2.49ms^{-1}

$T = \frac{mv^2}{L} + mgcos\theta$

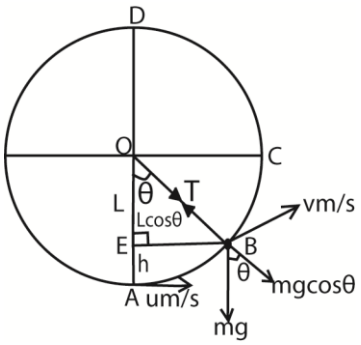
$T = \frac{5(2.49)^2}{1} + 5\ x\ 9.8cos60$ = 55.5N

T.M.E$_A$ = T.M.E$_B$

Example 4

A light rod of length 2m is pivoted at one end O and has a particle of mass 8kg attached at the other end. The rod is held vertically with the particle at A, directly below O and the particle is given an initial speed ums^{-1}; find

(a) Find an expression in terms of u and θ for the speed of the particle when at B where angle AOB = θ
(b) Restriction on u2 if the particle to perform complete oscillation



At B: $v^2 = u^2 + 2as$

a = -g, s = h = L –Lcosθ

$v^2 = u^2 – 2g(L –Lcos\theta)$

v = $\sqrt{u^2 - 2gL(1 - cos\theta)}$

If the particle to perform complete oscillation
v > 0 and θ = 180^0

$0^2 = u^2 – 2g(L –Lcos\theta)$

$u^2 – 2g(L –Lcos180) > 0$

$u^2 > 2g(L + L)$

$u^2 > 4gL$

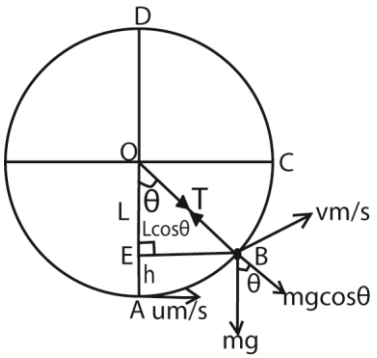
$u^2 > 4 \times 9.8 \times 2$

$u^2 > 78.4$

Example 5

A particle P of mass 8kg is suspended from a fixed point O by an inextensible string of length 2m. the particle is projected from its lowest position A with an initial speed ums^{-1}. Find

(a) An expression in terms of u and θ for the tension in the string when the particle is at B where angle AOB = θ
(b) Restriction on u2 if the particle is to perform complete oscillation



At equilibrium at B:

T – mgcosθ = $\frac{mv^2}{L}$

$T = \frac{mv^2}{L} + mgcos\theta$ ..... (i)

$v^2 = u^2 + 2ah$

a = -g, s = h = L –Lcosθ

$v^2 = u^2 – 2g(L – Lcos\theta)$

$v = \sqrt{u^2 - 2gL(1 - cos\theta)}$.....(ii)

Put (ii) into (i)

$T = \frac{m[u^2 - 2g(L - Lcos\theta)]}{L} + mgcos\theta$

$T = \frac{m[u^2 - 2g(L - Lcos\theta)] + mgLcos\theta}{L}$

$T = \frac{m}{L}(u^2 - 2gL + 3gLcos\theta)$

for the particle attached to circle when T > 0 and θ = 180^0

$T = \frac{m}{L}(u^2 - 2gL + 3gLcos\theta)$

$\frac{m}{L}(u^2 - 2gL + 3gLcos\theta) > 0$

$\frac{m}{L}(u^2 - 2gL + 3gLcos180) > 0$

$(u^2 - 2gL - 3gL) > 0$

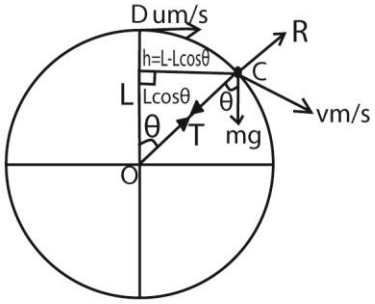
$u^2 > 5gL$

$u^2 > 5\ x\ 9.8\ x\ 2$

$u^2 > 98$

Particle in first quadrant

Consider a body of mass m rolled from the top of a sphere of radius L. The normal reaction R acts outwards



At equilibrium at C

mgcosθ – R = $\frac{mv^2}{L}$

R = mgcosθ - $\frac{mv^2}{L}$ .....(i)

$v^2 = u^2 + 2ah$

a = g, s = h = L –Lcosθ

$v^2 = u^2 + 2g(L – Lcos\theta)$ ..... (ii)

Putting (ii) into (i)

R = mgcosθ - $\frac{m[u^2 + 2g(L - Lcos\theta)]}{L}$

R = $\frac{mgLcos\theta - mu^2 - 2mgL + 2mgLcos\theta}{L}$

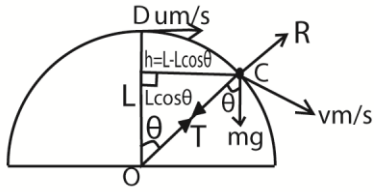
$R = \frac{m}{L}(3gLcos\theta - 2gL - u^2)$.....(iii)

Example 6.

A particle of mass 5kg is slightly disturbed from rest on the top of a smooth hemisphere, radius 4m and centre O, resting with its plane face on horizontal ground.

(a) Show that the particle leaves the surface of hemisphere at point C, where the angle between the radius OC and the upward vertical is $cos^{-1}\frac{2}{3}$

(b) Find the speed at C



At equilibrium at C

mgcosθ – R = $\frac{mv^2}{L}$

R = mgcosθ - $\frac{mv^2}{L}$ …….(i)

$v^2 = u^2 + 2ah$

u = 0, a = g, s = h = L –Lcosθ

v^2 = 2g(L –Lcosθ) ……. (ii)

Putting (ii) into (i)

R = mgcosθ - $\frac{m[2g(L-Lcos\theta)]}{L}$

$R = \frac{3mgLcos\theta - 2mgL}{L}$

When the particle leaves the surface of the sphere R = 0

$0 = \frac{3mgLcos\theta - 2mgL}{L}$

3cosθ =2

θ =$cos^{-1}\frac{2}{3}$

From eqn. (ii)

v^2 = 2g(L –Lcosθ) = 2g(L –L x $\frac{2}{3}$)

$v^2 = \frac{2}{3}gL = \frac{2}{3}\ x\ 9.8\ x\ 4$ = 26.1333

v = 5.1121ms^{-1}

Thank you

Dr. Bbosa Science

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Connected particles

Simple connections

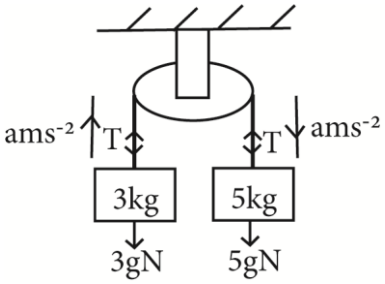
When two particles are connected by a light inextensible string passing over a smooth pulley and allowed to move freely, then as long as the string is tout, the following must be observed.

- acceleration of the particles is the same
- tension in the uninterrupted string is constant
- tensions in interrupted strings are different.

Example 1

Two particles of masses 5kg and 3kg are connected by a light inextensible string passing over a smooth fixed pulley. Find

(a) acceleration of the particle
(b) the tension in the string



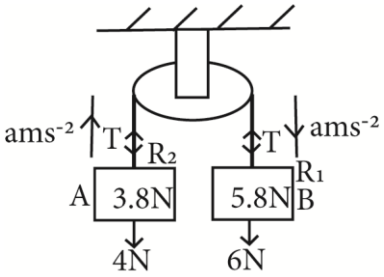
For 5kg mass: 5g – T = 5a …. (i)
For 3kg mass: T – 3g = 3a …. (ii)

(i) and (ii)
2g = 8a
a = $\frac{2 \times 9.8}{8}$ = 2.45ms^{-2}
(ii) tension in the string
T -3g = 3a
T = 3 x 2.45 + 3 x 9.8 = 36.78N
(iii) Force on the pulley
R = 2T = 3 x 36.78 = 73.56N

Example 2

An inextensible string attached to two scale A and B each of weight 20g passes over a smooth fixed pulley. Particles of weight 3.8N and 5.8N are placed in pans A and B respectively. If the system s released from rest (take g = 10ms^{-2}). Find the

(a) Tension in the string
(b) Reaction of the scale pan holding the 3.8N weight



Weight of the scale pan = $\frac{20}{1000} x\ 10 = 0.2N$

Total weight of A = 3.8 + 0.2 = 4N

Total weight of B = 5.8 + 0.2 = 6N

For 6N: 6 – T = 0.6a …….. (i)

For 4N: T – 4 = 0.4a …….. (i)

Adding (i) and (ii)

a = 2ms^{-2}

T = 4 + 0.4 x 2 = 4.8N

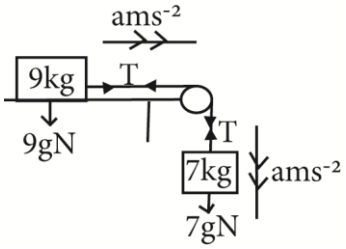
For scale pan A R_2 – 3.8 = 0.38a

R_2 = 3.8 + 2 x 0.38 x 2 = 4.56N

Example 3

A mass of 9kg resting on a smooth horizontal table is connected by a light inextensible string passing over a smooth pulley at the edge of the table to the pulley is a 7kg mass hanging freely 1.5m above the ground. Find

(a) common acceleration
(b) tension in the string
(c) force on the pulley when the system is allowed to move freely
(d) time taken for the 7kg mass to hit the ground



F = ma

For 7kg mass: 7g – T = 7a

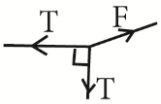
For 9kg mass: T = 9a

(i) + (ii): 7g = 16a

a = $\frac{7\ x\ 9.8}{16} = 4.29ms^{-2}$

(b) Tension: T = 9a = 9 x 4.29 = 38.61N

(c) The force on the pulley, F:



F =$\sqrt{T^2 + T^2} = T\sqrt{2}$ = 38.61$\sqrt{2}$ = 54.603N

(d) s = ut + $\frac{1}{2}at^2$

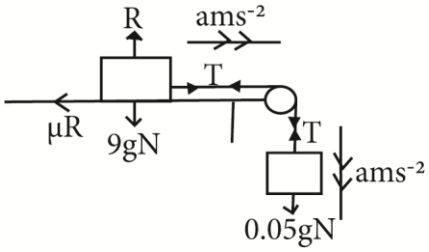
1.5 = 0 x t + $\frac{1}{2} x\ 4.29\ x\ t^2$

t = 0.84s

Example 4

A mass of 90g resting on a rough horizontal table is connected by a light inextensible string passing over a smooth pulley at the edge of the table attached to a 50g mass hanging freely. The coefficient of friction between the 90g mass and the table is $\frac{1}{3}$ and the system is released from rest, find

(a) common acceleration (b) the tension in the string



For 50g mass: 0.05g – T = 0.05a ….(i)

For 90g mass: T – μR = 0.09a

T - $\frac{1}{3}$ x 0.09 x 9.8 = 0.09a …. (ii)

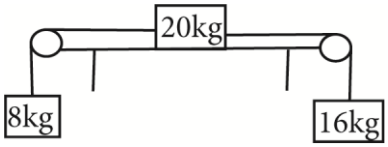
(i) + (ii): 0.05g - $\frac{1}{3}$ x 0.09 x 9.8 = 0.14a

a = $\frac{0.02g}{0.14}$ = 1.4ms^{-2}

(b) 0.05g – T = 0.05a

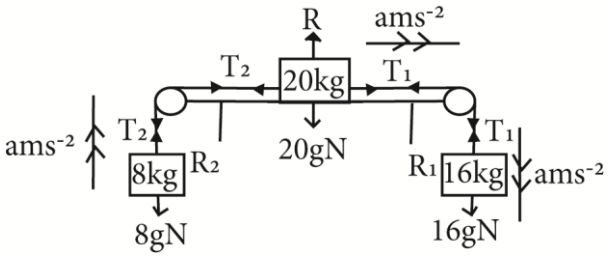
T = 0.05 x 9.8 – 0.05 x 1.4 = 0.42N

Example 5



The figure shows a block of mass 20kg resting on a smooth horizontal table. It is connected by light inextensible string which pass over fixed pulleys at the edges of the table to two loads of masses 8kg and 16kg which hang freely vertically. When the system is released freely calculate:

(a) Acceleration of 16kg mass
(b) tension in the string
(c) reaction on each pulley



For 16kg mass: 16g – T_1 = 16a …… (i)

For 20kg mass: T_1 – T_2 = 20a ……. (ii)

For 8kg mass: T_2 – 8g = 8a …… (iii)

(i) + (ii) + (iii): 8g = 44a

a = $\frac{8\ x\ 9.8}{44} = 1.782ms^{-2}$

(b) Tension in the string

16g – T_1 = 16a

T_1 = 16 x 9.8 – 16 x 1.782 = 128.288N

T_2 – 8g = 8a

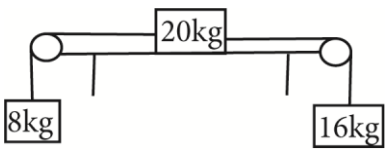
T_2 = 8 x 9.8 + 8 x 1.782 = 92.656N

(c) Reaction on each pulley

R_1 = $\sqrt{{T_1}^2 + {T_1}^2}$ = $T_1\sqrt{2}$ = $\sqrt{2}$ x 128.288 =181.427N

R_2 = $\sqrt{{T_2}^2 + {T_2}^2}$ = $T_2\sqrt{2}$ = $\sqrt{2}$ x 92.626 = 131 N

Example 6

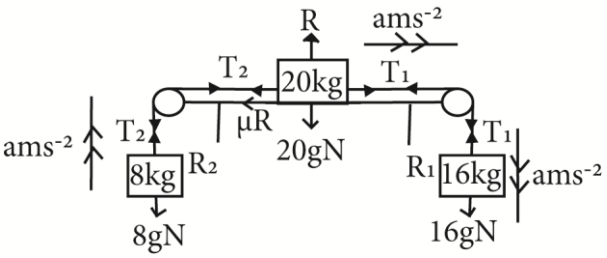


The figure shows a block of mass 20kg resting on a rough horizontal table of coefficient of friction 0.21. It is connected by light inextensible string which pass over fixed pulleys at the edges of the

table to two loads of masses 8kg and 16kg which hang freely vertically. When the system is released freely calculate:

(a) acceleration of the 16kg mass
(b) Tension in each string
(c) reaction on each pulley

Solution



For 16kg mass: $16g - T_1 = 16a$ …… (i)

For 20kg mass: $T_1 - T_2 - 20g\mu = 20a$ ……. (ii)

For 8kg mass: $T_2 - 8g = 8a$ …… (iii)

(i) + (ii) + (iii): $8g - 20g\mu = 44a$

$a = \frac{8\ x\ 9.8 - 20\ x\ 9.8\ x\ 0.21}{44} = 0.846 ms^{-2}$

(b) Tension in the string

$16g - T_1 = 16a$

$T_1 = 16 \times 9.8 - 16 \times 0.846 = 143.264N$

$T_2 - 8g = 8a$

$T_2 = 8 \times 9.8 + 8 \times 0.846 = 85.168N$

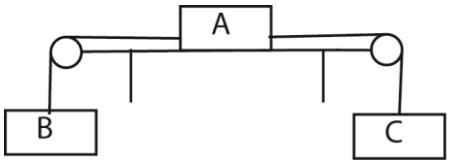
(c) Reaction on each pulley

$R_1 = \sqrt{{T_1}^2 + {T_1}^2} = T_1\sqrt{2} = 2 \times 128.291 = 202.606N$

$R_2 = \sqrt{{T_2}^2 + {T_2}^2} = T_2\sqrt{2} = \sqrt{2} \times 85.168 = 120.446N$

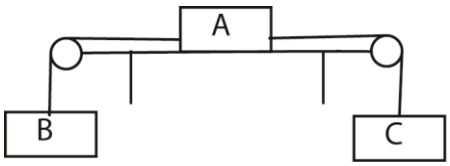
Revision exercise 1

1. Two particles of masses 7kg and 3kg are connected by a light inelastic string passing over a smooth fixed pulley. Find
   (a) acceleration of the particles [3.92ms^{-2}]
   (b) the tension in the string [41.16N]
   (c) the force on the pulley [82.32N]
2. Two particles of masses 6kg and 2kg are connected by a light inextensible string passing over a smooth fixed pulley. With the masses hanging vertically, the system is released from rest. Find
   (a) acceleration of the particles [4.9ms^{-2}]
   (b) the tension in the string [29.4N]
   (c) distance moved by the 6kg mass in the first 2s of motion [9.8m]
3. A man of mass 70kg and a bucket of bricks of mass 100kg are tied to the opposite ends of a rope which passes over a frictionless pulley so that they hang vertically downwards.
   (a) what is the tension in the section of the rope supporting the man [807.06N]
   (b) what is the acceleration of the bucket [1.73ms^{-2}]
4. Two particles of masses 200g and 300g are connected to a light inelastic string passing over a smooth pulley; when released freely find
   (i) common acceleration [1.96ms-2]
   (ii) the tension in the string [2.352N]
   (iii) the force on the pulley [4.704N]

5. The diagram below shows a particles of mass 8kg connected to a light scale pan by a light inextensible string which passes over a smooth fixed pulley. The scale pan holds two blocks A and B of mass 3kg and 4kg, with B resting on top of A. If the system is released from rest find
   (a) acceleration of the system [0.653ms^{-2}]
   (b) the reaction between A and B[41.813N]
6. A mass of 5kg is placed on a smooth horizontal table and connected by a light string to a 3kg mass passing over a smooth pulley at the edge of the table and hanging freely. If the system is allowed to move, calculate
   (a) the common acceleration of the masses [3.675ms^{-2}]
   (b) the tension in the string [18.375N]
   (c) the force acting on the pulley[26N]
7. A mass of 3kg on a smooth horizontal table is attached by a light inextensible sting passing over a smooth pulley at the edge of the table, to another mass of 2kg hanging freely 2.1m above the ground; find
   (a) common acceleration[3.92ms^{-2}]
   (b) the tension in the string [11.76N]
   (c) The force on the pulley in the system if it's allowed to move freely. [16.63N]
   (d) the velocity with which the 2kg mass hits the ground [4.06ms^{-1}]
8. A mass of 5kg rests on a rough horizontal table and is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a 3kg mass hanging freely. the coefficient of friction between the 5kg mass and the table is 0.25 and the system is released from rest find
   (a) common acceleration[2.144ms^{-2}]
   (b) tension in the string [22.97N]
9. A mass of 11kg rests on a rough horizontal table and is connected by a light inextensible string passing over a smooth pulley at the edge of the table to 500g mass hanging freely. The coefficient of friction between the 1kg mass and the table is 0.1 and the system is released from rest find
   (a) common acceleration [2.61ms^{-2}]
   (b) the tension in the string [3.593N]
10. The objects of mass 3kg and 5kg are attached to ends of a cord which passes over a fixed frictionless pulley placed at 4.5m above the floor. The objects are held at rest with 3kg mass touching the floor and the 5kg mass at 4m above the floor and then release, what is
    (a) the acceleration of the system[2.45ms^{-2}]
    (b) tension in the chord[36.75N]
    (c) the time that will elapse before the 5kg object hits the floor [1.81s]
11. 

    The diagram shows a particle A of mass 2kg resting on a rough horizontal table of coefficient of friction 0.5. It is attached to the particle B of mass 5kg and C of mass 3kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest find the
    (a) common acceleration [0.98ms^{-2}]
    (b) the tension of each string [12.37N, 44.15N]

12.

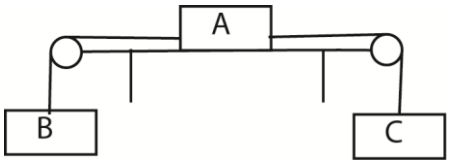


The diagram shows a particle A of mass 3kg resting on a rough horizontal table of coefficient of friction 0.5. It is attached to the particle B of mass 4kg and C of mass 6kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest find the

(a) common acceleration [0.75ms^{-2}]

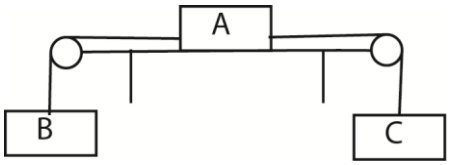
(b) the tension of each string [54.277N, 31.662N]

13.



The diagram shows a particle A of mass 5kg resting on a rough horizontal table of coefficient of friction 0.5. It is attached to the particle B of mass 3kg and C of mass 2kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest body B descend with an acceleration of 0.28ms-2, find the coefficient of friction between the body A and the surface of the table. [0.143]

14.



The diagram shows a particle A of mass 10kg resting on a smooth horizontal table. It is attached to the particle B of mass 4kg and C of mass 7kgby light inextensible strings hanging over smooth pulleys. If the system is released from rest find the

(c) common acceleration [1.4ms^{-2}]

(d) the tension of each string [44.8N, 58.8N]

Connected particles on inclined planes

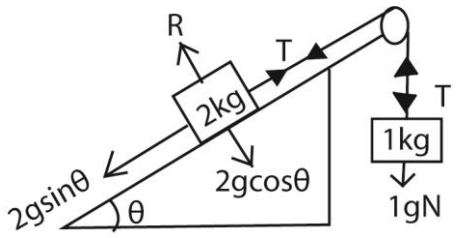
Example 7

A mass of 2kg lies on a smooth plane of inclination 1 in3. One end of a light inextensible sting is attached to this mass and the string passes up the line of greatest slope over a smooth pulley fixed at the top of the plane is freely suspended mass of 1kg at its end. If the system is released from rest , find the

(i) acceleration of the masses

(ii) tension in the string

(iii) distance each particle travels in the first 2s.



sinθ =$\frac{1}{3}$ F =ma

For 2kg mass: T – 2gsinθ = 2a….. (i)

For 1kg mass: 1g – T = 1a ……. (ii)

(ii) + (i): = 1g – 2gsinθ = 3a

a = $\frac{9.8-2\ x\ 9.8\ x\frac{1}{3}}{3}$ = 1.089ms^{-2}

(ii) Tension: 1g – T = 1a

T = 9.8 – 1.089 = 8.71N

(iii) s = ut + $\frac{1}{2}at^2$

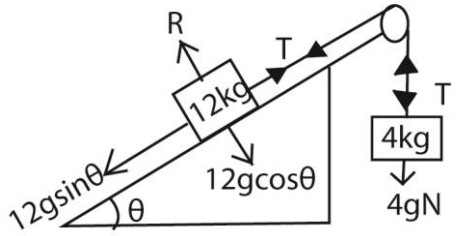
s = 0 x 2 + $\frac{1}{2}x\ 1.089\ x\ 2^2 = 2.178m$

Example 8

A mass of 12kg lies over a smooth incline plane 6m long and 1m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope over a smooth pulley fixed at the top of the plane to freely suspended mass of 4kg at its other end. If the system is released from rest, find the

(a) acceleration of the system
(b) velocity with which the 4kg mass hits the ground
(c) time the 4kg mass takes to hit the ground.

Solution



sinθ =$\frac{1}{6}$ F =ma

For 12kg mass: T – 12gsinθ = 2a….. (i)

For 4kg mass: 4g – T = 4a ……. (ii)

(ii) + (i): = 4g – 12gsinθ = 16a

a = $\frac{4\ x9.8-12\ x\ 9.8\ x\frac{1}{6}}{16}$ = 1.225ms^{-2}

(ii) Tension: 4g – T = 4a

T = 4 x 9.8 – 4 x 1.225 = 34.3N

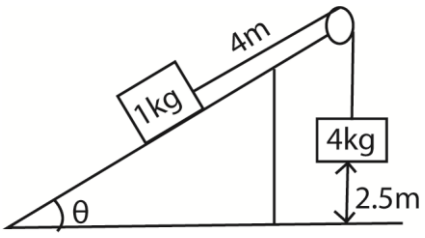
(iii) s = ut + $\frac{1}{2}at^2$

1 = 0 x t + $\frac{1}{2}x\ 1.225\ x\ t^2$

t = 1.28s

Example 9

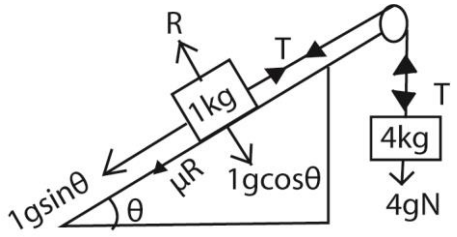
A mass of 1kg lies on a rough plane with coefficient of friction 0.25. One end of a light inextensible string is attached to 1kgmass and passes up the line of greatest slope over a smooth fixed pulley at the top of the plane and the other end of a string is tied to a mass of 4kg hanging freely.



The plane makes an angle θ with the horizontal where sinθ = $\frac{3}{5}$. When the system is released from rest, find:

(i) the acceleration of the system
(ii) tension in the string
(iii) velocity with which the 4kg mass hits the floor
(iv) velocity with which the 1kg mass hits the pulley

Solution



sinθ = $\frac{3}{5}$; cosθ = $\frac{4}{5}$ F =ma

For 12kg mass: T – 1gsinθ- 0.25R = 1a….. (i)

For 4kg mass: 4g – T = 4a ……. (ii)

(ii) + (i): = 4g – 1gsinθ - 0.25R = 5a

a = $\frac{4\ x9.8-1\ x\ 9.8\ x\frac{3}{5}-0.25\ x\ 1\ x\ 9.8\ \ x\frac{4}{5}}{5}$ =6.272ms^{-2}

(ii) Tension: 4g – T = 4a

T = 4 x 9.8 – 4 x 6.272 = 14 .112N

(iii) $v^2 = u^2 + 2as$ but u = 0

v = $\sqrt{2\ x\ 6.272\ x\ 2.5}$ = 5.6ms^{-1}

(iv) When a 4kg mass hits the floor, the 1kg mass has still to move 4 – 2.5 = 1.5m before hitting the pulley It will experience a retarding force due to gravity and friction

F = 1a = 1gsinθ + 0.25R

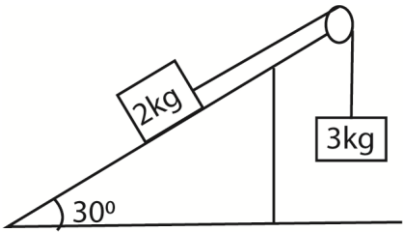
=$(1\ x\ 9.8\ x\ \frac{3}{5} + 0.25\ x\ 1\ x\ 9.8\ x\frac{4}{5})$

a = - 7.84ms^{-2}

$v^2 = u^2 + 2as$

v = $\sqrt{5.6^2 - 2\ x\ 7.84\ x\ 1.5}$ =2.8ms^{-1}

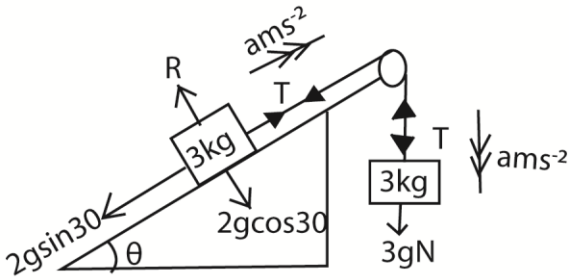
Example 10



A particle of mass 2kg on a rough plane inclined at 300 to the horizontal is attached by means of light inextensible string passing over a smooth pulley at the top edge of the plane to a particle of mass 3kg which hangs freely. If the system is released from rest with above parts of the strings taut, the 3kg mass travels a distance of 0.75m before attains a speed of 2.25ms^{-1}. Calculate

(a) acceleration
(b) coefficient of friction between the plane and 2kgmass
(c) reaction of the pulley on the string

Solution



(i) $v^2 = u^2 + 2as$

$a = \frac{2.25^2 - 0^2}{2\ x\ 0.75} = 3.375ms^{-2}$

(ii) F = ma

For 2kg mass: $T - 2gsin\theta - \mu R = 2x\ a$ ..... (i)

For 4kg mass: $3g - T = 3\ a$ ..... (ii)

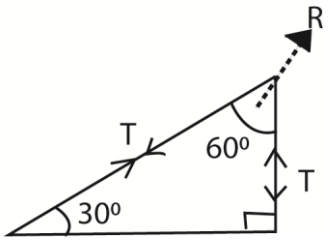
(ii) + (i): $3g - 2gsin\theta - \mu(2gcos\theta) = 5a$

$\mu = \frac{(3\ x\ 9.8) - (2\ x\ 9.8 sin30 + 5\ x\ 3.375)}{2\ x\ 9.8\ x\ cos30} = 0.161$

(iii)

Tension: $3g - T = 3a$

$T = 3 \times 9.8 - 3 \times 3.375 = 19.275N$



Using parallelogram law of force

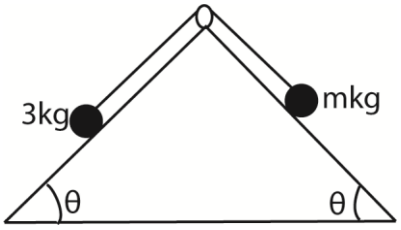
$R^2 = T^2 + T^2 + 2 \times T \cos 60 = 3T^2$

$R = 19.275\sqrt{3} = 33.4N$

Double inclined plane

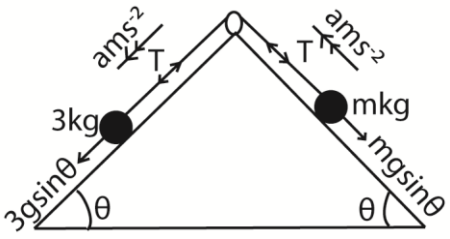
Example 11

The diagram below shows two smooth fixed slopes each inclined at an angle θ to the horizontal where sinθ = 0.6. Two particles of mass 3kg and mkg, where m< 3kg are connected by a light inextensible string passing over a smooth fixed pulley.



The particles are released from rest with a string taut. After travelling a distance of 1.08m, the speed of the particle is 1.8ms-1. Calculate

(i) acceleration
(ii) tension in the string
(iii) value of m



(i) $v^2 = u^2 + 2as$

$1.8^2 = 0^2 + 2 \times a \times 1.08$

$a = 1.5ms^{-2}$

(ii) F = ma

For 3kg mass: $3gsin\theta - T = 3a$

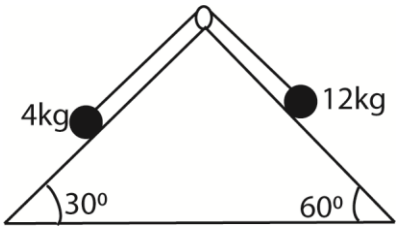
$T = 3 \times 9.8x\ 0.6 - 3 \times 1.5 = 13.14N$

(iii) For mkg mass; $T - mgsin\theta = ma$

$13.14 = m(9.8 + 1.5)$; $m = 1.78kg$

Example 12

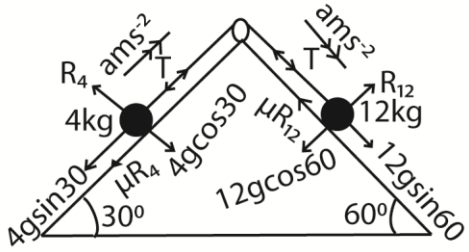
Two rough planes inclined at 300 and 600 to the horizontal and of the same height are placed back to back. Masses of 4kg and 12 kg are placed on the faces and connected by a light string passing over smooth pulley on the top of the planes.



If the coefficient of friction is 0.5 on both faces, find

(a) acceleration
(b) Tension in the strings

Solution



For 4kg mass: T -4gsin30 – 0.5 x 4gcos30 = 4a ……. (i)

For 12kg mass: 12gsin60 –T – 0.5 x 12gcos 60 = 12a ….. (ii)

(i) + (ii)

12gsin60 – 4gsin30 - 0.5(4gcos30 + 12gcos 60) = 16a

a = 2.25ms^{-2}

(b) For 4kgmass

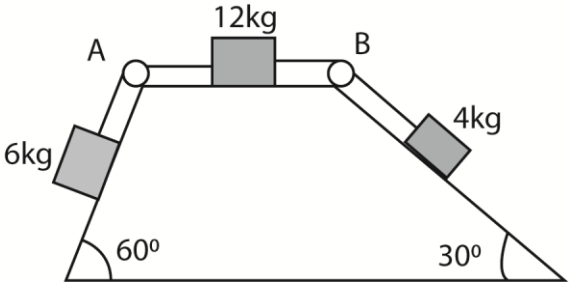
T -4gsin30 – 0.5 x 4gcos30 = 4a

T = 4gsin30 + 0.5 x 4gcos30 + 4x2.25

T = 45.54

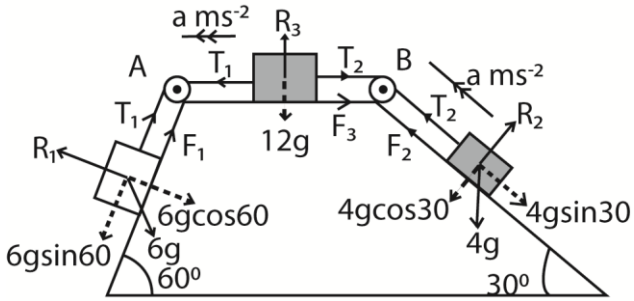
Example 13

1. The diagram below shows a 12kg mass on a horizontal rough plane. The 6kg and 4kg masses are on rough planes inclined at angles 600 and 300 respectively. The masses are connected to each other by light inextensible strings over light smooth pulleys A and B.



The planes are equally rough with coefficient of friction$\frac{1}{12}$. If the system is released from rest find the;

(a) Acceleration of the system (08marks)



For 6kg mass

6gsin60-($T_1 + \frac{1}{12}\ x\ 6gcos60) = 6a$

6gsin60 – T_1 - $\frac{1}{2}cos60 = 6a$ …… (i)

For 4kg mass

$T_2 – (\frac{1}{12}\ x\ 4gcos30 + 4gsin30) = 4a$

T_2 - $\frac{1}{3}gcos30 - 4gsin30$ = 4a …….. (ii)

For 12kg mass

$T_1 – (T_2 + \frac{1}{12}\ R_3)$ = 12a

$T_1 – (T_2 + \frac{1}{12}\ x\ 12g)$ = 12a

$T_1 – T_2$ - g = 12a…….. (iii)

Eqn. (i) + Eqn. (ii) + Eqn. (iii)

6gsin60 - $\frac{1}{2}gcos60 - \frac{1}{3}gcos30 - 4gsin\ 30 - g = 22a$

16.24327742 = 22a

a = $\frac{16.24327742}{22} = 0.73833ms^{-2}$

(b) Tensions in the strings. (04marks)

From equation (i)

$T_1 = 6gsin60 - \frac{1}{2}cos60 - 6a$

= $6gsin60 - \frac{1}{2}cos60 - 6\ x\ 0.73833$

= 44.0423N

From eqn. (ii)

$T_2 = \frac{1}{3}gcos30 + 4gsin30 + 4a$

= $\frac{1}{3}gcos30 + 4gsin30 + 4\ x\ 0.73833$

= 25.3823N

Revision exercise 2

1. A mass of 2kg lies on a smooth inclined plane 9m long and 3m high. One end of a light inextensible string is attached to this mass and the string passes up the line of greatest slope

over a smooth pulley fixed at the top of the plane is freely suspended mass of 1kg at its other end. If the system is released from rest, find

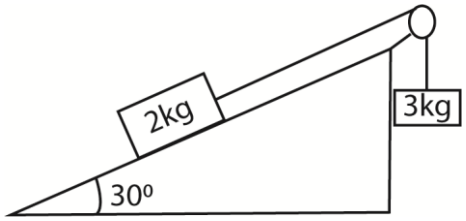
(i) acceleration of the system[1.089ms^{-2}]

(ii) tension in the string. [8.711N]

(iii) velocity with which the 1kg mass will hit the ground[2.556ms^{-1}]

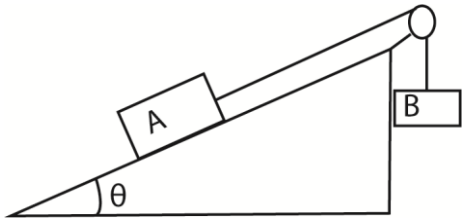
(iv) time the 1kg mass will hit the ground[2.347s]

2. A mass of 15kg lies on a smooth plane of inclination in 49. One end of a light inextensible string is attached to this mass and the string passes up a line of greatest slope, over a smooth pulley fixed at the top of the plane is freely suspended mass of 10kg at its other end. If the system is released from rest, find the acceleration of the masses and the distance each travel in the first 2s. [3.8ms^{-2}, 7.6m]
3. A mass of 2kg lies on a rough plane which is inclined at 300 to the horizontal. One end of a light inextensible string is attached to this mass and the string passes up a line of greatest slope, over a smooth pulley fixed at the top of the plane is freely suspended mass of 5kg at its other end. The system is released from rest as the 2kg mass accelerates up the slope, it experiences a constant resistance to motion of 14N down the slope due to friction. Find the tension of the string. [31N]
4. A mass of 10kg lies on a smooth plane which is inclined at θ to the horizontal. The mass is 5m from the top, measured along the plane. One end of a light inextensible string is attached to this mass and the string passes up a line of greatest slope, over a smooth pulley fixed at the top of the plane is freely suspended mass of 15kg at its other end. The 15kg mass is 4m above the floor. The system is released from rest and the string first goes slack $1\frac{3}{7}s$ latter. Find the value of θ. [30^0]
5. One of two identical masses lies on a smooth plane, which is inclines at $\sin^{-1}\left(\frac{1}{4}\right)$ to the horizontal and is 2m from the top. A light inextensible string attached to this mass passes along the line of greatest slope over a smooth pulley fixed at the top of the incline, the other end carries the other mass hanging freely 1m above the floor. If the system is released from rest, find the time taken for the hanging mass to reach the floor. [0.663s]
6. A particle of mass 2kg on a smooth plane inclined at 300 to the horizontal is attached by means of a light inextensible string passing over a smooth pulley at the edge of the plane to a particle of mass 4kg which hangs freely.



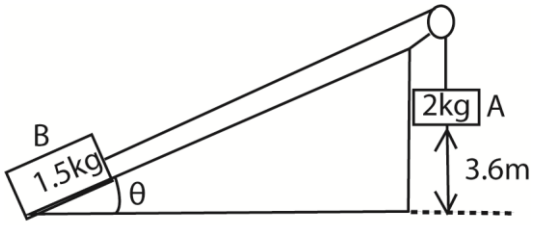
If the system is released from rest with above parts of the string taut, find the speed acquired by the particles when both have moved a distance of 1m [2.8ms^{-1}}

7. A body A of mass 13kg lying on a rough inclined plane, coefficient of friction, μ. From A, a light inextensible string passes up the line of greatest slope and over a smooth fixed pulley to a body B of mass mkg hanging freely, the plane makes an angle θ with the horizontal where sinθ = $\frac{5}{13}$.



When m = 1kgand the system is released from rest, B has upward acceleration of a ms^{-2}. When m = 11kg and the system released from rest, B has downward acceleration of ams^{-2}. Find a and μ. [$1.96ms^{-2}$, 0.1]

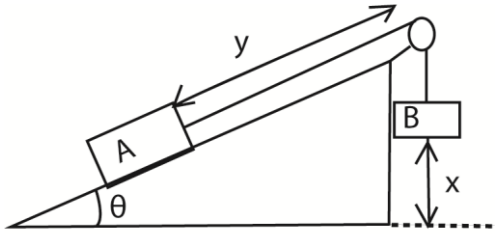
8. A particle A of mass 2kg and B of mass 1.5kg are connected by light inextensible string passing over a smooth pulley. The system is released from rest with A at height of 3.6m above the horizontal ground and B at the foot of a smooth slope inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{6}$. Take g = $10ms^{-2}$.



Calculate

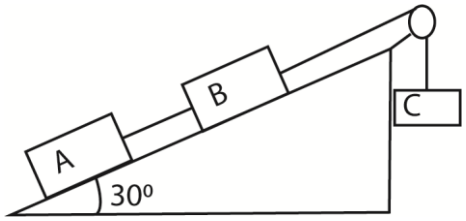
(i) the magnitude of the acceleration of particles [$6ms^{-1}$]
(ii) the speed with which A reaches the ground[$5ms^{-2}$]
(iii) the distance B moves up the slope before coming to instantaneous rest. [14.4m]

9. A mass A of 4kg and a mass B of 3kg are connected by a light inextensible string passing over a smooth pulley. The system is released from rest and mass accelerates up along a smooth slope inclined at an angle θ to horizontal where $\theta = 30^0$.



If y = 3m and x = 2.8m, calculate the velocity with which A hits the pulley [$2.42ms^{-1}$]

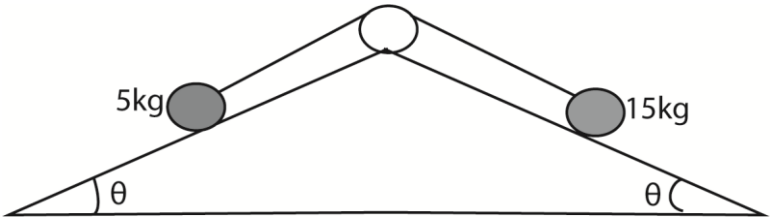
10. The diagram below shows particles A, B and C of masses 10kg, 8kg and 2kg respectively connected by a light inextensible strings. The string connecting B and C passes over a smooth light pulley fixed at the top of the plane.



If the coefficient of friction between the plane and particles A and B are 0.22 and 0.25 respectively, calculate

(i) acceleration of the system [$1.6477ms^{-2}$]
(ii) tension in the strings[22.89N, 13.851N]

11. The diagram below shows two smooth fixed slopes each inclined at an angle θ to the horizontal where $\sin\theta = \frac{3}{5}$. Two particles of mass 5kg and 15kg are connected by a light inextensible string over a smooth fixed pulley.

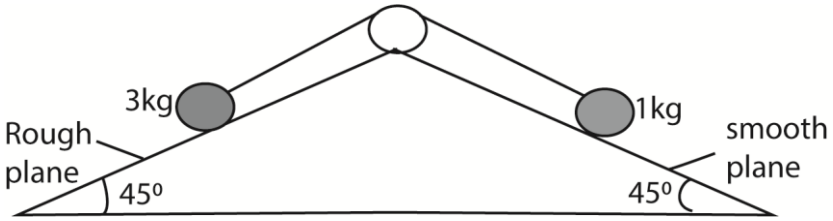


The particles are releases from rest with a string taut calculate

(i) Acceleration of the particles

(ii) Tension in the string

12. The diagram below shows two smooth plane and a rough plane both inclined at 450 to the horizontal. Two particles of mass of mass 1kg and 3kg are connected by light inextensible string passing over a smooth fixed pulley.



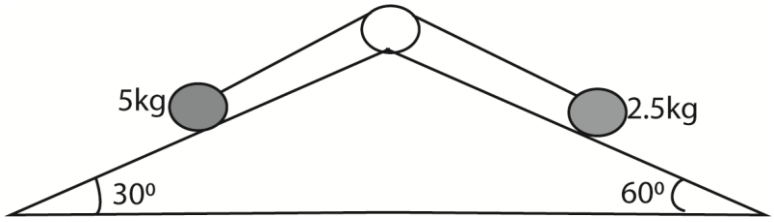
The particle are released from rest with a string taut. Calculate

(i) acceleration of the particle [1.4ms^{-2}]

(ii) tension in the string [.48N

(iii) coefficient of friction [0.4]

13. Two equally rough planes inclined at 30^0 and 60^0 to the horizontal and of the same height are placed back to back. Masses of 5kg and 2.5kg are placed on the faces and connected by a light string passing over a smooth pulley at the top of the planes.

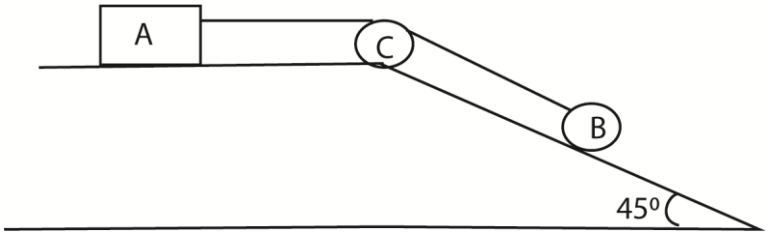


If the string is taut and 5kg is just about to slip downwards find the

(i) coefficient of friction[0.06]

(ii) tension in the string [21.9538N]

14. In the diagram, particle A and particle B are masses of 10kg and 8kg respectively and rest on planes as shown below. They are connected by a light inextensible string passing over a smooth pulley C.

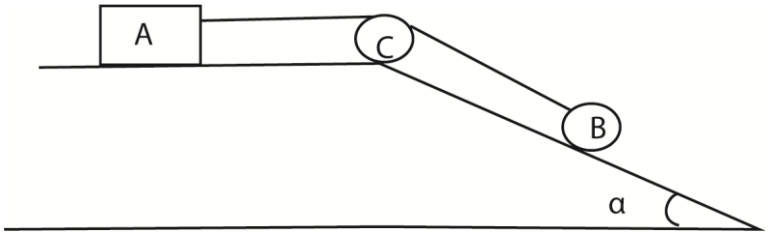


Find the acceleration in the system and the tension in the string if

(i) the particles are in contact with smooth planes [3.08ms^{-2}, 30.N]

(ii) the particles are in contact with rough planes with coefficient of friction 0.25. [0.95ms^{-2}, 33.98N]

15. In the diagram particles A and B are of masses mkg and 5mkg respectively and rest on the planes as shown below. They are connected by a light inextensible string passing over a smooth fixed pulley at C

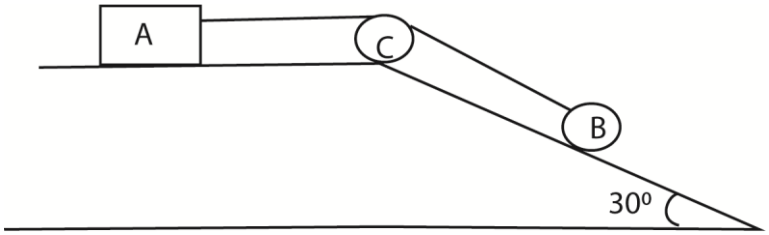


Find the acceleration of the system and the tension in the string if sinα = $\frac{4}{5}$ when;

(i) the particles are in contact with smooth plane[6.533ms-2, 6.533N]

(ii) the particles are in contact with rough plane of coefficient of friction $\frac{1}{3}$. [4.356ms^{-2}, 7.622N]

16. In the diagram particles A and B of masses 2.4kg and 3.6kg respectively. A rests on a rough horizontal plane (coefficient of friction 0.5), it is connected by a light inextensible string passing over a smooth pulley C to particle B resting on smooth plane inclined at 30^0 to the horizontal.



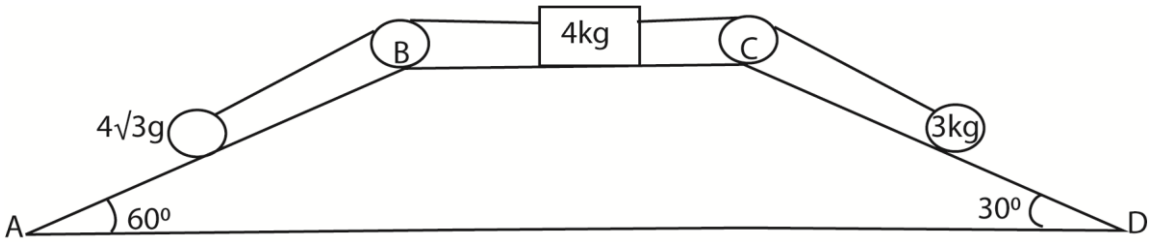
When the system is released from rest find

(i) acceleration of the system and tension in the string [0.98ms^{-2}, 14.112N]

(ii) the force on the pulley C [7.3049N]

(iii) the velocity of A mass after 2 seconds[1.96ms^{-2}]

17. The diagram below shows a 4kg mass on a horizontal rough plane with coefficient of friction 0.25. The $4\sqrt{3}kg$ mass rests on a smooth plane inclined at angle 600 to the horizontal while the 3kg mass rests on a rough plane inclined at an angle 300 to the horizontal and coefficient of friction $\frac{1}{\sqrt{3}}$. the masses are connected to each other by a light inextensible strings over light smooth fixed pulleys B and C.



Find the

(i) acceleration of the system[1.407ms-2]

(ii) tension in the string [49.051N, 33.622N]

(iii) work done against frictional force when the particles each moved 0.5m [12.25J]

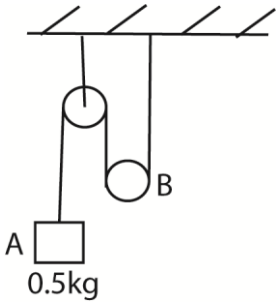
Multiple connections

- Acceleration of a particle moving between to portions of the string is equal to half the net acceleration of the particle (s) attached to the end of the string
- The tension in uninterrupted string is constant
- The tensions in interrupted strings are different.

Case I: A pulley moving between two portions of a string

Example 15

The diagram below shows particle A of mass 0.5kg attached to one end of alight inextensible string passing over a fixed pulley and under a movable light pulley B. The other end of the string is fixed as shown

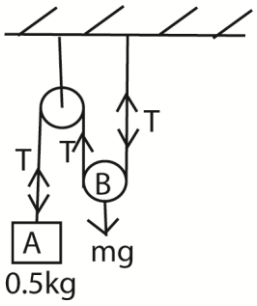


(i) What mass should be attached at B for the system to be in equilibrium
(ii) If B is 0.8kg what are the accelerations of particles A and pulley B?
(iii) find the tension in the string in (ii)

Solution

(i) Let T = tension in string

m = mass at B



For the system to be in equilibrium upward forces are equal to downward force. by resolving vertically

For mass A: T= 0.5g …… (i)

For pulley B: 2T = mg

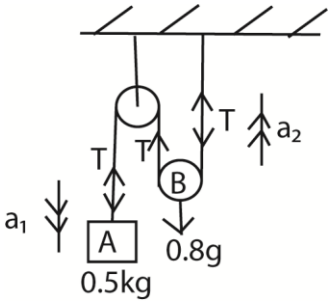
$$T = \frac{mg}{2} \text{ …… (i)}$$

equating (i) to (ii)

$\frac{mg}{2} = 0.5g$; m = 1kg

(ii) Let a_1 = acceleration of A

a_2 = acceleration of B



For mass A: $0.5g - T = 0.5a_1$…… (i)

For pulley B: $2T - 0.8g = 0.8a_2$ but $a_2 = ½ a_1$

⇨ $2T - 0.8g = 0.4a_1$

$T - 0.4g = 0.2a_1$ …….. (i)

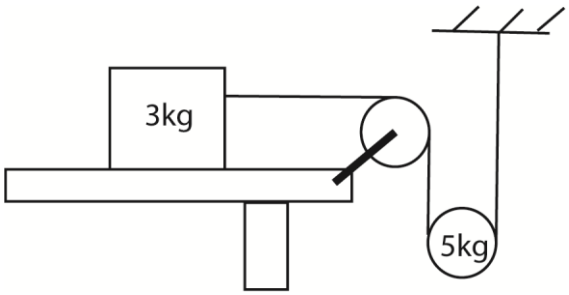
(i) + (ii) a1 = $\frac{9.9}{7}$ = 1.4ms^{-2} and a2 = 0.7 ms^{-2}

From eqn. (i)

T = 0.5 x 9.8 - 0.5 x 1.4 = 4.2N

Example 16

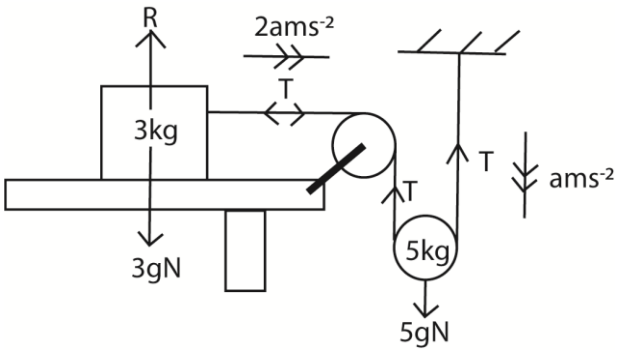
A particle of mass 3kg on a smooth horizontal table is tied to one end of the string which passes over a fixed pulley at the edge and then under a movable pulley of mass 5kg, its other end being fixed so that the string beyond the table are vertical.



Find

(i) acceleration of 3kg and 5g
(ii) Tension in the string

Solution



F = ma

For 3kg: T = 3 x 2a …… (i)

For 5kg: 5g – 2T = 5a …… (ii)

(ii) + 2 x (i)

5x 9.8 = 17a

a = $\frac{49}{17}$ = 2.8824ms^{-2}

Acceleration of 5kg: = 2.8824ms^{-2}

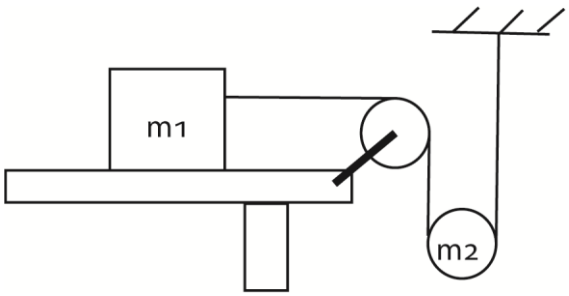
Acceleration of 3kg: = 2.8824 x 2ms^{-2}

= 5.7648ms^{-2}

T = 6a = 2.8824 x 6 = 17.2944N

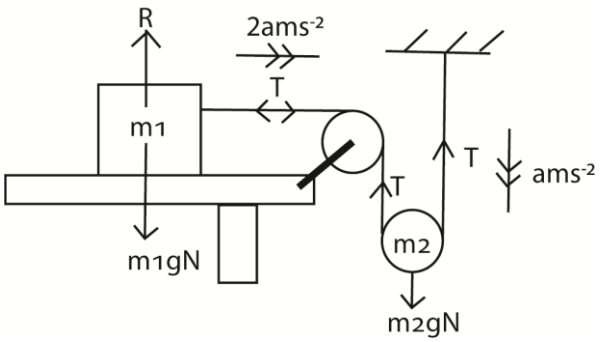
Example 17

A particle of mass m_1 on a smooth horizontal table is tied to one end of the string which passes over a fixed pulley at the edge and then under a movable pulley of mass m_2, its other end being fixed so that the parts of the string beyond the table is vertical.



Show that m_2 descends with acceleration $\frac{m_2 g}{4m_1+m_2}$

Solution



F = ma

For m_1 kg mass: T = m_1 x 2a …… (i)

For m_2 kg mass: m_2g – 2T = m_2a … (ii)

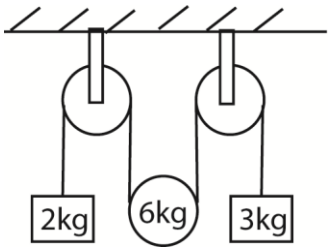
(ii) + 2 x (i)

$m_2g = 4m_1a + m_2a$

$a = \frac{m_2 g}{4m_1+m_2}$

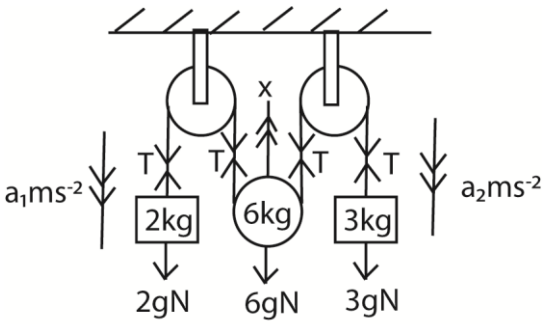
Example 18

A string has a load of mass 2kg attached at one end. The string passes over a smooth fixed pulley then under a movable pulley of mass 6kg and over another fixed pulley and has a load of mass 3kg attached to its end.



Find the acceleration of the movable pulley and the tension in the string

Solution



$x = \frac{a_1 + a_2}{2}$

For 2kg mass: 2g – T = $2a_1$ …… (i)

For 3kg mass: 3g – T = $3a_2$ …… (ii)

For 6kg mass: 2T – 6g = $6\ x\ \frac{1}{2}(a_1 + a_2)$…. (iii)

(ii) – (i): g = ($3a_2$ – $2a_1$) …… (iv)

2 x (ii) + (iii): 0 = $9a_2$ + $3a_1$ …… (v)

3 x (iv)- (v): 3g = $-9a_1$

a1 = $\frac{-g}{3}$ = -3.267ms^{-2}

From (v): 0 = $9a_2$ + $3a_1$

0 = $9a_2$ + 3(-3.267)

a_2 = 1.089ms^{-2}

Acceleration of pulley = $\frac{1}{2}(a_1 + a_2)$

= $\frac{1}{2}(-3.267 + 1.089)$

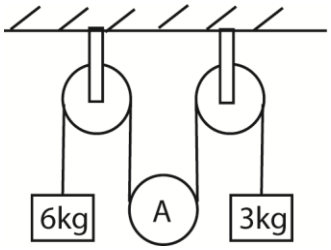
= -1.089ms^{-2}

Tension: T = 2g – $2a_1$

T = 2 x 9.8 – 2 x -3.267 = 26.134N

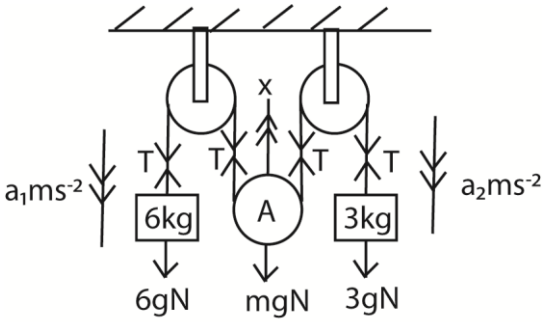
Example 19

In the pulley system below, A is a heavy pulley which is free to move



Find the mass of pulley A if it does not move upwards of downwards when the system is released from rest.

Solution



$$x = \frac{a_1 + a_2}{2}$$

For 2kg mass: $6g - T = 6a_1$ ..... (i)

For 3kg mass: $3g - T = 3a_2$ ..... (ii)

For mkg mass: $2T - mg = 0$..... (iii)

$a_1 = -a_2$

$6g - T = -6a_2$ ..... (iv)

$3g - T = 3a_2$ ..... (v)

(iv) – (v)

$3g = 9a_2$

$a_2 = \frac{-g}{3} = -3.267 ms^{-2}$

$3g - T = 3a2$

$T = 3 \times 9.8 - 3(-3.267) = 39.201$

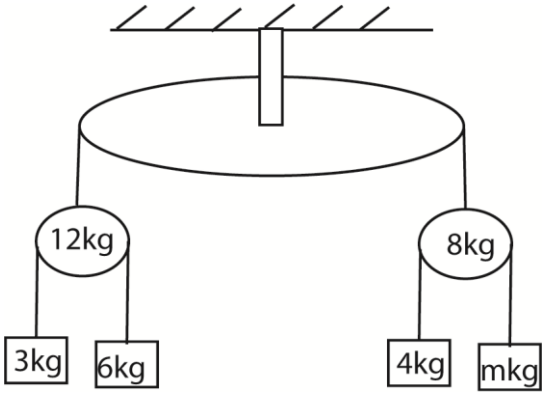
$2T - mg = 0$

$m = \frac{2\ x\ 39.201}{9.8} = 8kg$

Case 2: A pulley moving on one portion of a string

Example 20

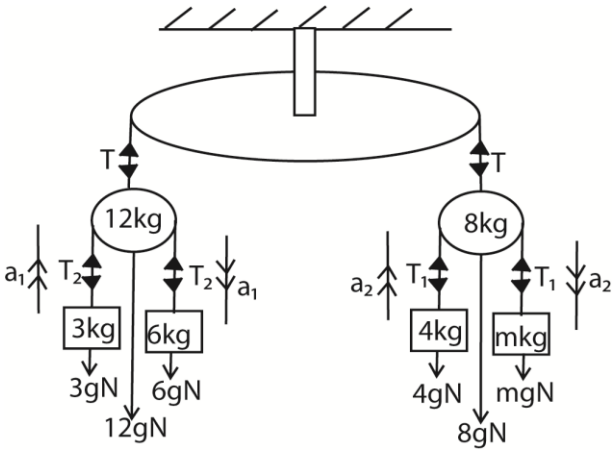
The diagram below shows two pulley to pulleys of masses 8kg and 12kg connected by a light inextensible string hanging over a fixed pulley.



The hanging portions of the strings are vertical. Given that the string of the fixed pulley remains stationary, find the

(i) tensions in the string
(ii) value of m

Solution.



For 3kg mass: $T_2 - 3g = 3a_2$ ..... (i)

For 6kg mass: $6g - T_2 = 6a_2$ ..... (ii)

For 4kg mass: $T_1 - 4g = 4a_1$ ..... (iii)

For mkg mass: $mg - T_1 = ma_1$ .....(iv)

For 8kg mass: $2T_1 + 8g = T$ ..... (v)

For 12kg mass: $2T_2 + 12g = T$ ..... (vi)

eqn. (i) + eqn. (ii): $3g = 3a_2$

$a_2 = \frac{3\ x\ 9.8}{9} = 3.2667ms^{-2}$

eqn. (i): $T_2 - 3g = 3a_2$

$T_2 = 3 \times 9.8 + 3 \times 3.2667 = 39.2001N$

eqn. (vi): $2T_2 + 12g = T$

$T = 2 \times 39.2001 + 12 \times 9.8 = 196.0002N$

eqn. (v): $2T_1 + 8g = T$

$T_1 = \frac{196.0002 - 8\ x\ 9.8}{2} = 58.8001N$

eqn. (iii): $T_1 - 4g = 4a_1$

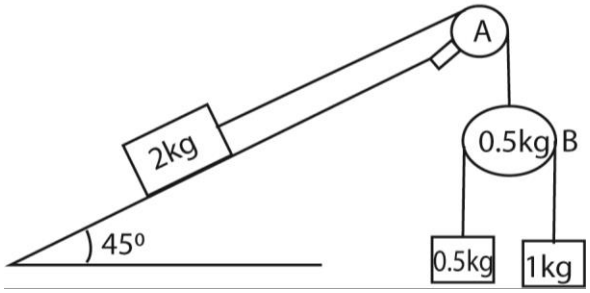
$a_1 = \frac{58.8001 - 4\ x\ 9.8}{4} = 4.9ms^{-2}$

eqn. $mg - T_1 = ma_1$

$m = \frac{58.8001}{9.8 - 4.9} = 12kg$

Example 21

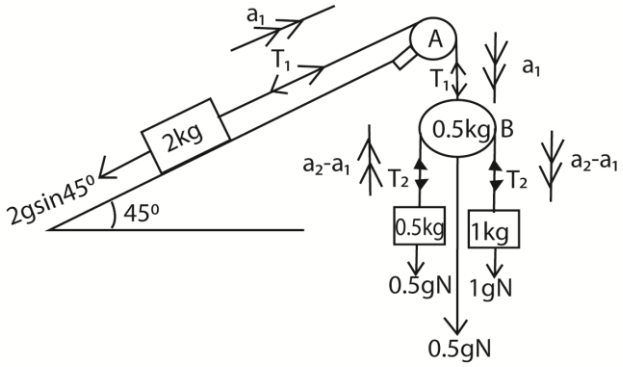
The diagram shows a particle of mass 2kg on a smooth plane inclined at 450 to the horizontal and attached by means of a light inextensible string over a smooth pulley, A at the top of the plane to pulley B of mass 0.5kg which hangs freely. Pulley B carries to particles of mass 0.5kg and 1kg on either side



Find

(a) acceleration of 2kg, 0.5kg and 1kg mass
(b) the tension in the strings

Solution



For 2kg mass: $T_1 - 2g\sin 45 = 2a_1$ ..... (i)

For 0.5kg mass: $T_2 - 0.5g = (a_2 - a_1)$ ..... (ii)

For 1kg mass: $gN - T_2 = 1(a_1 + a_2)$ ..... (iii)

For pulley B: $2T_2 + 0.5g - T_1 = 0.5a_1$ ..... (iv)

eqn. (ii) + eqn (iii): $0.5g = 1.5a2 + 0.5a1$

$9.8 = 3a_2 + a_1$ ..... (v)

eqn. (i) + eqn. (iv): $2T_2 - 2g\sin 45 + 0.5g = 2.5a_1$

$2T_2 - 8.9593 = 2.5a_1$ ..... (vi)

2 x eqn. (iii) + eqn. (vi): $10.6407 = 4.5a_1 + 2a_2$

$5.3204 = 2.5a_1 + a2$..... (vii)

eqn. (vii) – eqn. (v): $5.75a_1 = 6.1612$

$a_1 = \frac{6.1612}{5.75} = 1.0715 ms^{-2}$

from eqn. (v): $9.8 = 3a_2 + a_1$

$a_2 = \frac{9.8 - 1.0715}{3} = 2.9095 ms^{-2}$

Acceleration of 2kg mass = $1.0715 ms^{-2}$

Acceleration of 0.5kg mass = $2.9095 ms^{-2}$

Acceleration of 1kg = 2.9095 + 1.0715

$= 3.981 ms^{-2}$

From eqn. (i): $T_1 - 2g\sin 45 = 2a_1$

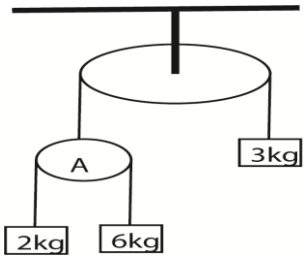
$T1 = 2 \times 1.0715 + 2 \times 9.8\sin 45 = 16.0023N$

from eqn. (iv): $2T_2 + 0.5g - T_1 = 0.5a_1$

$T_2 = \frac{0.5\ x\ 1.0715 + 16.0023 - 4.9}{2} = 5.8190N$

Example 22

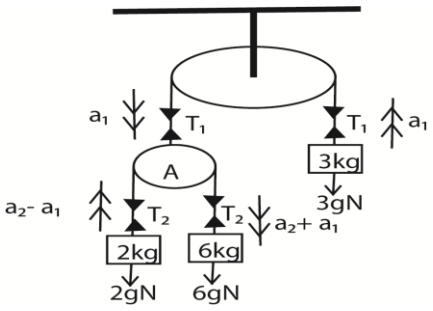
The diagram shows a fixed pulley carrying a string which has a mass of 3kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 6kg at one and a mass of 2kg at the other end.



Find

(a) acceleration of pulley A
(b) acceleration of 2kg, 6kg and 3kg masses
(c) tension in the string

solution



For 3kg mass: $T_1 - 3g = 3a_1$ ..... (i)

For 6kg mass: $6g - T_2 = 6(a_2 + a_1)$ ..... (ii)

For 2kg mass: $T_2 - 2g = 2(a_2 - a_1)$ ..... (iii)

For pulley A: $2T_2 - T_1 = 0 \times a_1$ ..... (iv)

eqn. (ii) and eqn. (iii): $4g = 8a_2 + 4a_1$ ..... (v)

eqn. (i) + eqn. (iv): $2T_2 - 3g = 3a_1$…… (vi)

2 x eqn. (iii) - eqn. (vi)

$-g = 4a_2 - 7a_1$ …….. (vii)

2eqn. (vii) – eqn. (v)

$-18a_1 = -6g$

$a_1 = \frac{6\ x\ 9.8}{18} = 3.27ms^{-2}$

$4g = 8a_2 + 4a_1$

$a_2 = \frac{9.8-3.27}{2} = 3.27ms^{-2}$

Acceleration of pulley A $=3.27ms^{-2}$

Acceleration of 2kg = $3.27ms^{-2} - 3.27ms^{-2} = 0$

Acceleration of 6kg = $3.27ms^{-2} + 3.27ms^{-2}$

$= 6.54ms^{-2}$

Acceleration of 3kg $=3.27ms^{-2}$

$T_1 - 3g = 3a_1$

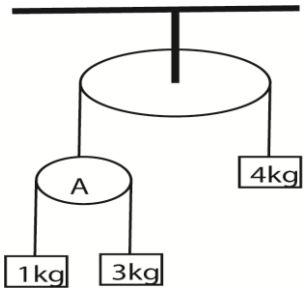
$T_1 = 3 \times 3.27 + 3 \times 9.8 = 39.21N$

$2T_2 - T_1 = 0 \times a_1$

$T_2 = \frac{39.21}{2} = 19.61N$

Example 23

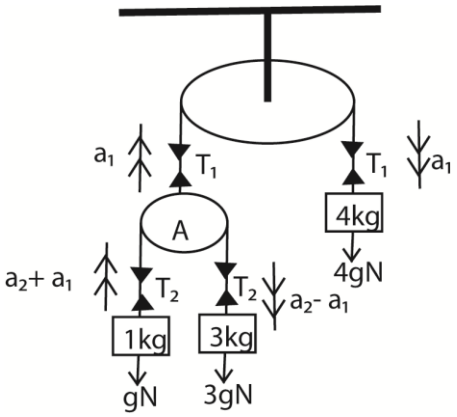
The diagram shows a fixed pulley carrying a string which has mass of 4kg attached at one end and a light pulley A at the other end. Another string passes over pulley A and carries a mass of 3kg at one end and a mass of 1kg at the other end.



Find

(a) acceleration of pulley A
(b) acceleration of 1kg, 3kg and 4kg masses
(c) tension in the string

Solution



For 4kg mass: $4g - T_1 = 4a_1$ …….. (i)

For 3kg mass: $3g - T_2 = 3(a_2 - a_1)$ …… (ii)

For 1kg mass: $T_2 - g = (a_2 + a_1)$ …… (iii)

For pulley A: $T_1 - 2T_2 = 0 \times a_1$ …….. (iv)

eqn. (ii) and eqn. (iii): $g = 2a_2 - a_1$ ….. (v)

eqn. (i) + eqn. (iv): $4g - 2T_2 = 4a1$ ……. (vi)

2 x eqn.(iii) + eqn. (v): $2g = 2a_2 + 6a_1$ ….. (vii)

eqn. (vii) – eqn. (i): $7a_1 = g$

$a_1 = \frac{9.8}{7} = 1.4ms^{-2}$

$g = 2a_2 - a_1$

$a_2 = \frac{9.8+1.4}{2} = 5.6ms^{-2}$

Acceleration of pulley A = $1.4ms^{-2}$

Acceleration of1kg mass = 5.6 + 1.4 = $7ms^{-2}$

Acceleration of 3kg mass = 5.6 – 1.4 = $4.2ms^{-2}$

Acceleration of 4kg mass = $1.4ms^{-2}$

$4g - T_1 = 4a_1$

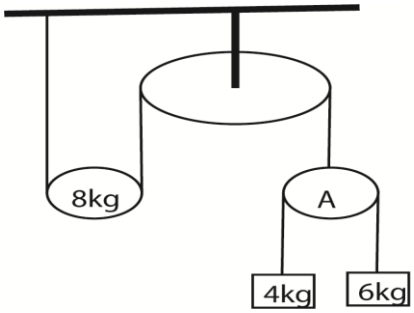
$T_1 = 4 \times 9.8 - 4 \times 1.4 = 33.6N$

$T_1 - 2T_2 = 0 \times a_1$

$T_2 = \frac{33.6}{2} = 16.8N$

Example 24

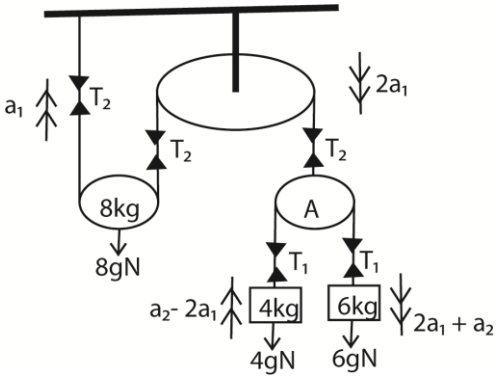
The diagram below shows a fixed pulley carrying a movable pulley of mass 8kg at one end and a light pulley A attached at the other end. A string passes over pulley A and carries a mass of 4kg at one end and a mass of 6kg at the other end.



Find

(a) acceleration of pulley A
(b) acceleration of 8kg, 6kg and 4kg masses
(c) tension in the string

Solution



For 8kg mass: $2T_2 - 8g = 8a_1$ ..... (i)

For 4kg mass: $T_1 - 4g = 4(a_2 - 2a_1)$ ..... (ii)

For 6kg mass: $6g - T_1 = 6(2a_1 + a_2)$ ..... (iii)

For pulley A: $2T_1 - T_2 = 0 \times a_1$ ..... (iv)

eqn. (ii) and eqn. (iii): $2g = 10a_2 + 4a_1$

$4.9 = 2.5a_2 + a_1$ ..... (v)

eqn. (i) + 2 x eqn. (iv): $4T_1 - 8g = 8a_1$.... (vi)

4 x eqn. (iii) + eqn. (vi): $16g = 56a_1 + 24a_2$

$2g = 7a_1 + 3a2$ ..... (vii)

7 x eqn. (v) – eqn. (vii): $14.5a_2 = 14.7$

$a_2 = \frac{14.7}{14.5} = 1.0138ms^{-2}$

eqn. (v); $4.9 = 2.5a_2 + a_1$

$a_1 = 4.9 - 2.5 \times 1.0138 = 2.3655ms^{-2}$

Acceleration of pulley = $2a_1 = 2 \times 2.3655$

$= 4.731ms^{-2}$

Acceleration of 6kg = 4.731 + 1.0138

$= 5.7448ms^{-2}$

Acceleration of 3kg = $a_2 - 2a_1$

$1.0138 - 4.731 = -3.7172ms^{-2}$

From eqn. (i): $2T_2 - 8g = 8a_1$

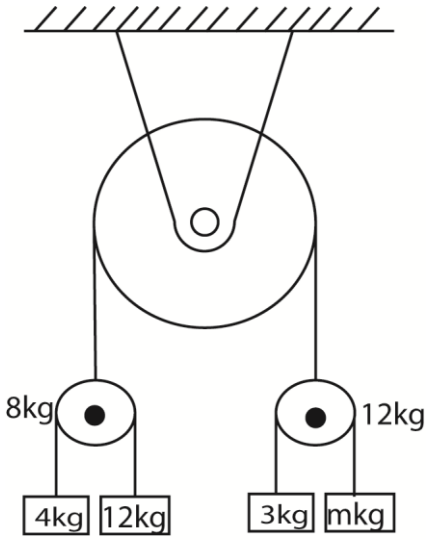
$T_2 = \frac{8 \times 4.731 + 8 \times 9.8}{2} = 58.124N$

From eqn. (iv)

$T_1 = \frac{T_2}{2} = \frac{58.124}{2} = 29.062N$

Example 25

The diagram below shows two pulleys of mass 8kg and 12kg connected by a light inextensible string hanging over a fixed pulley.



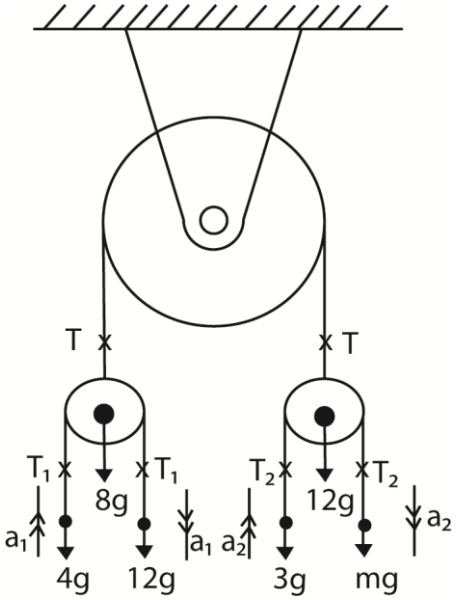
The acceleration of 4kg and 12kg masses are $\frac{g}{2}$ upward and $\frac{g}{2}$ downward respectively. The acceleration of the 3kg and m masses are $\frac{g}{3}$ upwards and $\frac{g}{3}$ downwards respectively. The hanging portions of the strings are vertical. Given that the string of the fixed pulley remains stationary, find the

(a) tensions in the strings (09marks)

Let T = tension n the string joining masses 8kg and 12kg

T_1 = tension in the string joining masses 4kg and 12kg

T_2 = tension in string joining masses 3kg and mkg



Since the string of the fixed pulley remains stationary, this means the pulleys of the 8kg and 12kg are stationary or fixed

Resolving vertically

$2T = 8g + 12g$

$T = 10g = 10 \times 9.8 = 98N$

For 4kg mass; resultant force = $T_1 - 4g$

$4a_1 = T_1 - 4g$

$T_1 = 4g + 4a_1 = 4g + 4 \times \frac{g}{2} = 6g$

$= 6 \times 9.8 = 58.8N$

For 3kg mass; resultant force = $T_2 - 3g$

$\Rightarrow \quad 3a_2 = T_2 - 3g$

$T_2 = 3g + 3a_2 = 3g + 3 \times \frac{g}{3} = 4g$

$= 4 \times 9.8 = 39.2N$

Hence the tensions in the strings are 98N, 58.8N and 39.2N

(b) value of m. (03marks)

For the m kg mass

Resultant force = $mg - T_2$

$ma_2 = mg - T_2$

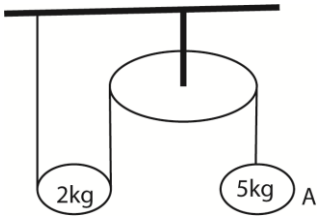
$m(\frac{g}{3}) = mg - 4g$

$\frac{2}{3}mg = 4g$

$m = \frac{12}{2} = 6kg$

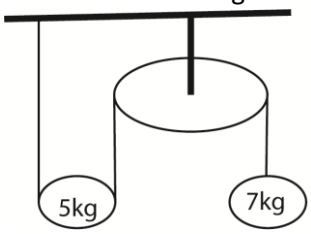
Revision exercise 3

1. A string with one end fixed passes under a movable pulley of mass 2kg, over a fixed pulley and carries a 5kg mass at its end



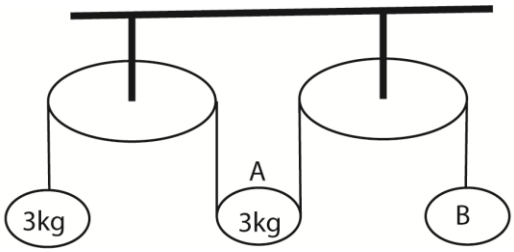
Find the acceleration of the movable pulley and the tension in the string. [3.56ms^{-2}, 13.36N]

2. a string with one end fixed passes under a movable pulley of mass 5kg, over a fixed pulley and carries a mass of7kg at its other end.



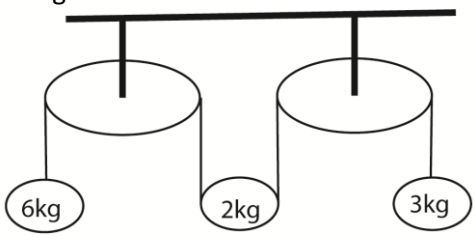
Find the acceleration of the movable pulley and the particle [2.673ms^{-2}, 5.146ms^{-2}]

3. In the pulley system below, A is a heavy pulley which is free to move.



Find the mass B, if it does not move upwards or downwards when the system is released from rest. [1kg]

4. Two particles of mass 3kg and 6kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a heavy smooth movable pulley of mass 2kg, the portions of the string not in contact are vertical

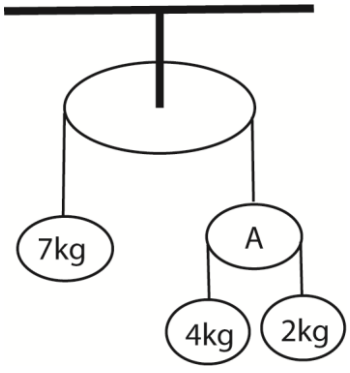


If the system is released from rest, find

(a) acceleration of movable pulley [5.88ms^{-2}]

(b) tension in the string [15.6N]

5. The diagram shows a fixed pulley carrying a string which has a mass of 7kg attached at one end and a light pulley A attached at the other end. Another string passes over the pulley A and caries a mass of 4kg at one end and a mass of 2kg at the other end.

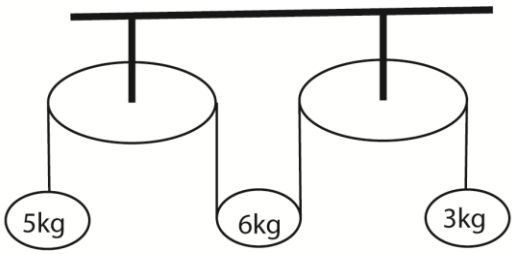


If the system is released from rest, find

(a) acceleration of 4kg mass [2.38ms^{-2}]

(b) tension in the strings [59.33N, 29.66N]

6. Two particles of mass 3kg and 5kg are connected by a light inextensible string passing over two fixed smooth pulleys and under a heavy smooth movable pulley of mass 6kg, the portions of the string not in contact are vertical

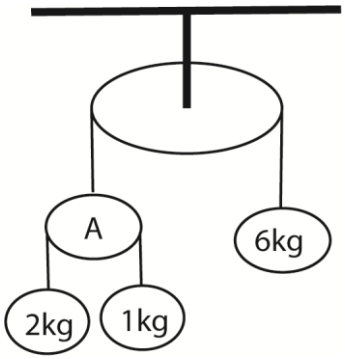


If the system is released from rest, find

(a) acceleration of movable pulley [1.089ms^{-2}]

(b) tension in the string [32.667N]

7. The diagram shows a fixed pulley carrying a string which has a mass of 7kg attached at one end and a light pulley A attached at the other end. Another string passes over pulley A and carries a mass of 4kg at one end and a mass of 2kg at the other end.

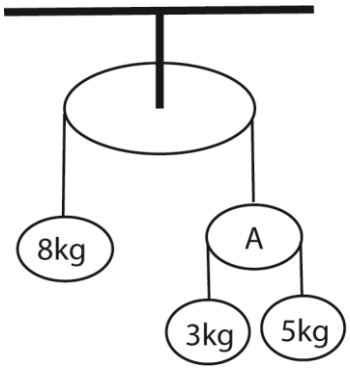


If the system is released from rest, find

(a) acceleration of 1kg mass [8.2923ms^{-2}]

(b) tension in the strings
[18.0923N, 36.1846N]

8. The diagram shows a system of masses and pulleys.

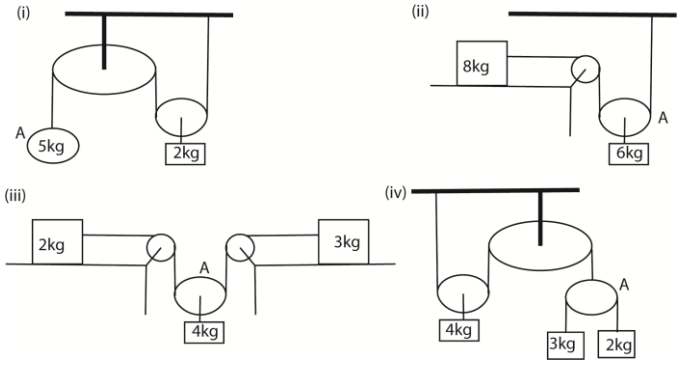


If the system is released from rest, find

(a) acceleration of 5kg mass [2.8451ms^{-2}]

(b) tension in the strings
[75.8712N, 37.9356N]

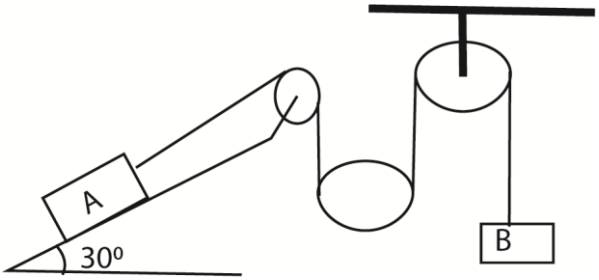
9. For each of the systems below: all the strings are light and inextensible, all pulleys are light and smooth and all surface are smooth. In each case find the acceleration of A and the tension in the string.



(i) [7.127ms^{-2}, 13.364N] (ii) [1.547ms^{-2}, 24.758N]

(iii) [3.564ms^{-2}, 10.691N] (iv) [4.731ms^{-2}, 12.166N, 24.331N]

10. Two particles A and B of mass 4kg and 2kg respectively and a movable pulley c of mass 6kg are connected by a light inextensible string as shown below



Given that the coefficient of friction between A and the plane is 0.2 and the system is released from rest, find the acceleration of A, B, C and the tension in the string.
[A = -0.25ms^{-2}, B = 2.9ms^{-2}, C = 1.325ms^{-2}]

Thank You

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Nurture your dreams

Coplanar forces (Rigid bodies)

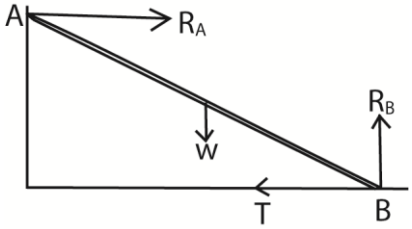
A rigid body is one in which distances between its various parts remain fixed.

Equilibrium of a rigid body

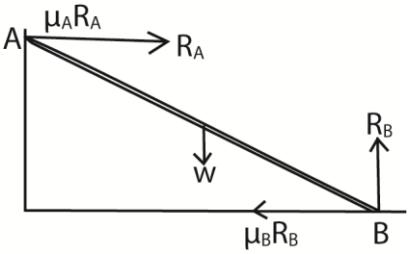
(i) Sum of forces acting in one direction is equal to the sum of force acting in opposite direction
(ii) Sum of clockwise moments about a point is equal to sum of anticlockwise moments about the same point.

Points to note

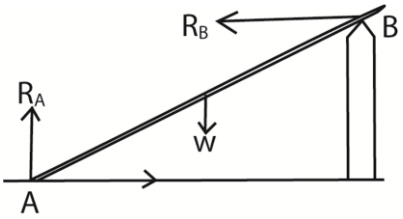
- When a rigid body rests in contact with a smooth wall and a string tied at the base the reaction on the body is perpendicular the surface



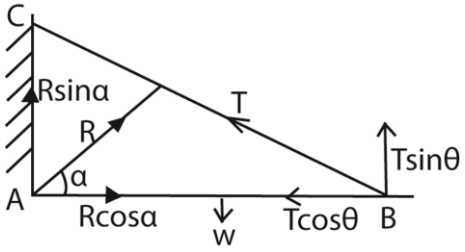
- When a rigid body rests in contact with a Rough wall and ground



- A rigid body resting against a smooth peg or bar, the reaction on the rigid body is perpendicular



- When a rod is hinged, the reaction at the hinge acts at an angle to either the horizontal or vertical



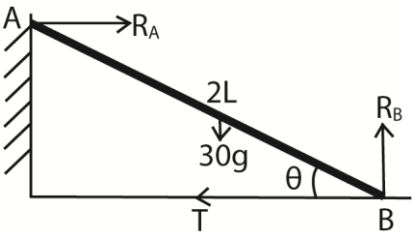
Smooth contacts at a ladder

Example 1

A uniform ladder AB of mass 30kg rests with its upper end A against a smooth vertical wall and the lower end B on a smooth horizontal ground. A light inextensible string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 60^0 to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. Find

(i) Tension in the string (ii) Normal reaction at point A and B

Let the length of the ladder = 2L



(↑) R_B = 30g = 30 x 9.8 = 294

(→) RA = T …… (i)

B) R_A x 2Lsinθ = 30g x Lcosθ ……. (ii)

R_A x 2Lsin60 = 30g x Lcos60

RA = 84.87N

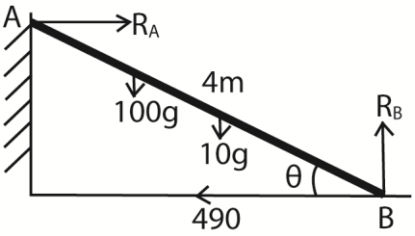
T = 84.87N

Example 2

A uniform ladder AB of mass 10kgand length 4m rests with its upper end A against a smooth vertical wall and lower end B on smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium at an angle $\tan^{-1} 2$ to the horizontal. The vertical plane containing the ladder and the string is at right angles to the wall. A man of mass 100kg ascends the ladder.

(i) If the string will break when the tension exceeds 490N, find how far up the ladder the man can go before this occurs

(ii) What tension must the string be capable of withstanding if the man is to reach the top of the ladder



Let the distance a man ascends be y

(→) R_A = 490N

B) R_A x 4sinθ = 10g x 2cosθ + 100gycosθ

R_A x 4tanθ = 20g + 100gy

490 x 4tanθ = 20g+ 100gy

y = $\frac{490\ x\ 4\ x\ 2-29\ x\ 9.8}{100\ x\ 9.8}$ = 3.8m

(ii)

B) R_A x 4sinθ = 10g x 2cosθ + 100g x 4cosθ

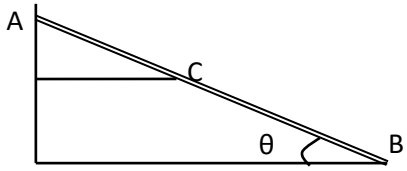
R_A x 4tanθ = 20g + 400g

RA = 514.5N

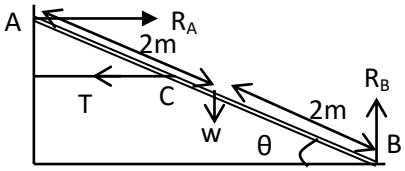
T= 514.5N

Example 3

A uniform ladder AB of weight W and length 4m rests with its upper end A against a smooth vertical wall and the lower end B on a smooth horizontal ground. The ladder is kept in equilibrium inclined at an angle θ to the ground by a light horizontal string attached to the wall and point C on the ladder. The vertical plane containing the ladder and the string is at right angle to the wall.



If tanθ = 2, find the tension in the string when BC is of length 3m



(↑) R_B = W ……. (i)

(→) R_A = T …… (ii)

B↻ R_A x 4sinθ = w x 2cosθ + T x 3sinθ … (iii)

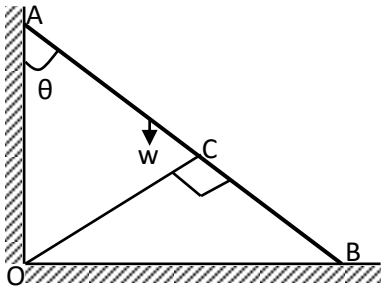
R_A x 4tanθ = 2w + T x 3tanθ

$R_A = \frac{2w+T\ x\ 3\ x\ 2}{4\ x\ 2} = \frac{w+3T}{4}$

$\frac{w+3T}{4} = T$; T= w

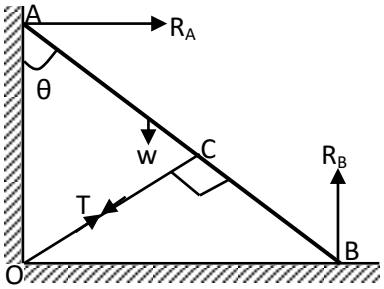
Example 4

The diagram below shows a uniform rod AB of weight w and length L resting at an angle θ against a smooth vertical wall at A. The other end B rests at a smooth horizontal table. The rod is prevented from slipping by an inelastic string OC. C being a point on AB such that AC is perpendicular to AB and O is the point of intersection of the wall and the table. Angle AOB is 90^0.



Find (i) tension in the string (ii) reaction at A and B in terms of θ and w.

Solution



(↑) R_B = w + Tsinθ ……(i)

(→) R_A = Tcosθ …… (ii)

Taking moments at O

R_B x Lsinθ = w$\frac{L}{2}$sinθ + R_A x Lcosθ

(w + Tsinθ)Lsinθ = w$\frac{L}{2}$sinθ + Tsinθ x Lcosθ

(i) Tension in the string

T($cos^2\theta - sin^2\theta$) = $\frac{w(2sin\theta - sin\theta)}{2}$

T = $\frac{wsin\theta}{2(cos^2\theta - sin^2\theta)} = \frac{w}{2}tan^2\theta$

(ii)Reaction at A and B in terms of θ and w

$R_A = Tcos\theta = \left[\frac{wsin\theta}{2(cos^2\theta - sin^2\theta)}\right]cos\theta$

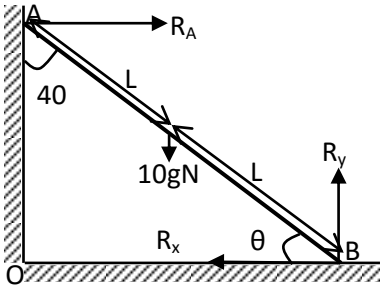
$$R_A = \frac{wsin\theta cos\theta}{2(cos^2\theta - sin^2\theta)} = \frac{wtan^2\theta}{4}$$

$R_B = w + Tcos\theta = w + \left[\frac{wsin\theta}{2(cos^2\theta - sin^2\theta)}\right]sin\theta$

$$R_B = \frac{w(2cos^2\theta - sin^2\theta)}{2(cos^2\theta - sin^2\theta)} = \frac{w}{2}tan^2\theta$$

Example 5

A uniform rod AB of mass 10kg is smoothly hinged at B and rests in a vertical plane with the end A against a smooth vertical wall. If the rod makes an angle 40^0 with the wall, find the reaction on the wall and the magnitude of the reaction at B



Let length of the ladder be 2L

$\theta = 90^0 - 40^0 = 50^0$

Taking moments at B

(→) R_A x 2Lsinθ = 10g x Lcosθ

R_A x 2Lsinθ = 10x 9.8 x Lcosθ

R_A = 41.12N

(↑) R_y = 10gN = 10 x 9.8 = 98N

(→) $R_x = R_A$ = 41.12N

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

$R = \sqrt{(41.12)^2 + 98^2}$ =106.28N

$$\alpha = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{98}{41.12}\right) = 67.24^0$$

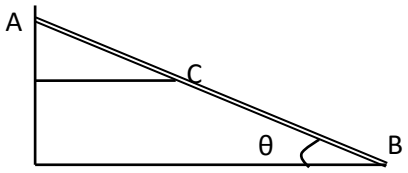
Reaction at B is 106.24 at 67.24^0 to the beam

Revision exercise A

1. A uniform ladder AB of mass 10kg and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 40^0 to the horizontal. The vertical plan containing the ladder and the string is at the right angles to the wall. Find
   (i) Tension in the string [58.4]
   (ii) Normal reaction at points A and B [R_A= 58.4N, R_B= 98N]
2. A uniform ladder AB of mass 30kg and length 6m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 70^0 to the horizontal. The vertical plan containing the ladder and the string is at the right angles to the wall. Find
   (i) Tension in the string [53.3N]
   (ii) Normal reaction at points A and B [R_A= 53.3, R_B= 294N]

3. A uniform ladder AB of mass 30kg rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium inclined at 80^0 to the horizontal. The vertical plan containing the ladder and the string is at the right angles to the wall. Find
   (i) Tension in the string [25.92N]
   (ii) Normal reaction at points A and B [R_A= 25.92, R_B= 294N]
4. A uniform ladder AB of mass 8kg and length 6m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The base of the ladder is 1m from the floor and the top of the ladder is 2m from the floor. A light horizontal string, which has one end attached to B and the other end attached to the wall vertically below the top of the ladder and 1m above the floor, keeps the ladder in equilibrium. Find the tension in the string [27.7N]
5. A uniform ladder AB of weight w and length 2L rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. A light horizontal string, which has one end attached to B and the other end attached to the wall, keeps the ladder in equilibrium The vertical plan containing the ladder and the string is at the right angles to the wall. Find the tension in the string $\left[\frac{w}{2\sqrt{3}}\right]$

6. A uniform ladder AB of weight w and length 4m rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The ladder is kept in equilibrium inclined at an angle θ to the ground by a light inextensible string attached to the wall and to a point C on the ladder. The vertical plane containing the ladder and the string is at right angles to the wall

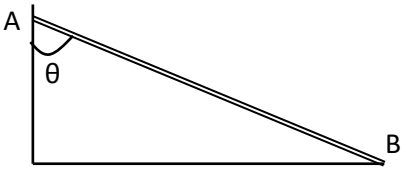


If tanθ = 2, find the tension in the string when BC is of length

(i) 1m $\left[\frac{w}{3}N\right]$

(ii) 2m $\left[\frac{w}{2}N\right]$

7. A uniform ladder AB of mass mkg and length 2L rests with its upper end A against a smooth vertical wall and lower end B on a smooth horizontal ground. The ladder is kept in equilibrium inclined at an angle θ to the vertical by a light inextensible string attached B and to the wall vertically below the top of the ladder. The vertical plane containing the ladder and the string is at right angles to the wall



(a) A man of mass 4m kg stands on the ladder at a distance $\frac{L}{2}$ from the bottom of the ladder, find

(i) The tension in the string $\left[\left(\frac{3mgtan\theta}{2}\right)N\right]$

(ii) Normal reaction at the bottom of the ladder [5mg N)

(iii) Normal reaction at the top of the ladder $\left[\left(\frac{3mgtan\theta}{2}\right)N\right]$

(b) If the maximum tension which the string can bear without breaking is 4mgtanθ, find how far up the ladder the man can safely climb. $\left[\frac{7L}{4}\right]$

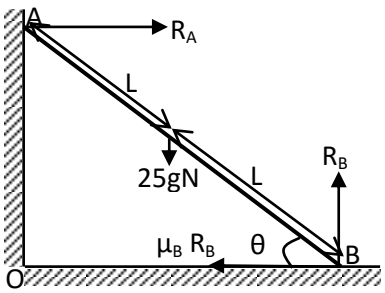
Rough contact at the foot and smooth contact at the top

Example 6

A uniform ladder of mass 25kg rests in limiting equilibrium with the top end against a smooth vertical wall and its base on a rough horizontal floor. If the ladder makes an angle 75^0 with the horizontal, find:

(i) Magnitude of the normal reaction at the floor

(ii) Coefficient of friction between the floor and the ladder



(i) (↑) $R_B = 25g = 25 \times 9.8 = 245N$

(→) $R_A = \mu_B R_B = 245\ \mu_B$

Taking moments about B

$R_A \times 2Lsin\theta = 25g \times Lcos\theta$

R_A x 2Lsin75 =25g x Lcos75

R_A = 32.824N

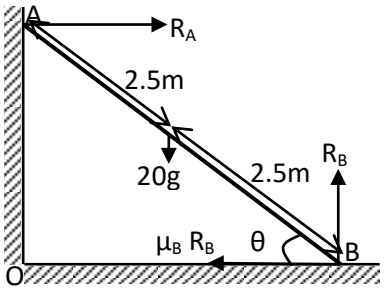
R_A =32.824 = 245 μ_B

μ_B =0.134

Example 7

A uniform ladder which is 5m long and mass of 20kg leans with its upper end against a smooth vertical wall and its lower end on a rough ground. The bottom of the ladder is 3m from the wall. Calculate

(i) The frictional force between the ladder and the ground
(ii) The coefficient of friction



θ= $\cos^{-1}\frac{3}{5} = 53.13^0$

(i) (↑) R_B = 20g = 20 x 9.8 = 196N

(→) R_A = μ_B R_B = 196 μ_B

Taking moments about B

R_A x 5sinθ =20g x 2.5cosθ

R_A x 5sin53.13 =20g x 2.5cos53.13

R_A = 73.5N

Frictional force = 73.5N

(ii) R_A = μ_B R_B = 196 μ_B

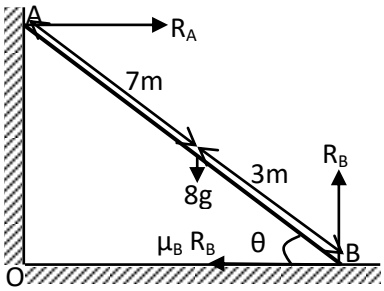
73.5 = 196 μ_B

μ_B = 0.375

Example 8

A non-uniform ladder AB 10m long and 8kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground and the upper end resting against a smooth vertical wall. If the centre of gravity of the ladder is 3m from the foot of the ladder and the ladder makes an angle of 30^0 with the horizontal, find the

(i) Coefficient of friction between the ladder and the ground
(ii) Reaction at the wall



(i) (↑) R_B = 8g = 8 x 9.8 = 78.4N

(→) R_A = μ_B R_B = 78.4 μ_B

Taking moments about B

R_A x 10sinθ =8g x 3cosθ

R_A x 10sin30 =8g x 3cos 30

R_A = 40.738N

Reaction at the wall = 40.738N

R_A = μ_B R_B = 78.4 μ_B

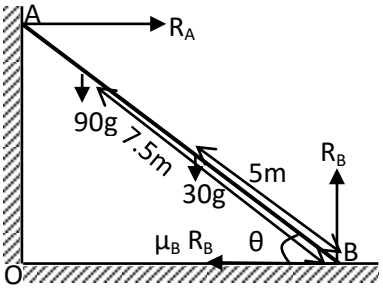
78.4 μ_B = 40.738

μ_B = 0.5196

Example 9

A uniform ladder AB 10m long and mass 30kg lies in a limiting equilibrium with its lower end resting on a rough horizontal ground and its upper end resting against a smooth vertical wall. If the ladder makes an angle of 60^0 with the horizontal, with a man of mass 90kg standing on the ladder at a point 7.5m from its base, find

(i) Magnitude of normal reaction and friction force at the ground
(ii) The minimum value of the coefficient of friction between the ladder and the ground that would enable the man to climb to the top of the ladder



(i) (↑) $R_B = 30g + 90g = 120 \times 9.8 = 1176N$

(→) $R_A = \mu_B R_B = 1176\ \mu_B$

Taking moments about B

$R_A \times 10\sin60 = 30g \times 5\cos60 + 90g \times 7.5\cos60$

$R_A = 466.788N$

$\mu_B R_B = R_A$

$1176\ \mu_B = 466.788$

$\mu_B = 0.397$

(ii) $R_A \times 10\sin60 = 30g \times 5\cos60 + 90g \times 10\cos60$

$R_A = 594.093N$

$\mu_B R_B = R_A$

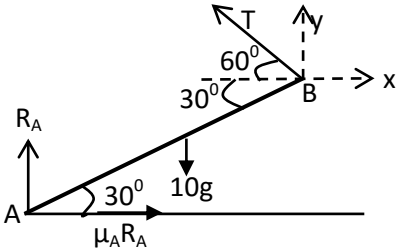
$1176\ \mu_B = 594.093$

$\mu_B = 0.5052$

Rough contact at the root of the ladder

Example 10

A uniform pole AB of mass 10kg has its lower end A on a rough horizontal ground and being raised to a vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is 30^0 to the horizontal, find (i) tension in the rope (ii) coefficient of friction on the ground



(i) Taking moments about A

T x 2L = 10g x Lcos30

T = 42.44N

(ii) (↑) $R_A + T\sin60 = 10g$

RA + 42.44sin60 = 10 x 9.8

RA = 61.25N

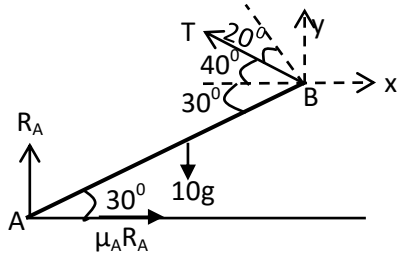
(→) $T\cos60 = \mu_A R_A$

$42.44\cos60 = 61.25\mu_A$

$\mu_A = 0.346$

Example 11

A uniform pole AB of mass 10kg has its lower end A on a rough horizontal ground and being raised to a vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is makes angle of 70^0 the pole and the pole is 30^0 to horizontal, find (i) tension in the rope (ii) coefficient of friction on the ground



(i) Taking moments about A

2L x Tcos20 = 10g x Lcos30

T = 45.157N

(ii) (↑) R_A + Tsin40 = 10g

R_A + 45.157sin40 = 10 x 9.8

R_A = 68.974N

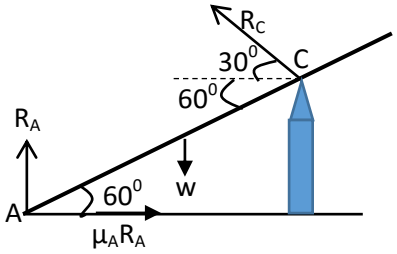
(→) Tcos40 = $\mu_A R_A$

45.157cos40 = 68.974μ_A

μ_A = 0.5015

Example 12

A uniform ladder AB of length 2L rests in limiting equilibrium with its lower end A resting on a rough horizontal ground. A point C on the beam rests against a smooth support. AC is of length $\frac{3L}{2}$ with C higher than A and AC makes an angle of 60^0 with the horizontal. Find the coefficient of friction between the ladder and the ground



Taking moments about A

$R_C = \frac{2wcos60}{3}$ .....(i)

(↑) R_A + $R_C$sin30 = w .....(ii)

$R_A + \frac{2wcos60}{3} sin30 = w$

$R_A = \frac{(3-2sin30cos60)w}{3}$

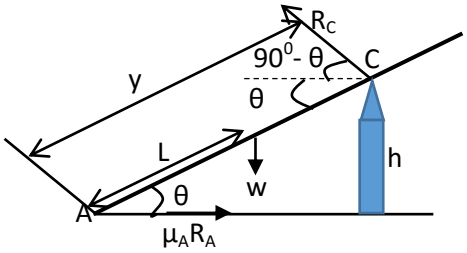
(→)$R_C$cos30 = $\mu_A R_A$

$\frac{2wcos60}{3}$cos30 = $\frac{(3-2sin30cos60)w}{3}\mu_A$

$\mu_A = \frac{2cos30cos60}{(3-2sin30cos60)} = 0.346$

Example 13

A uniform rod of length 2L inclined at an angle θ to the horizontal rests in vertical plane against a smooth horizontal bar at a height h above the ground. Given that the lower end of the rod is on a rough ground and the rod is about to slip; show that the coefficient of friction between the rod and the ground is $\frac{Lsin^2\theta cos\theta}{h-Lcos^2\theta sin\theta}$.



Taking moments about A

R_C x y = wLcosθ

$R_C = \frac{wLcos\theta}{y}$ .....(i)

(↑) $R_A + R_C sin(90-\theta) = w$ .....(ii)

$R_A = \frac{(y-lcos^2\theta)w}{y}$

(→) $R_C cos(90 - \theta) = \mu_A R_A$ .... (iii)

$\frac{wLcos\theta}{y} sin\theta = \mu_A \frac{(y-lcos^2\theta)w}{y}$

$\mu_A = \frac{Lcos\theta sin\theta}{(y-lcos^2\theta)}$

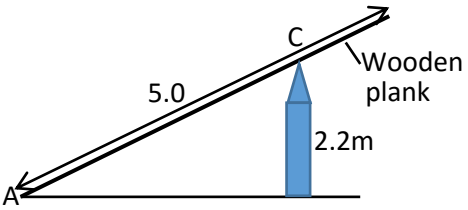
But $sin\theta = \frac{h}{y}$ => $y = \frac{h}{sin\theta}$

$\mu_A = \frac{Lcos\theta sin^2\theta}{\left(\frac{h}{sin\theta}-lcos^2\theta\right)}$

$\mu_A = \frac{Lcos\theta sin\theta}{(h-lcos^2\theta sin\theta)}$

Example 14

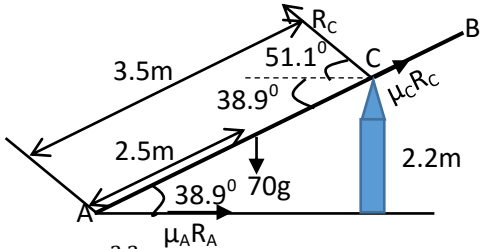
The diagram below shows a uniform wooden plank of mass 70kg and length 5m. The end A rests on a rough horizontal ground. The plank is in contact with the top of a rough pillar at C. the height of the pillar is 2.2m and AC = 3.5m



Given that the coefficient of friction at the ground id 0.6 and the plank is just about to slip, find

(i) Angle the plank makes with the ground at A
(ii) Normal reaction at A and normal reaction at C.
(iii) Coefficient of friction at C

Solution



$\theta = \sin^{-1}\frac{2.2}{3.5} = 38.9^0$

Taking moments about A

R_C x 3.5 = 70g x 2.5cos38.9

R_C = 381.34N

(↑) $R_A + \mu_C R_C sin38.9 + R_C sin51.1 = 70g$

$R_A + \mu_C 381.34 sin38.9 + 381.34 sin51.1 = 70g$

$R_A = 389.2248 - 239.4674\ \mu_C$

(→) $R_C cos51.1 = 0.6R_A + \mu_C R_C sin38.9$

$381.34 cos51.1 = 0.6(389.2248 - 239.4674\ \mu_C) +$

μ_C x 381.34 x sin38.9

$\mu_C = \frac{5.93252}{153.0948} = 0.0388$

R_A = 389.2248 – 239.4674 x 0.0388 = 379.933N

Revision Exercise 2

1. A non- uniform ladder AB 10m long and mass 8kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground of angle of friction 17^0 and the upper end resting against a smooth vertical wall. if the centre of gravity is at point C and the ladder makes an angle of 63^0 with horizontal, find the length AC. [6m]

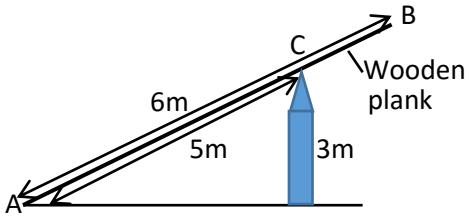
2. A uniform ladder of mass 30kg lies in limiting equilibrium with its lower end resting on a rough horizontal ground of coefficient of friction 0.4 and the upper end resting against a smooth vertical wall. If the ladder makes an angle of 60^0 with the horizontal, find the magnitude of the frictional force at the ground. [158.4N]

3. A uniform pole AB of mass 100kg has its lower end A on a rough horizontal ground and being raised to vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is 20^0 to the horizontal, find

   (i) Normal reaction on the ground [157N)
   (ii) Frictional force on the ground [547N]

4. A ladder 12m long and weighing 200N is placed 60^0 to the horizontal with one end B leaning against a smooth vertical wall and the other end A on a rough horizontal ground. Find:
   (a) Reaction at the wall [57.7N]
   (b) Reaction at the ground [208.2N at 73.9^0 to the horizontal]
5. A uniform pole AB has its lower end A on a rough horizontal ground of friction λ and being raised to vertical position by a rope attached to B. The rope and the pole lie in the same vertical plane and A does not slip across the ground. If the rope is at right angles to the pole and the pole is at an angle θ to the horizontal. Show than $tan\lambda = \frac{sin2\theta}{3-cos2\theta}$.
6. A uniform ladder of length 10m and weight w lies in limiting equilibrium with its lower end resting on a rough horizontal ground of coefficient of friction $\frac{1}{3}$ and the upper end resting against a smooth vertical wall. The ladder makes an angle of θ with the horizontal where tanθ = 1.7. A man of weight 2w starts to climb the ladder. Find
   (i) How far up the ladder the man can climb before slipping can occur. [6m]
   (ii) Find in terms of w, the magnitude of the frictional force at the ground to enable the man reach the top.$\left[\frac{8w}{17}\right]$
7. A uniform ladder of length 5m and weight 80N lies in limiting equilibrium with its lower end resting on a rough horizontal ground and the upper end resting against a smooth horizontal rail fixed 4m vertically above the ground. If the ladder makes an angle of θ with the vertical where tanθ ≤ 0.75 with the horizontal.
   (a) Find expressions in terms of θ for
      (i) Vertical reaction R of the ground [$80-50\sin^2\theta\cos\theta$]
      (ii) Friction F at the ground [$50\cos^2\theta\sin\theta$]
      (iii) Normal reaction N at the rail [$50\sin\theta\cos\theta$]
   (b) Given that the ladder does not slip, show that F is maximum when tanθ = $\frac{1}{\sqrt{2}}$ and find its maximum value $\left[\frac{100}{3\sqrt{3}}N\right]$

8. A uniform ladder AB of weight w rests in limiting equilibrium with its lower end A resting on a rough horizontal ground, coefficient of friction μ. A point C on the ladder rests against a smooth peg. AC is of length $\frac{1}{4}AB$ from end B and a height h from the ground. If AB makes an angle of θ with the ground , show that the frictional force is $\frac{\mu w}{3}(2-3cos^2\theta)$
9. A smooth horizontal rail is fixed at a height of 3m above the horizontal rough ground. A straight uniform pole AB of length 6m and mass 20kg rests in limiting equilibrium at a point Con the rail with its lower end A resting on the ground. The vertical plane containing the pole is at right angles to the rail. The distance AC is 5m



Calculate the

(i) Magnitude of the force exerted by the rail on the pole [94N]

(ii) Coefficient of friction between the pole and the ground [0.47]

(iii) Magnitude of the force exerted by the ground on the pole [133N]

10. A uniform ladder lies in limiting equilibrium with its lower end resting on a rough horizontal ground of coefficient of friction $\frac{5}{8}$ and the upper end resting against a smooth vertical wall. If the ladder makes an angle of 45^0 with the vertical.

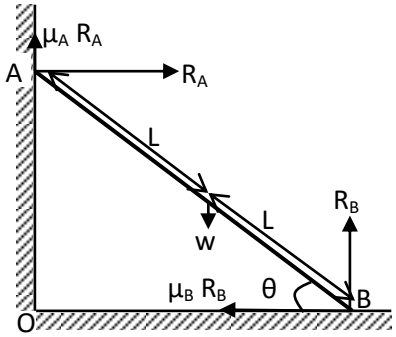
(i) Show that a man whose weight is equal to that of the ladder can ascend $\frac{3}{4}$ of the length of the ladder

(ii) Find what weight must be placed on the bottom of the ladder to enable the man ascend to the top $\left[\frac{2w}{5}\right]$

Rough contacts both at the top and foot of a ladder

Example 15

A uniform ladder rests with one end on a rough horizontal ground and the other against a rough vertical wall, the coefficient of friction being respectively $\frac{3}{5}$ and$\frac{1}{3}$. Find the inclination of the ladder to the vertical when it is about to slip



(↑) $R_B + \frac{1}{3}R_A = w$ ..... (i)

(→) $R_A = \frac{3}{5}R_B$ => $R_B = \frac{5}{3}R_A$

Substituting for R_B in equation (i)

$\frac{5}{3}R_A + \frac{1}{3}R_A = w$

$R_A = \frac{w}{2}$

Taking moments about B

R_A x 2Lsinθ + $\frac{1}{3}R_A$ x 2Lcosθ = w x Lcosθ ...(ii)

Substituting for R_A in eqn. (ii)

$\frac{w}{2}$ x 2Lsinθ + $\frac{1}{3}x$ $\frac{w}{2}$ x 2Lcosθ = w x Lcosθ

Sinθ = $\frac{2}{3}cos\theta$

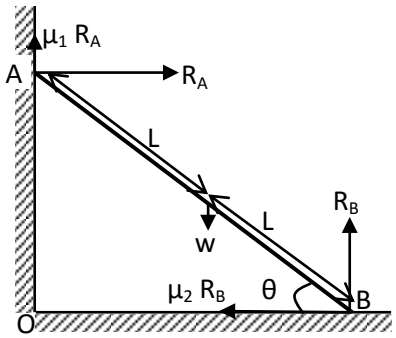
tanθ = $\frac{2}{3}$

θ = 33.7^0

Angle to the vertical = 90 - 33.7 = 56.3^0

Example 16

A uniform ladder rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction μ_1 and its base on a rough horizontal floor with coefficient of friction μ_2.If the ladder makes an angle of θ with the floor, prove that tanθ = $\frac{1-\mu_1\mu_2}{2\mu_2}$



(↑) $R_B + \mu_1 R_A$ = w …….. (i)

(→) $R_A = \mu_2 R_B$ …….(ii)

$R_B = \frac{1}{\mu_2}R_A$

Substituting R_B in eqn. (i)

$\frac{1}{\mu_2}R_A + \mu_1 R_A = w$

$R_A = \frac{\mu_2 w}{1+\mu_1\mu_2}$

Taking moments about B

R_A x 2Lsinθ + $\mu_1 R_A$ x 2Lcosθ = w x Lcosθ …….(iii)

$\frac{\mu_2 w}{1+\mu_1\mu_2}$ x 2Lsinθ + μ_1x $\frac{\mu_2 w}{1+\mu_1\mu_2}$ x 2Lcosθ = w x Lcosθ

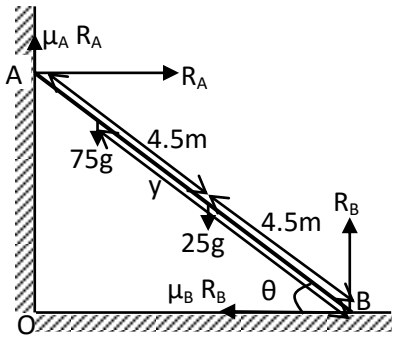
$\frac{\mu_2 w}{1+\mu_1\mu_2}$ x 2Lsinθ = w x Lcosθ - μ_1x $\frac{\mu_2 w}{1+\mu_1\mu_2}$ x 2Lcosθ

$\frac{2\mu_2}{1+\mu_1\mu_2}$ sinθ = $\frac{1+\mu_1\mu_2-2\mu_1\mu_2}{1+\mu_1\mu_2}$ cosθ

tanθ = $\frac{1-\mu_1\mu_2}{2\mu_2}$

Example 17

The foot of a ladder length of 9m and mass 25kg rests on a rough horizontal surface while the upper end rests in contact with a rough vertical wall. The ladder is in vertical plane perpendicular to the wall. If the first rug is 30cm from the foot and the rest at the interval of 30cm, find the highest rug to which a man of mass 75kg can climb without causing the ladder to slip, when the ladder is inclined at 600 to the horizontal and the coefficient of friction at each end is 0.25.



Let the man ascend a distance = ym

(↑) $R_B + \frac{1}{4}R_A = 25g + 75g$ …… (i)

(→) $R_A = \frac{1}{4}R_B$ => $R_B = 4R_A$ ….. (ii)

Substituting for R_B in eqn. (i)

$4R_A + \frac{1}{4}R_A = 25g + 75g$

$R_A = 226.154N$

Taking moments about B

R_A x 9sinθ + $\frac{1}{4}R_A$ x 9cosθ = 25 x 4.5cosθ +75 x 4.5cosθ

$226.154N$ x 2Lsinθ + μ_1x $226.154N$ x 2Lcosθ

= 25 x 4.5cos60 +75 x 4.5cos60

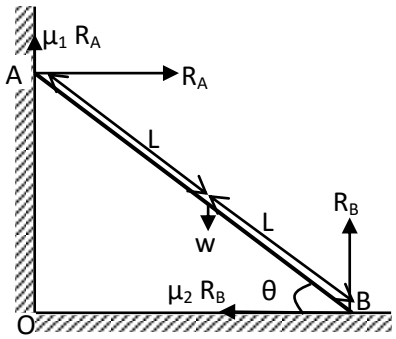
Y = 4m

Number of rugs = $\frac{4}{0.3}$ = 13

Example 18

A uniform ladder of length 2L and weight w rests in a vertical plane with one end on a rough horizontal ground and the other against a rough vertical wall, the angle of friction being respectively $\tan^{-1}\left(\frac{1}{3}\right)$ and $\tan^{-1}\left(\frac{1}{2}\right)$.

(a) Find the inclination of the ladder to the horizontal when it is in limiting equilibrium at either end



$\mu_A = \frac{1}{3}$ and $\mu_B = 2$

(↑) $R_B + \frac{1}{3} R_A = w$ …….. (i)

(→) $R_A = \frac{1}{2} R_B$

$R_B = 2R_A$ ……..(ii)

Substituting R_B in eqn. (i)

$2R_A + \frac{1}{3}R_A = w$

$R_A = \frac{3w}{7}$

Taking moments about B

R_A x 2Lsinθ + $\frac{1}{3}R_A$ x 2Lcosθ = w x Lcosθ ..(iii)

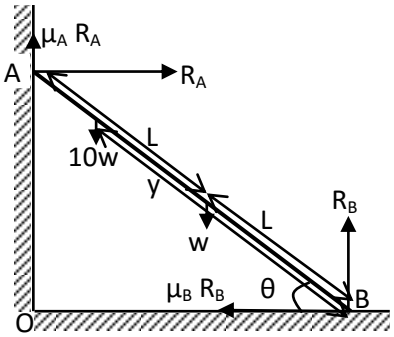
$\frac{3w}{7}$ x 2Lsinθ + $\frac{1}{3}$ x $\frac{3w}{7}$ x 2Lcosθ = w x Lcosθ

$\frac{6}{7}$ sinθ = $\frac{5}{7}$ cosθ

tanθ = $\frac{5}{6}$

θ = 39.8^0

(b) A man of weight 10times that of the ladder begins to ascend it. How far will he climb before the ladder slips



Let the man ascend a distance = ym

(↑) $R_B + \frac{1}{3}R_A = 10w + w$ …… (i)

(→) $R_A = \frac{1}{2}R_B$ => $R_B = 2R_A$ ….. (ii)

Substituting for R_B in eqn. (i)

$2R_A + \frac{1}{3}R_A = 11w$

$R_A = \frac{33}{7}w$

Taking moments about B

R_A x 2Lsinθ + $\frac{1}{3}R_A$ x 2Lcosθ = wLcosθ + 10wycosθ

$\frac{33}{7}w$ x 2Lsinθ + $\frac{1}{3}$x $\frac{33}{7}w$ x 2Lcosθ

= wLcosθ + 10wycosθ

$\frac{66}{7}Lsin\theta + \frac{15}{7}Lcos\theta = 10ycos\theta$

$\frac{66}{7}Ltan\theta + \frac{15}{7}L = 10y$

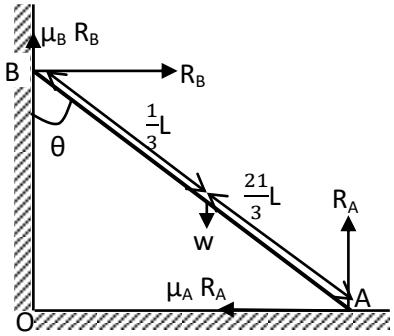
10L = 10y => y = Lm

Example 19

A non-uniform ladder AB is in limiting equilibrium with its lower end A resting on a rough horizontal ground and the upper end B resting against a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The centre of gravity of the ladder is t a G = $\frac{2}{3}AB$. The coefficient of friction between the ladder and the wall is twice that between the ladder and the floor. If the ladder makes an angle θ with the wall and the angle of friction between the ladder and the floor is λ;

(i) Show that 4tanθ = 3tan2λ

(ii) How far can a man of equal mass as the ladder ascend without the ladder slipping given that θ = 45^0 and coefficient of friction between the ladder and the floor is $\frac{1}{2}.\left[\frac{2}{3}AB\right]$



$\mu_A = tan\lambda \; and \; \mu_B = 2tan\lambda$

(↑) $R_A + 2tan\lambda = w$.....(i)

(→) $R_B = tan\lambda$ R_A..... (ii)

Substituting R_B in eqn. (i)

R_A + 2tanλtanλ$R_A = w$

$R_A = \frac{w}{1+ tan^2\lambda}$

Taking moments about B

R_A x Lsinθ = w$\frac{L}{3}$sinθ + tanλR_ALcosθ

$\frac{w}{1+ tan^2\lambda}$ x Lsinθ - w$\frac{L}{3}$sinθ = tanλ$\left(\frac{w}{1+ tan^2\lambda}\right)$ Lcosθ

$\frac{2-2tan^2\lambda}{3(1+2tan^2\lambda)} sin\theta = \frac{tan\lambda}{1+2tan^2\lambda} cos\theta$

$tan\theta = \frac{3tan\lambda}{2-2tan^2\lambda} =$

$tan\theta = \frac{3}{2}x\frac{2}{2}\left[\frac{3tan\lambda}{2(1-tan^2\lambda)}\right] = \frac{3}{4}\left[\frac{2tan\lambda}{(1-tan^2\lambda)}\right] = \frac{3}{4}tan2\lambda$

4tanθ = 3tan2λ

Revision exercise 3

1. A uniform ladder rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction 0.25 and its base on a rough horizontal floor with coefficient of friction μ. If the ladder makes an angle of 300 with the vertical; find the value of μ [0.269]
2. A non-uniform ladder AB of length 6m is in limiting equilibrium with its lower end A resting on rough horizontal ground with coefficient of friction $\frac{1}{3}$ and the upper end B resting against a rough vertical wall with coefficient of friction $\frac{1}{4}$. The ladder is in a vertical plane perpendicular to the wall. The centre of gravity of the ladder is at C where AC = 4m. If the ladder makes an acute angle θ with the ground. Show that tanθ =$\frac{23}{12}$.
3. A uniform ladder AB is of weight 2w and length 10m rests in limiting equilibrium with the top end against a rough vertical wall with coefficient of friction $\frac{1}{3}$and its base on a rough horizontal floor with coefficient of friction $\frac{1}{3}$. If the ladder makes an angle an angle of θ with horizontal, such that tanθ = $\frac{16}{17}$. A man of weight 5w starts to climb the ladder.
   (a) How far up the ladder can a man climb before slipping [9m]
   (b) When a boy of weight Y stands on the bottom rung of the ladder at A, the man is just able to climb to the top safely. Find Y in terms of W $\left[\frac{7w}{11}\right]$

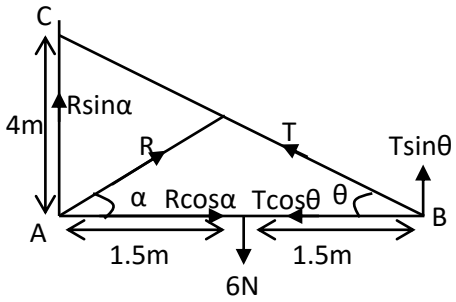
4. A non-uniform ladder AB of length 12m and mass 30 is in limiting equilibrium with its lower end A resting on a rough horizontal ground with coefficient of friction 0.25 and upper end B resting against a rough vertical wall with coefficient of friction 0.2. The ladder is in a vertical plane perpendicular to the wall. The centre of gravity of the ladder is at its trisection of the length nearer to A. The ladder makes an angle θ with the horizontal such that tanθ =$\frac{9}{4}$. A straight horizontal string connects A to a point at the base of the wall vertically below B. A man of mass 9okg begins to climb the ladder
   (i) How far up the ladder can the man climb without causing tension in the string [8m]
   (ii) What tension must the string be capable of withstanding if the man is to reach the tip of the ladder safely [126N]

Beams hinged and maintained in a horizontal position

Example 20

A uniform beam AB, 3.0m long and weight 6N is hinged at a wall at A and is held stationary in a horizontal position by a rope attached to B and joined to a point C on the wall, 4.0m vertically above A. find

(i) The tension T in the rope
(ii) The magnitude and direction of the reaction R at the hinge.



θ = $\tan^{-1}\frac{4}{3}$ = 53.13^0

Taking moments at A at equilibrium

(Tsin53.13) x 3 = 6 x 1.5

T = 3.75N

(↑) $R_A$sinα + Tsionθ = 6

$R_A$sinα = 6 – 3.75sin53.13

$R_A$sinα = 3……..(i)

(→)Rcosα = Tcosθ

Rcosα = 3.75cos 53.13

Rcosα = 2.238 ……(ii)

$(i) \div (ii)$

Tanα $\frac{3}{2.238}$ α = 53.3^0

Substituting α into eqn. (i)

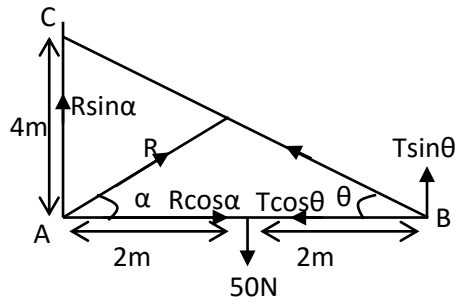
$R_A$sin53.3 = 3;

R_A = 3.74N at 53.3^0 to the beam

Example 21

A uniform beam AB of length 4m and weight 50N is freely hinged at A to a vertical wall and is held horizontally in equilibrium by a string which has one end attached to B and the other end attached to a point C on the wall 4m above A. Find

(i) The tension in the rope
(ii) The magnitude and direction of the reaction R at the hinge



θ = $\tan^{-1}\frac{4}{4}$ = 45^0

Taking moments at A at equilibrium

(Tsin45) x 3 = 50 x 2

T = 35.36N

(↑) $R_A$sinα + Tsionθ = 50

Rsinα = 50 – 35.36sin45

Rsinα = 24.997……..(i)

(→)Rcosα = Tcosθ

Rcosα = 35.36cos 45

Rcosα = 25 ……(ii)

$(i) \div (ii)$

Tanα $\frac{24.997}{25}$ α = 45^0

Substituting α into eqn. (ii)

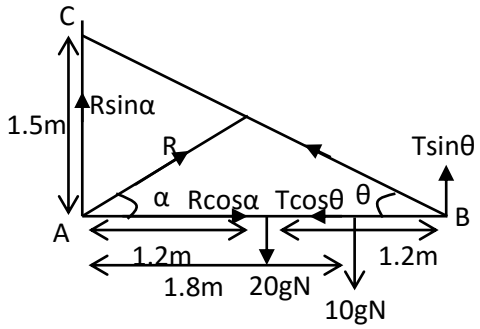
Rcos45 = 25;

R = 35.36N at 45^0 to the horizontal

Example 22

A uniform beam AB of length 2.4m and weight 20N is freely hinged at A to a vertical wall and is maintained in horizontally in position by a chain which has one end attached to B and the other end attached to a point C on the wall 1.5m above A. If the beam carries a load of 10N at a point 1.8m from A. Calculate

(i) The tension in the chain
(ii) The magnitude and direction of the reaction R at the hinge



θ = $\tan^{-1}\frac{1.5}{2.4}$ = 32.01^0

Taking moments at A at equilibrium

(Tsin32.01) x 2.4 = 20g x 1.2 + 10g x 1,8

Tension in chain, T = 323.87N

(↑) $R_A$sinα + Tsionθ = 20gN +10gN

Rsinα = 30g – 323.87sin32.01

Rsinα = 122.63……..(i)

(→)Rcosα = Tcosθ

Rcosα = 323.87cos 32.01

Rcosα = 274.63 ……(ii)

$(i) \div (ii)$

Tanα $\frac{122.63}{274.63}$ α = 24.06^0

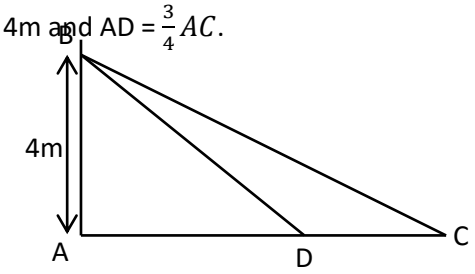
Substituting α into eqn. (ii)

Rcos24.06 = 274.63;

R = 300.85N at 24.07^0 to the horizontal

Example 23

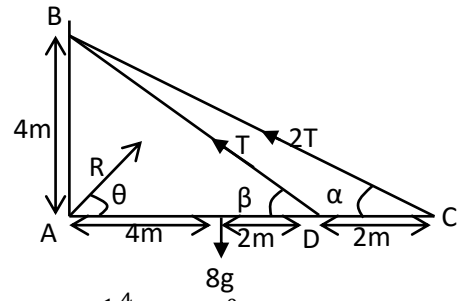
A uniform beam AC of mass 8kg and length 8m is hinged at A and maintained in equilibrium by two strings attached to it at points A and D as shown below. The tension in BC is twice that in AB. AB = 4m and AD = $\frac{3}{4}AC$.



Find:

(i) Tension in the string BC

(ii) Magnitude and direction of the resultant force at the hinge



$\alpha = \tan^{-1}\frac{4}{8} = 26.6^0$

$\beta = \tan^{-1}\frac{4}{6} = 33.7^0$

Taking moments about A

6 x Tsinβ + 8 x Tsinα = 8g x 4

6 x Tsin33.7 + 8 x Tsin26.6 = 8g x 4

T= 29.886N

Tension in BC = 2 x 29.886 = 59.772N

(↑) Rsinθ + Tsinβ+ 2Tsinα = 8g

Rsinθ = 35.0545 …….. (i)

(→) Rcosθ = Tcosβ+ 2Tcosα

Rcosθ = 78.3092 ……..(ii)

(i)÷ (ii) tanθ = $\frac{78.3092}{35.0545}$; θ = 24.1^0

Substituting θ in eqn. (ii)

R = $\frac{78.3092}{cos24.1}$ = 85.8N

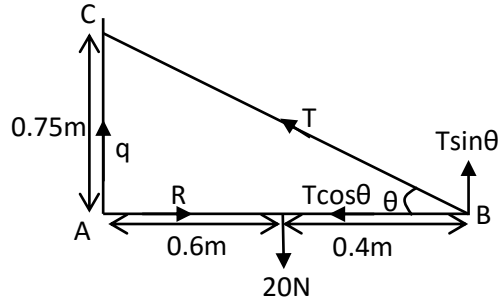
∴ R = 85.8N at 24.1^0 to horizontal

Example 24

A rod 1m long has weight of 20N and its centre of gravity 60cm from A. It rests horizontally with A against a rough vertical wall. A string BC is fastened to the wall at 75cm vertically above A. Find the

(i) Normal reaction and friction at A if friction is limiting the coefficient of friction.

(ii) Tension in the string



$\theta = \tan^{-1}\frac{0.75}{1} = 36.9^0$

taking moments about A

Tsinθ x 0.4 = 20 x 0.6

Tsin36.9 x 0.4 = 20 x 0.6

T = 19.99N

(↑) q + Tsinθ = 20

q + 19.99sin36.9 = 20

q =8N

(→) R = Tcosθ;

R= 19.99cos36.9 = 15.99N

q =μR

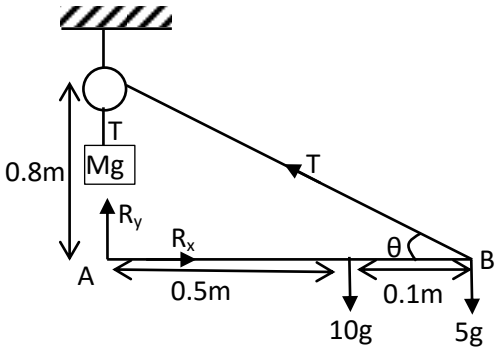
$\mu = \frac{8}{15.99} = 0.5$

Example 25

A rod of length 0.6m and mass 10kgis hinged at A. Its centre of gravity is 0.5m from A, a light inextensible string attached at B passes over a fixed pulley 0.8m above A and supports a mass M hanging freely. If a mass of 5kg is attached at B so as to keep the rod horizontal, find the

(i) Value of N

(ii) Reaction at the hinge



$tan\theta = \frac{0.8}{0.6} = \frac{4}{3}; sin\theta = \frac{4}{5}; cos\theta = \frac{3}{5}$

For Mkg mass: T = Mg…….(i)

For the beam: taking moments a bout A

0.6 x Tsinθ = 5g x 0.6 + 10g x 0.5

0.6 x T x $\frac{4}{5}$ = 5g x 0.6 + 10g x 0.5

T = $\frac{50x\ 9.8}{3} = 163.33N$

From eqn. (i)

T = Mg

M = $\frac{163.33}{9.8} = 16.67kg$

(↑) Tsinθ + R_y = 10g + 5g

163.33 x $\frac{4}{5}$ + R_y = 15 x 9.8

R_y = 16.336N

(→) Tcosθ = Rx

Rx = 163.33 x $\frac{3}{5}$ =97.998N

R = $\sqrt{(R_x)^2 + (R_y)^2}$=$\sqrt{(97.998)^2 + (16.336)^2}$

R = 99.35N

$\alpha = \tan^{-1}\frac{16.336}{97.998} = 9.34^0$

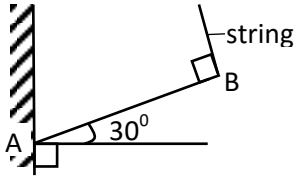
Reaction at B is 99.35N at 9.34^0 to horizontal

Beam Hinged and maintained at an angle

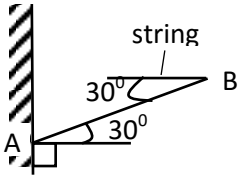
Example 26

Each of the following diagrams shows a uniform rod of mass 5kg and length 6m freely hinged at A to a vertical wall. A string attached to B keeps the rod in equilibrium. For each case, find the tension in the string and the magnitude and direction of the reaction at the hinge

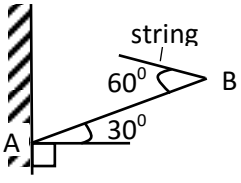
(a)



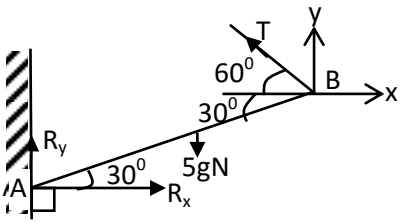
(b)



(c)



Solution



Taking moments about A

T x 2L = 5g x Lcos30

T = 21.2176N

(↑) Tsin60 + R_y = 5g

21.2176sin60 + R_y = 5g

R_y = 30.625N

(→) Rx = Tcos60 = 21.2176cos 60 = 10.6088N

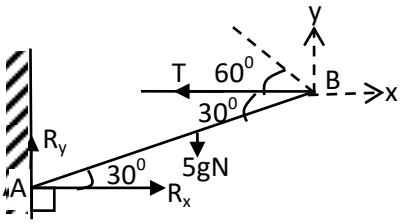
R = $\sqrt{(R_x)^2 + (R_y)^2}$ = $\sqrt{(10.6088)^2 + (30.625)^2}$

R = 32.41N

$\alpha = \tan^{-1}\frac{30.625}{10.6088} = 70.9^0$

Reaction at A is 32.41N at 70.9^0 to horizontal

(ii)



Taking moments about A

T x 2L = 5g x Lcos60

T = 42.4352N

(↑)R_y = 5g = 49N

(→) Rx = T = 42.432N

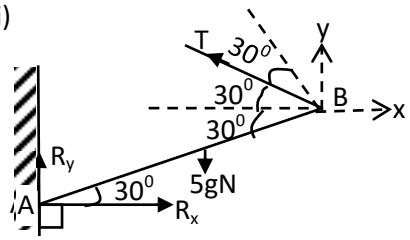
R = $\sqrt{(R_x)^2 + (R_y)^2}$ = $\sqrt{(42.432)^2 + (49)^2}$

R = 64.8209N

$\alpha = \tan^{-1}\frac{49}{42.432} = 49.1^0$

Reaction at A is 64.8209N at 49.1^0 to horizontal

(iii)



Taking moments about A

Tcos30 x 2L = 5g x Lcos30

T = 24.9N

(↑)Tsin30 + R_y = 5g

24.5sin30 + Ry = 5g

Ry = 36.75N

(→) Rx = Tcos30 = 24.5cos30 =21.2176

R = $\sqrt{(R_x)^2 + (R_y)^2}$ = $\sqrt{21.2176^2 + (24.5)^2}$

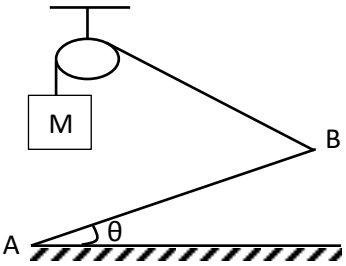
R = 42.4352N

$\alpha = \tan^{-1}\frac{36.75}{21.2176} = 60^0$

Reaction at A is 64.8209N at 49.1^0 to horizontal

Example 27

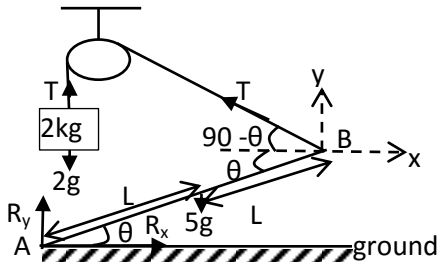
A uniform rod AB of mass 5kg is smoothly hinged on the ground at point A. The rod making an angle θ with the horizontal ground is kept in equilibrium by a light inelastic string attached to B. The string which makes 90^0 with the rod passes over a smooth fixed pulley and carries a stationary mass m of 2kg at the other end



Show that

(a) cosθ = $\frac{4}{5}$

(b) the magnitude of reaction at the hinge is $\frac{49}{5}\sqrt{13}N$



For 2kg mass: T = 2g ..... (i)
For beam taking moments about A
T x 2L = 5g x Lcosθ ..... (ii)
2g x 2L = 5g x Lcosθ
cosθ = $\frac{4}{5}$
(↑) Tsin(90-θ) + R_y = 5g
Tcosθ + R_y = 5g

2g x $\frac{4}{5}$ + Ry = 5g

Ry = $\frac{17}{5}g = \frac{17}{5}\ x\ 9.8 = 33.32$ N

(→) Tcos(90 – θ) = Rx

Rx = Tsinθ = 2g x $\frac{3}{5}$ =11.76N

R = $\sqrt{(R_x)^2 + (R_y)^2}$=$\sqrt{(33.32\ )^2 + (11.76)^2}$

R = 35.3344N

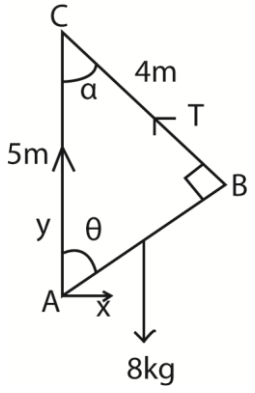
$\alpha = \tan^{-1}\frac{33.32}{11.76} = 70.56$

Reaction at A is 35.3344N at 70.56^0 to horizontal

Example 28

A uniform rod AB of length 3m and mass 8kg is freely hinged to a vertical wall at A. A string BC of length 4m attached at b and to point C on the wall, keeps the rod in equilibrium. If C is 5m vertically above A, find the

(a) Tension in the string (03marks)



$AB^2 + 4^2 = 5^2$

AB = $\sqrt{(25-16)}$ = 3

Let T be tension in the string, from the diagram

cos θ = $\frac{3}{5}$, cos α = $\frac{4}{5}$

Equation of moment about A

T x 3 = 8g x 1.5cosα

3T = 8 x 9.8 x $\frac{4}{5}$; T = 31.36N

∴ tension in the string is 31.36N

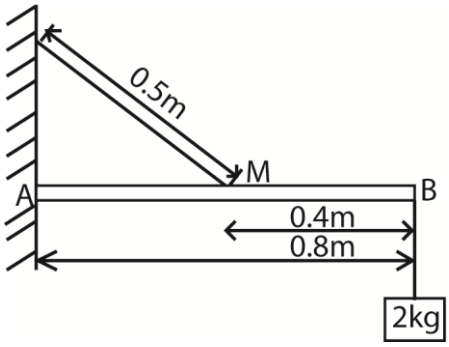
(b) Magnitude of the normal reaction at A. (02marks)

x = Tcosθ = 31.36 x $\frac{3}{5}$ = 18.816N

∴ the magnitude of normal reaction at A is 18.816N

Example 29

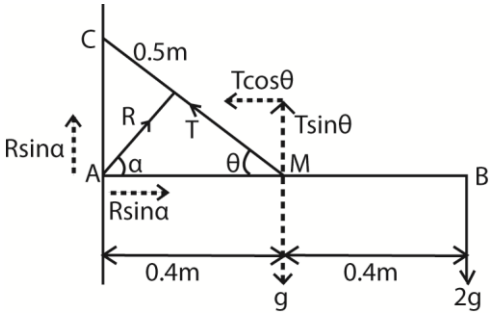
The figure below shows a uniform beam of length 0.8metres and mass 1kg. the beam is hinged at A and has a load of mass 2kg attached at B



The beam is held in a horizontal position by a light inextensible string of length 0.5metres. the string joins the mid-point M of the beam to a point C vertically above A.

Find the

(a) Tension in the string(08marks)



$AC^2 = (05)^2 – (0.4)^2$

AC = 0.3

cosθ = $\frac{0.4}{0.5} = 0.8$

sinθ = $\frac{0.3}{0.5} = 0.6$

Taking moments at A

(9.8 x 0.4) + (2 x 9.8 x 0.8) = T x 0.4sinθ

(9.8 x 0.4) + (2 x 9.8 x 0.8) = T x 0.4 x 0.6

T = 81.667N

(b) Magnitude and direction of the force exerted by the hinge. (04marks)

Resolving forces

(→); Rcosα = Tcosθ = $\frac{245}{3}\ x\ 0.$

Rcosα = $\frac{196}{3}$ ..... (i)

(↑); Rsinα + Tsinθ = g + 2g

Rsinα = 3 x 9.8 - $\frac{245}{3}\ x\ 0.6$

Rsinα = -19.6 ..... (ii)

Equn (ii) ÷ Eqn (i)

$\frac{Rsin\ \alpha}{Rcos\ \alpha} = \frac{-19.6}{\frac{196}{3}}$ = -0.3

tan α = -0.3

α = -16.7^0

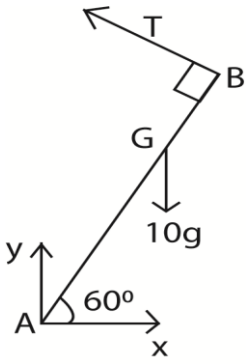
Hence the direction of force at the hinge is 16.7^0 with the beam

From eqn (i)

Rcos $16.7^0 = \frac{196}{3}$; R = 68.21N

Example 30

A non-uniform rod AB of mass 10k has its centre of gravity a distance ¼ AB. The rod is smoothly hinged at A. it is maintained in equilibrium at 60^0 above the horizontal by a light inextensible string tied at B and at a right angle to AB. Calculate the magnitude and direction of the reaction at A. (12marks)



Taking moments about point A

T x (AB) = $10g\left(\frac{3}{4}ABcos\ 60^0\right)$

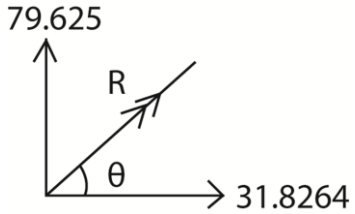
T = $\frac{15g}{4} = \frac{15\ x\ 9.8}{4}$ = 36.75N

Resolving forces horizontally

X = $Tcos30^0 = 36.75cos30^0 = 31.8264N$

Y = $10g - 36.75sin\ 30^0 = 79.625N$

$|R| = \sqrt{(31.8264)^2 + (79.625)^2}$ = 85.75N



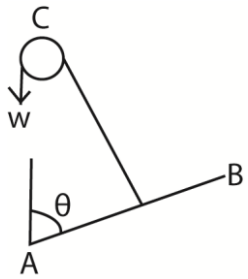
θ = $\tan^{-1}\left(\frac{79.625}{31.8264}\right)$= 68.2^0

The direction of resultant force is 68.2^0 or E68.2^0N or N21.8^0E

Revision Exercise 4

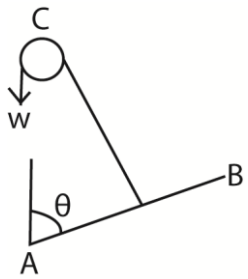
1. A uniform rod AB of mass 5kg is freely hinged at A to a vertical wall and held horizontal in equilibrium by a string which has one end attached to a point C on the wall above A. the string makes an angle of 30^0 with AB, find
   (a) The tension in the rope [49N]
   (b) The magnitude and direction of the reaction at A [49N at 30^0 with AB
2. A non-uniform beam AB of weight 20N and of length 4m is freely hinged at A to a vertical wall and is held horizontal in equilibrium by a string which has one end attached at B and the other end attached to point C on the wall above A. The string makes an angle of 600 with AB. If the tension in the string is 12N, find
   (a) The magnitude and direction of reaction at the hinge [11.3N at 58^0 with AB
   (b) Distance from A to the centre of gravity. [2.08m]
3. One end of a uniform plank of length 4m and weight 100N is hinged to a vertical wall. An inelastic rope, tied to the other end of the plank is fixed at a point 4m above the hinge. Find
   (a) tension in a rope [388.9N]
   (b) the reaction of the wall on the plank [302.1N at 24.4^0 to horizontal]
4. A uniform diving board AB, of length 4m and maintained in a horizontal position by means of a light strut DC. D is a point on the wall 1m below A and C is a point on the board where AC = 1m. An object of mass 60kg is placed at end B
   (a) Find the position of the centre of mass of the 60kg mass and the mass of the board combined. [80 cm from B]
   (b) Determine
      (i) Thrust in the strut [452.548N]
      (ii) Magnitude of reaction at A [388.33N]

5. Two light strings are perpendicular to each other and support a particle of weight 100N. The tension in one string is 40.0N. Calculate the angle this string with the vertical and the tension in the other string. [66.4^0, 91.7N]
6. A uniform pole AB of weight 5W and length 8a is suspended horizontally by two vertical strings attached to it at C and D where AC = BD = a.. A body of weight 9W hangs from the pole at E where ED = 2a. calculate the tension in each string [5.5W, 8.5W]
7. AB is a uniform rod of length 1.4m. it is pivoted at C where AC = 0.5m, and rests in horizontal equilibrium when weights of 16N and 8N are applied at A and B respectively. Calculate
   (a) Weight of the rod [4N]
   (b) The magnitude of the reaction at the pivot [28N]
8. A uniform rod AB of length 4a and weight W is smoothly hinged at its upper end A. the rod is held at 30^0 to the horizontal by a string which is at 90^0 to the rod and attached to it at C where AC = 3a. Find
   (a) The tension in the string [0.58W]
   (b) Reaction at A [0.578W]
9. A sphere of weight 40N and radius 30cm rests against a smooth vertical wall. The sphere is supported in this position by a string of length 20cm attached to a point on the sphere and to the point on the wall, find
   (a) Tension in the string [50N]
   (b) Reaction due to the wall[30N at 90^0 to the wall]
10. A smooth uniform rod AB of length 3a and eight 2w is pivoted at A so that it can rotate in a vertical plane. A light ring is free to slide over the rod. A light inextensible string is attached to the ring and passes over a fixed smooth peg at a point C a height 4a above A and carries a particle of weight w hanging freely



   (a) Show that the angle θ that the rod makes to the vertical in equilibrium is given by $\tan\theta = \frac{4}{3}$.
   (b) Find the magnitude of the force of the pivot on the rod at A in terms of w. $\left[\frac{3w\sqrt{5}}{3}\right]$.

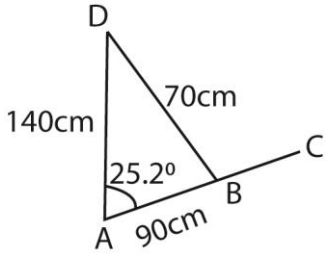
11. A smooth uniform rod AB of length 3a and weight W is pivoted at A so that it can rotate in a vertical plane. A light ring is free to slide over the rod. A light inextensible string is attached to the ring and passes over a fixed smooth peg at point C a height 4a above A and carries a particle of weight w hanging freely.



(a) Show that the angle θ that the rod makes to the vertical in equilibrium is given by tanθ= $\frac{8w}{3W}$.

(b) Find the smallest value of the ratio $\frac{w}{W}$ for which equilibrium is possible. $\left[\frac{\sqrt{7}}{8}\right]$

12. A uniform rod AC of mass 2kg and length 120cm hangs at rest in a vertical plane with end A in contact with a vertical wall. An inelastic string of length 70cm is attached to a point B on AC such that AB is 90cm. The other end of the string is attached to the wall at a point D, 140cm vertically above A



If the string is taut and angle DAC is 25.2^0, find

(a) Angle DBA [121.6^0]

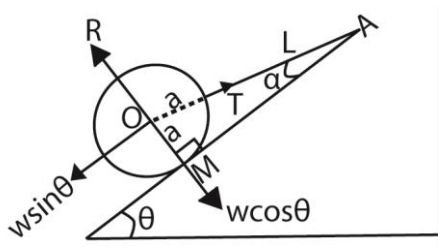
(b) The tension in the string [6.53N]

Beams on inclined plane

Example 31

A sphere of radius a and weight w rests on a smooth inclined plane supported by a string of length with one end attached to appoint on the surface of the sphere and the other end fastened to a point on the plane. If the angle of inclination of the plane to the horizontal is θ. Prove that the tension of the string is $\frac{w(a+L)sin\theta}{\sqrt{L^2+2aL}}$

Solution



$AM^2 = OA^2 - OM^2$

AM = $\sqrt{(a+L)^2 - a^2}$

AM = $\sqrt{L^2 + 2aL}$

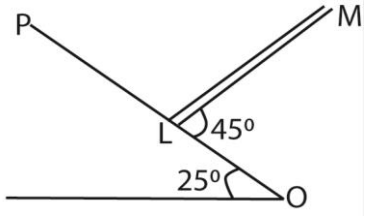
$cos\beta = \frac{\sqrt{L^2+2aL}}{(L+a)}$

Also wsinθ = Tcosα

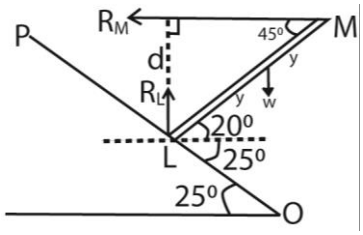
$T = \frac{wsin\theta}{cos\alpha} = \frac{w(a+L)sin\theta}{\sqrt{L^2+2aL}}$

Example 32

A uniform rod LM weight w rests with L on a smooth plane PO of inclination 250 as shown in the diagram below



The angle between LM and the plane is 450. What force parallel to PO applied at M will keep the rod in equilibrium?



R_M x d = w x ycons20

But d = 2ysin45

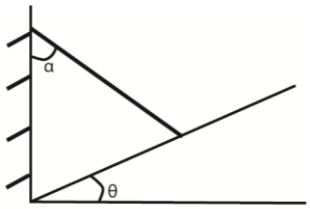
R_M x 2ysin45 = w x ycos20

R_M = 0.6645w

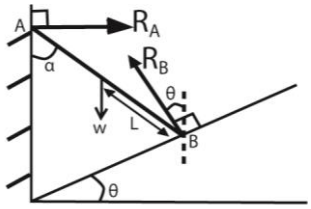
Taking moments about L

Example 33

The diagram shows a uniform ladder resting in equilibrium with its top end against a smooth vertical wall and its base on a smooth inclined plane. The plane makes an angle of θ with the horizontal and the ladder makes an angle α with the wall.



Prove that tanα = 2tanθ



Taking moments about B

R_A x 2Lcosα = w x Lsinα

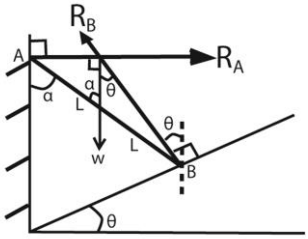
$R_A = \frac{wtan\alpha}{2}$ ..... (i)

(↑) $R_B$cosθ = w ..... (ii)

(→)$R_B$sinθ = R_A .....(iii)

(iii)÷ (ii)$\frac{R_B sin\theta}{R_B cos\theta} = \frac{w}{R_A}$; $R_A = wtan\theta$

$wtan\theta = \frac{wtan\alpha}{2}$; tanα = 2tanθ



Using cotangent rule for triangle

Lcotθ – Lcot90^0 = 2Lcotα

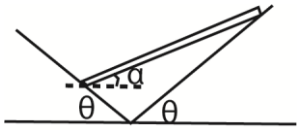
cot θ =2cotα

$\frac{1}{tan\theta} = \frac{2}{tan\alpha}$

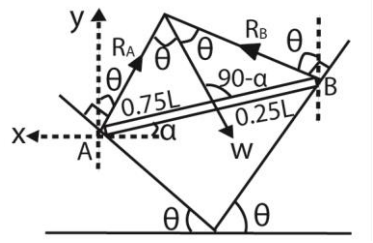
tanα = 2tanθ

Example 34

A rod AB of length L has its centre of gravity at a point G where AG = $\frac{1}{4}L$. The rod rests in equilibrium in a vertical plane at an angle α to the horizontal with its ends in contact with two inclined planes whose line of intersection is perpendicular to the rod. If the planes are smooth and equally inclined at an angle θ to the horizontal



Show that 2tanθtanα = 1 and the reaction on each plane is $\frac{w}{1+cos\theta}$



Using cotangent rule for triangle

$\frac{3Lcot\theta}{4} - \frac{L}{4cot\theta} = \left(\frac{3L}{4} + \frac{L}{4}\right)\cot(90^0 - \alpha)$

0.5cotθ = tanα

2tanθtanα = 1

$(\uparrow)R_A cos\theta + R_B cos\theta$= w

$R_B = w - R_A cos\theta$ .....(i)

$(\rightarrow)R_A cos\alpha + R_B cos\alpha$

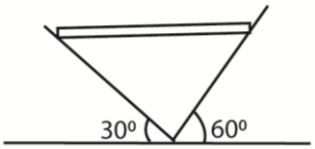
$R_B = R_A$ .....(ii)

$R_B = w - R_B cos\theta$

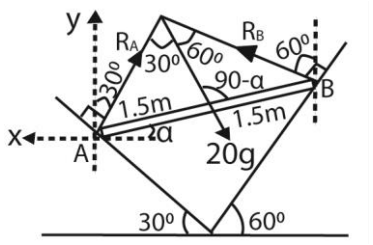
$R_B = R_A = \frac{w}{1+cos\theta}$

Example 35

A uniform road 3m long and of mass 29kg is placed on two smooth planes inclined at 30^0 and 60^0 to the horizontal



Find the reaction on each plane and the inclination of the rod to horizontal when it is in equilibrium.



$(\uparrow)\ R_A cos30 + R_B \cos 60 = 20g$

$R_B = 40g - R_A\sqrt{3}$.....(i)

$(\rightarrow)R_B \sin 60 = R_A \cos 30$

$R_{B\sqrt{3}} = R_A$ .....(ii)

$R_B = 40g - R_B\sqrt{3}\ x\ \sqrt{3}$

$R_B = 98N$

$R_A = 98\sqrt{3} = 169.74N$

Using cotangent rule for triangles

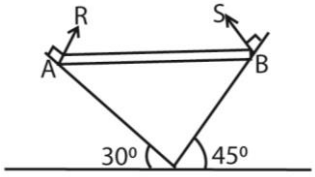
1.5cot30 – 1.5cot60 = 3cot(90 – α)

$1.5\sqrt{3} - 1.5\frac{1}{\sqrt{3}} = 3\tan\alpha$

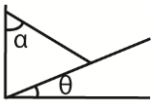
$\alpha = 30^0$

Revision exercise 5

1. A uniform rod of length 2a and mass m is placed on two smooth inclined planes at 30^0 and 60^0 to the horizontal. The normal reactions at the ends of the rod have magnitude R and S.



   (a) Show that R = $S\sqrt{2}$
   (b) Prove that the inclination of the rod to horizontal is $\cot^{-1}(1+\sqrt{3})$
2. A uniform rod rests with its end on two smooth planes inclined at 30^0 and 60^0 to the horizontal. A weight equal to twice that of the beam can slide along its length. Find the position of the sliding weight when the beam rests in horizontal position $\left[\frac{1}{8} of\ the length\ from\ 30^0 plane\right]$
3. The diagram shows a uniform ladder resting with its top end against a smooth vertical wall and its base on a smooth inclined plane. The plane makes an angle of θ with the horizontal and the ladder makes an angle of α with the wall.



   Find α when θ is
   (i) 10^0 [19.4^0]
   (ii) 30^0 [49.1^0]

Thank you

Dr. Bbosa Science

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A-level math paper 2: Correlations and scatter diagrams

Correlation refers to **the degree of correspondence or relationship between two variables**. Correlated variables tend to change together. If one variable gets larger, the other one systematically becomes either larger or smaller.

The degree of correlation/association is determined by rank correlation coefficients

There are two types

1. Spearman rank correlation coefficient (ρ)
2. Kenddall's rank correlation coefficient (τ

Interpretation of the rank correlation coefficients

A rank correlation coefficient measures the degree of similarity between two rankings

The table below is used

Correlation coefficient	Interpretation
0-0-19	Very low correlation
0.2-0.39	Low correlation
0.4-0.59	Moderate correlation
0.6 – 0.79	High correlation
0.8 – 1.0	Very high correlation

Note: the positive or negative signs indicate positive or negative relationships respectively. Or they the relationships are directly or inversely related.

The closer to zero the lower the relationship

Spearman rank correlation (ρ)

It is given by ρ = 1- $\left[\frac{6\sum d^2}{n(n^2-1)}\right]$

Where d = difference between ranks

n = total number of pairs

Example 1

Two examiners marked the scripts of 8 candidates. The table shows the marks awarded by two examiners x and y.

x	72	60	56	76	68	52	80	64
y	56	44	60	74	66	38	68	52

Calculate the rank correlation coefficient and comment on your reults

Solution

Rx	Ry	d	d^2
3	5	2	4
6	7	1	1
7	4	3	9
2	1	1	1
4	3	1	1
8	8	0	0
1	2	-1	1
5	5	1	1
			$\sum d^2 = 18$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 18}{8(8^2-1)}\right] = 0.786$

There is a high positive correlation between x and y

Example 2

The following shows the marks obtained by 10 students in mathematics and physics exams.

Mathematics	80	80	70	60	65	80	68	90	95	50
Physics	50	45	70	80	70	90	70	80	70	95

Calculate the ranks correlation coefficient and comment on your results

Solution

R_M	R_P	d	d^2
4	9	-5	25
6	6.5	0.5	0.25
9	3.5	5.5	30.25
8	6.5	1.5	2.25
4	2	2	4
7	6.5	0.5	0.25
2	3.5	1.5	2.25
1	6.5	-5.5	30.25
10	1	9	81
			$\sum d^2 = 211.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 211.5}{10(10^2-1)}\right] = -0.282$

There is a low negative correlation between mathematics and physics

Example 3

The following table gives the order in which six candidates were ranked in two tests x and y

x	E	C	B	F	D	A
y	F	A	D	E	C	C

Calculate the rank correlation coefficient and comment on your results

Solution

Rx	Ry	d	d^2
5	6	1	1
3	1	2	4
2	4	2	4
6	5	1	1
4	2.5	1.5	2.25
1	2.5	-1.5	225
			$\sum d^2 = 14.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 14.5}{6(6^2-1)}\right] = -0.586$

There is a moderate negative correlation between x and y

Kendall's Rank correlation coefficient

It is a coefficient that represents the degree of concordance/agreement between toe columns of ranked data. The greater the 'inversions' the smaller the coefficient will

Kendall's Rank correlation coefficient, τ (tau) = $\frac{C-D}{C+D}$

Where C = number of concordant pairs or pairs in agreement

D = number of disconcordant pairs or pairs in disagreement

The Tau correlation coefficient returns a value of 0 to 1, where: 0 is no relationship, 1 is a perfect relationship

- Concordant pairs are the number of observed ranks below a particular rank which are larger than that particular rank.
- Disconcordant pairs are the number of observed rank below a particular rank which are smaller than that particular rank

Example 4

Two examiners marked the scripts of 8 candidates. The table shows the marks awarded by two examiners x and y.

x	72	60	56	76	68	52	80	64
y	56	44	60	74	66	38	68	52

Calculate the rank correlation coefficient and comment on your results

Solution

Ranking values

The subscripts are the ranks

x	72_3	60_6	56_7	76_2	68_4	52_8	80_1	64_5
y	56_5	44_7	60_4	74_1	66_3	38_8	68_2	52_6

- The ranks of x and y are filled in the table as below those of x in ascending order and those of y correspondingly as shown in the table below.
- The values of C are bigger values in the column Ry bigger than and below a particular value in that column. While the values of D are sallerr values in the column Ry bigger than and below a particular value in that column

- For instance, the first value of C in the table below is the number of values bigger than and below 2 in column Ry i.e. (5, 3, 6, 7, 4, 8) =6; the first value D = the number of values smaller than and below 2 in column Ry; i.e. (1) = 1

Rx	Ry	C	D
1	2	6	1
2	1	6	0
3	5	3	2
4	3	3	0
5	6	2	1
6	7	1	1
7	4	1	0
8	8		
		$\sum C = 22$	$\sum D = 5$

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{22-5}{22+5}$ = 0.63

moderate positive correlation

Example 5

The height (cm) an ages (years) of random sample of ten farmers are given in the table below

Height (cm)	156	151	152	160	146	157	149	142	158	140
Ages (years)	47	38	44	55	46	49	54	52	45	30

(a)(i) Calculate the Kendall rank correlation coefficient

(ii) comment on your result (06marks)

Let the farmers be A, B, C, D, E, F, G, H, I, J (the subscripts are the rank)

Farmers	A	B	C	D	E	F	G	H	I	J
Height	156_4	151_6	152_5	160_1	146_8	157_3	149_7	142_9	158_2	140_{10}
Age	47_5	38_9	44_8	55_1	46_6	49_4	54_2	52_3	45_7	30_{10}

By re-arranging the findings we have

Farmers	D	I	F	A	C	B	G	E	H	J
Height	1	2	3	4	5	6	7	8	9	10
Age	1	7	4	5	8	9	2	6	3	10
Agreements (C)	9	3	5	4	2	1	3	1	1	=29
Disagreements (D)	0	4	2	2	3	3	0	1	0	=15

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{29-15}{29+15}$ = 0.32

low positive correlation

Significance of ranks correlation coefficients

The calculated ranks correlation coefficients to be statistically significant; the calculated value should be greater than that from the table critical values associated with the various sample sizes and significance levels (α).

That is

- ⇨ If the $|\rho_c| > |\rho_T|$ or $|\tau_c| > |\tau_T|$, a significant relationship exists
- ⇨ If the $|P_c| < |P_T|$ or $|\tau_c| < |\tau_T|$, no significant relationship exists

Where ρ_c = calculated Spearman's correlation coefficient

ρ_T = Table Spearman's correlation coefficient at either 1% (α=0.01) or 5% (α =0.05)

τ_c = calculated Kendall's correlation coefficient

τ_T = Table Kendall's correlation coefficient at either 1% (α=0.01) or 5% (α =0.05)

Example 6

The following shows the marks obtained by 8students in mathematics and physics exams

Mathematics	65	65	70	75	75	80	85	85
Physics	50	55	58	55	65	58	61	65

Calculate the ranks correlation coefficient and comment of the significance of your results at 5% level (Spearman's $\rho = 0.71$), $Kendall's\ \tau = 0.64$

(a) Using Spearman's correlation coefficient

R_M	R_P	d	d^2
7.5	8	0.5	0.25
7.5	6.5	1	1
6	4.5	1.5	2.25
4.5	6.5	-2	4
4.5	1.5	3	9
3	4.5	-1.5	2.25
1.5	3	-1.5	2.25
1.5	1.5	0	0
			$\sum d^2 = 21$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 21}{8(8^2-1)}\right] = 0.75$

Since $\rho_C(0.75) > \rho_T$ (0.71), a significant relationship exist

Using Kendall's correlation coefficient

R_M	R_P	C	D
1.5	1.5	6	0
1.5	3	5	1
3	4.5	3	1
4.5	1.5	4	0
4.5	6.5	1	1
6	4.5	2	0
7.5	6.5	1	0
7.5	8	=22	3

τ (tau) = $\frac{C-D}{C+D}$

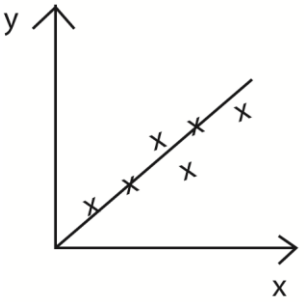
τ (tau) = $\frac{22-3}{22+3} = 0.76$

Since $\tau_C(0.75) > \tau_T$ (0.64), a significant relationship exist

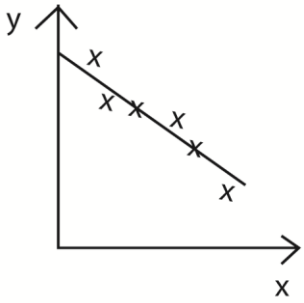
Scatter graphs

They are graphs showing the relationship between two variables

Positive relationship



Negative relationship



Example 7

The heights and masses of ten students are given in the table below

Height (cm)	156	152	152	146	160	157	149	142	158	68
Mass (kg)	62	58	63	58	70	60	55	57	68	56

(a)(i) Plot the data on a scatter diagram

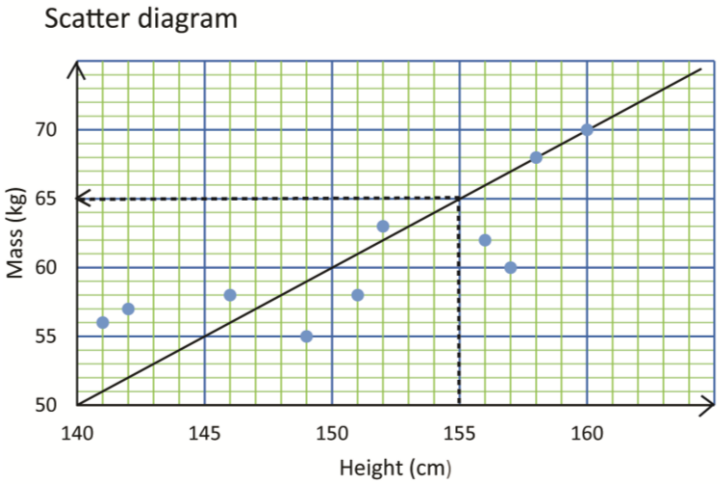
(ii) Draw the line of the best fit. Hence estimate the mass corresponding to height of 155cm

(b) (i) Calculate the rank correlation coefficient for the data.

(ii) Comment on the significance of the heights on the masses of the students.[Spearman's ρ = 0.79 and Kendall's τ = 0.64at 1% level of significance based on 10 observations]

Solution

(a)(i)



The weight corresponding to height 155cm is 65kg

Note: this value may vary from 63 to 67kg depending on how one has drawn the line of the best fit.

(b)(i) Using Spearman' ran correlation coefficient

Heights (x)	Mass (y)	Rx	Ry	d	d^2
156	62	4	4	0	0
151	58	6	6.5	0.5	0.25
152	63	5	3	2	4
146	58	8	6.5	1.5	2.25
160	70	1	1	0	0
157	60	3	5	-2	4
149	55	7	10	-3	9
142	57	9	8	1	1
158	68	2	2	0	0
141	56	10	9	1	1
					$\sum d^2 = 21.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 21.5}{10(10^2-1)}\right] = 0.8697$

Since $\rho_C(0.87) > \rho_T(0.79)$, a significant relationship exist between height and weight of students at 1% level

Using Kendall's method

By naming the pairs we have

A(156, 62), B(151, 58), C(152, 63), D(146, 58), E(160, 70), F(157, 60), G149, 55), H(142, 57), I(158, 68), J(141, 56)

	E	I	F	A	C	B	G	D	H	J
x	1	2	3	4	5	6	7	8	9	10
y	1	2	5	4	3	6.5	10	6.5	8	9
C	9	8	5	5	5	3	0	2	1	=38
D	0	0	2	1	0	0	3	0	0	=6

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{38-6}{38+6}$ = 0.73

Since $\tau_C(0.73) > \tau_T(0.64)$, a significant relationship exist between the heights and masses of student.

Revision questions

1. UNEB 1990/212
   Eight candidate seeing admission to a University sat for written and oral test. The scores were shown below

Written (x)	55	54	35	62	87	53	71	50
Oral (y)	57	60	47	65	83	56	74	63

   (a) Plot the result on a scatter diagram. Comment on the relationship between the written test and oral test
   (b) Draw the line of the best fit on your graph and use it to estimate y when x = 70.
   (c) Calculate the rank correlation coefficient. Comment on your results
2. UNEB 1990/2/12
   The pairs of observation have been made on two random variables x and y. the ten (x, y) are (0, 20), (-7, 12), (-10, 15), (-12, 22), (-17, 5), (-30, -5), (-32, 13), (10, 30), (15, 40), and (-12, 8).
   (a) Draw the results on a scatter diagram
   (b) Draw the line of the best fit

(c) Estimate the expected value of y corresponding to x = -7

(d) Calculate the rank correlation coefficient and comment on the significance of the results at 1% significance level. ($\rho = 0.894$, $\tau = 0.778$)

3. UNEB 1991/2/12

Three examiners X, Y and Z each marked scripts of ten candidates who sat for mathematics examination. The table below shows the examiner's ranking of candidates.

	A	B	C	D	E	F	G	H	I	J
X	8	5	9	2	10	1	7	6	3	4
Y	5	3	6	1	4	7	2	10	8	9
Z	6	3	7	2	5	4	1	10	9	8

Calculate the coefficient of rank correlation of the rankings

(i) X and Y

(ii) Y and Z

(iii) Comment on the significance of each at 5% significant level

4. UNEB 1992/2/13

Three weighing scales from three different shops W, X and Y in a market were used to weigh 10 bags of beans (A, B, C……) and the results in (kg) were given in the table below

	A	B	C	D	E	F	G	H	I	J
W	65	68	70	63	64	62	73	75	72	78
X	63	68	68	60	65	60	72	73	70	66
Y	63	74	78	75	64	73	79	70	67	79

Determine the rank correlation coefficient for the performance of the scales

(i) W and X

(ii) X and Y

(iii) Which of the three scales W, X and Y were in good working conditions

5. UNEB 1994/2/14

(a) In many government institution, officers complain about typing errors. A test was designed to investigate the relationship between typing speed and errors made. Twelve typist A, B, C …L were picked at random to type a text. The table below shows the rankings of the typist according to speed and errors made. (N.B lowest ranking in error implies the least errors

Typist	A	B	C	D	E	F	G	H	I	J	K	L
Speed	3	4	2	1	8	11	10	6	7	12	5	9
Errors	2	6	5	1	10	9	8	3	4	12	7	11

(i) Calculate the coefficient of rank correlation

(ii) Comment on the significance at 1% significance level

(b) The cost of travelling at a certain distance away from the city centre is found to depend on the route and distance a given place is away from the centre. The table below gives average rates of travel charged for distances to be travelled away from the city centre

Distance (s km)	9	12	14	21	24	30	33	45	46	50
Rate charged (r shs)	750	1000	1150	1200	1350	1250	1400	1750	1600	2000

(i) Plot the above data on a scatter diagram and draw a line of best fit through the points of the scatter diagram

(ii) Estimate the expected value r corresponding to s = 40km

6. UNEB/1995/2/13

In a certain commercial institution, a speed and error typing examination was administered to 12 randomly selected candidates A, B, C, …, L of the institution. The table below shows their speed (y) in seconds and the number of errors in their typed scripts (x)

	A	B	C	D	E	F	G	H	I	J	K	L
No. of errors (x)	12	24	20	10	32	30	28	15	18	40	27	35
Speed (y) in seconds	130	136	124	120	153	160	155	142	145	172	140	157

(i) Calculate the coefficient of rank correlation of the ranking

(ii) Comment on your results

(iii) Plot the above data on a scatter diagram and draw a line of the best fit through the points of the scatter diagram

7. UNEB 1996/2/16

The following table gives the marks obtained in calculus, physics and statistics by seven students

Calculus	72	50	60	55	35	48	82
Physics	61	55	70	50	30	50	73
Statistics	50	40	62	70	40	40	60

Determine the rank correlation coefficient for the performance of students in

(i) Calculus and physics

(ii) Calculus and statistics

8. UNEB/1999/2/8

Given the table below

x	80	75	86	60	75	92	86	50	64	75
y	62	58	60	45	68	68	81	48	50	70

Determine the rank correlation coefficient between the variable x and y, comment on your results

9. UNEB2003/2/15

The table below shows the percentage of sand y in the soils at different depth x (in cm)

Soil depth (x)(cm)	35	65	55	25	45	75	20	90	51	60
% of sand, y	86	70	84	92	79	68	96	58	86	77

(a) Plot the results on a scatter diagram. Comment on the relationship between the depth of the soil and the percentage of sand in the soil

(b) Draw the line of the best fit on you graph and use it to estimate

(i) The percentage of sand in the soil at a depth of 31cm

(ii) Depth of the soil with 54% sand

(iii) Calculate the rank correlation coefficient

10. UNEB 2004/2/7

Eight applicants for a certain job obtained the following marks in aptitude and written test

Applicants	A	B	C	D	E	F	G	H
Aptitude test	33	45	16	42	45	35	40	48
Written test	57	60	40	75	68	48	54	68

(i) Calculate the coefficient of rank correlation of applicant's performance in the two tests

(ii) Comment on your results

11. UNEB 2005/2/7

The table below shows the marks scored by students in mathematics and fine art tests

students	A	B	C	D	E	F	G	H	I	J
Mathematics,	40	48	79	26	55	35	37	70	60	40
Fine art	59	62	68	47	46	39	63	29	55	67

Calculate the coefficient of rank correlation for the students' performance in the two subjects and comment on your results.

12. UNEB 2007/2/12

Below are marks scored by 8 students A, B, C … H in Mathematics, Economics and geography in the end of term examination.

	A	B	C	D	E	F	G	H
Math	52	75	41	60	81	31	65	52
Economics	50	60	35	65	66	45	60	48
Geography	59	62	68	54	63	40	55	72

Determine the rank correlation coefficient for the performance of students in

(i) Math and economics

(ii) Geography and math

(iii) Comment on the significance of the math in performance of economics and geography. ($\rho = 0.86$, $\tau = 0.79$ based on 8 observations at a 1% level of significance)

13. UNEB 2011/2/12

The heights and ages of ten students are given in the table below

Height, cm	156	151	152	146	160	157	149	142	158	140
Mass, kg	62	58	63	58	70	60	55	57	68	56

(a) Plot the data on a scatter diagram

(b) Draw the line of best fit on you graph and use it to estimate the mass corresponding to a height of 155cm

(c) Calculate the rank correlation coefficient for the data. Comment on the significance of the height on masses of students ($\rho = 0.79$, τ 0.64 based on 10 observations at 1% level of significance.

14. UNEB 2013/2/9

The heights and ages of ten farmers are given in the table below

Height, cm	156	151	152	160	146	157	149	142	158	140
Age, years	47	38	44	55	46	49	45	30	45	20

(a) Plot the data on a scatter diagram

(b) Draw the line of best fit on your diagram and use it to estimate

(i) Age when height = 147

(ii) Height when the age is 43

(c) Calculate the rank correlation coefficient for the data. Comment on your results (ρ=0.752, τ= 0.6)

15. UNEB 2015/2/12

The table gives the points awarded to eight schools by three judges, J_1, J_2 and J_3 during a music competition. J_1 was the chief judge.

J_1	72	50	50	55	35	38	82	72
J_2	60	55	70	50	50	50	73	70
J_3	50	40	62	70	40	48	67	67

(a) Determine the rank correlation coefficients between the judges of

(i) J_1 and J_2

(ii) J_1 and J_3

(b) Who of the two judges had a better correlation with the chief judge? Give a reason.

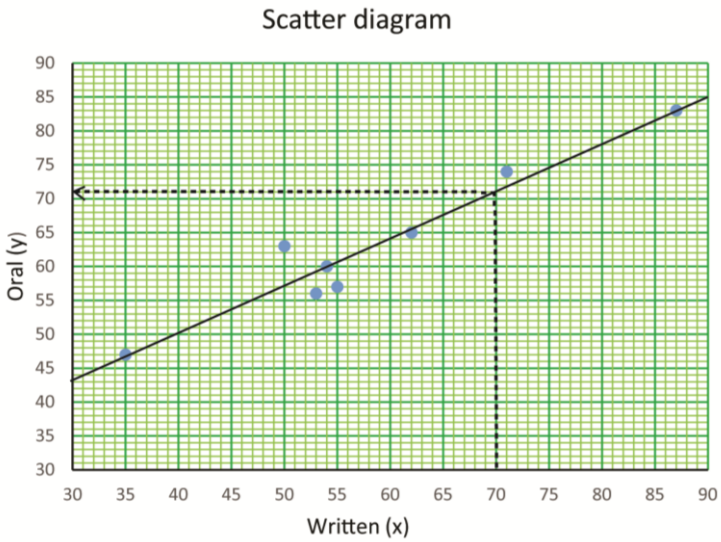
Answers to the revision exercise

1. UNEB 1990/212

Eight candidate seeing admission to a University sat for written and oral test. The scores were shown below

Written (x)	55	54	35	62	87	53	71	50
Oral (y)	57	60	47	65	83	56	74	63

(a) Plot the result on a scatter diagram. Comment on the relationship between the written test and oral test



(b) Draw the line of the best fit on your graph and use it to estimate y when x = 70. The value of y when x = 70 is 71

(c) Calculate the rank correlation coefficient. Comment on your results

Using Spearman's rank correlation method

x	y	Rx	Ry	d	d^2
55	57	4	6	-2	4
54	60	5	5	0	0
35	47	8	8	0	0
62	65	3	3	0	0
87	83	1	1	0	0
53	56	6	7	-1	1
71	74	2	2	0	0
50	63	7	4	3	9
					$\sum d^2$ =14

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 14}{8(8^2-1)}\right] = 0.833$

There is very high correlation between x and y

Using Kendall' Rank correlation coefficient method

Rx	Ry	C	D
1	1	7	0
2	2	6	0
3	3	5	0
4	6	3	1
5	5	2	1
6	7	1	1
7	4	1	0
8	8	=25	=3

$\tau = \frac{C-D}{C+D} = \frac{25-3}{25+3} = 0.786$

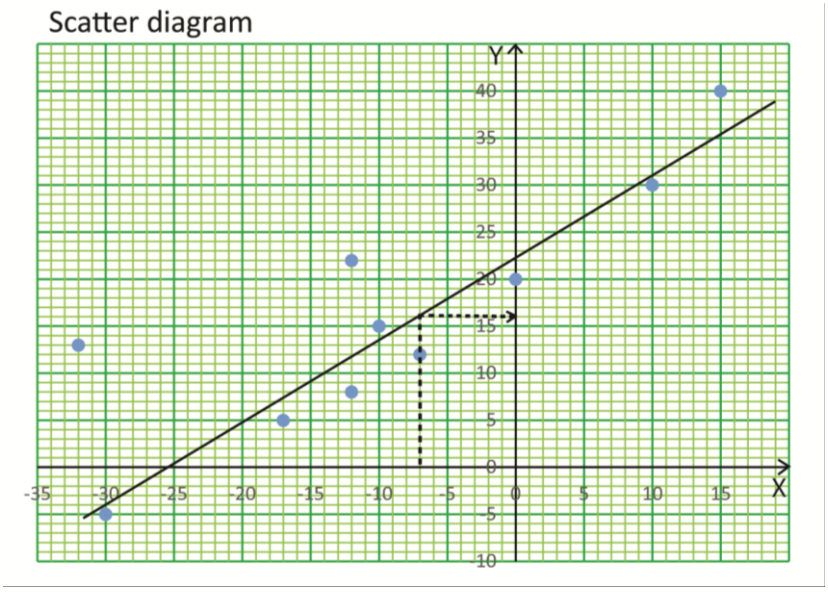
there is high correlation between x and y

2. UNEB 1990/2/12

The pairs of observation have been made on two random variables x and y. the ten (x, y) are (0, 20), (-7, 12), (-10, 15), (-12, 22), (-17, 5), (-30, -5), (-32, 13), (10, 30), (15, 40), and (-12, 8).

(a) Draw the results on a scatter diagram

(b) Draw the line of the best fit



(c) Estimate the expected value of y corresponding to x = -7

The value of y corresponding to x = -7 is 15.7

(d) Calculate the rank correlation coefficient and comment on the significance of the results at 1% significance level. (ρ = 0.894, τ= 0.778)

Using Spearman's rank correlation coefficient

X	Y	Rx	Ry	d	d^2
0	20	3	4	-1	1
-7	12	4	7	-3	9
-10	15	5	5	0	0
-12	22	6.5	3	3.5	12.25
-17	5	8	9	-1	1
-30	-5	9	10	-1	1
-32	13	10	6	4	16
10	30	2	2	0	0
15	40	1	1	0	0
-12	8	6.5	8	-1.5	2.25
					$\sum d^2$ =42.5

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 42.5}{10(10^2-1)}\right] = 0.742$

Since ρ_C (0.742)<ρ_T (0.894)

There is no significant relationship between X and Y at 1% significance level

Using Kendall's rank correlation coefficient

Rx	Ry	C	D
1	1	9	0
2	2	8	0
3	4	6	1
4	7	3	3
5	5	4	1
6.5	3	4	0
6.5	8	2	1
8	9	1	1
9	10	0	1
10	6	=37	=8

$\tau = \frac{C-D}{C+D} = \frac{37-8}{37+8} = 0.644$

Since τ_C (0.644)<τ_T (0.778)

There is no significant relationship between X and Y at 1% significance level.

3. UNEB 1991/2/12

Three examiners X, Y and Z each marked scripts of ten candidates who sat for mathematics examination. The table below shows the examiner's ranking of candidates.

	A	B	C	D	E	F	G	H	I	J
X	8	5	9	2	10	1	7	6	3	4
Y	5	3	6	1	4	7	2	10	8	9
Z	6	3	7	2	5	4	1	10	9	8

Calculate the coefficient of rank correlation of the rankings

(i) X and Y

Spearman's rank correlation

Rx	Ry	d	d^2
8	5	3	9
5	3	2	4
9	6	3	9
2	1	1	1
10	4	6	36
1	7	-6	36
7	2	5	25
6	10	-4	16
3	8	-5	25
4	9	-5	25
			$\sum d^2$ =186

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 186}{10(10^2-1)}\right]$ = -0.127

There is low negative relationship between X and Y

Kendall's rank correlation coefficient

Rx	Ry	C	D
1	7	3	6
2	1	8	0
3	8	2	5
4	9	1	5
5	3	4	1
6	10	0	4
7	2	3	0
8	5	1	1
9	6	0	1
10	4	=22	=23

$\tau = \frac{C-D}{C+D} = \frac{22-23}{22+23}$ = -0.022

There is very low negative relationship between X and Y at 1% significance level.

(ii) Y and Z

Spearman's rank correlation coefficient

Ry	Rz	d	d^2
5	6	-1	1
3	3	0	0
6	7	-1	1
1	2	-1	1
4	5	-1	1
7	4	3	9
2	1	1	1
10	10	0	0
8	9	-1	1
9	8	1	1
			$\sum d^2$ =16

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 16}{10(10^2-1)}\right]$ = 0.903

There is high positive relationship between Y and Z

Kendall's rank correlation coefficient

Rz	Ry	C	D
1	2	9	1
2	1	8	0
3	3	7	0
4	7	3	3
5	4	5	0
6	5	4	0
7	6	3	0
8	9	1	1
9	8	1	0
10	10	41	5

$\tau = \frac{C-D}{C+D} = \frac{41-5}{41+5} = 0.7826$

There is very high relationship between X and Y

(iii) Comment on the significance of each at 5% significant level

(From the tables of critical values at 5% level of significance based on 10 observations, ρ = 0.648 , τ = 0.467)

(i) For X and Y since $|\rho_C(-0.127)| < |\rho_T(0.648)|$ and $|\tau_C(-0.022)| < |\tau\rho_T(0.467)|$; there is no significant relationship between X and Y at 5% significance level.

(ii) For Y and Z since $|\rho_C(0.903)| > |\rho_T(0.648)|$ and $|\tau_C(0.7826)| > |\tau\rho_T(0.467)|$; there is significant relationship between Y and Z at 5% significance level.

4. UNEB 1992/2/13

Three weighing scales from three different shops W, X and Y in a market were used to weigh 10 bags of beans (A, B, C……) and the results in (kg) were given in the table below

	A	B	C	D	E	F	G	H	I	J
W	65	68	70	63	64	62	73	75	72	78
X	63	68	68	60	65	60	72	73	70	66
Y	63	74	78	75	64	73	79	70	67	79

Determine the rank correlation coefficient for the performance of the scales

(i) W and X (ρ = 0.8)

Spearman's rank correlation

W	X	RW	RX	d	d^2
65	63	7	8	-1	1
68	68	6	4.5	1.5	2.25
70	68	5	4.5	0.5	0.25
63	60	9	9.5	-0.5	0.25
64	65	8	7	1	1
62	60	10	9.5	0.5	0.25
73	72	3	2	1	1
75	73	2	1	1	1
72	70	4	3	1	1
78	66	1	6	-5	25
					$\sum d^2 = 33$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 33}{10(10^2-1)}\right] = 0.8$

There is high positive relationship between Y and Z

Kendall's correlation coefficient

Rw	Rx	C	D
1	6	4	5
2	1	8	0
3	2	7	0
4	3	6	0
5	4.5	4	0
6	4.5	4	0
7	8	2	1
8	7	2	0
9	9.5	0	0
10	9.5	=37	=6

$\tau = \frac{C-D}{C+D} = \frac{37-6}{37+6} = 0.721$

There is high relationship positive between X and Y

(ii) X and Y (ρ = 0.185)

Spearman's correlation coefficient

X	Y	Rx	Ry	d	d^2
63	63	8	10	-2	4
68	74	4.5	4	0.5	0.25
68	78	4.5	3	1.5	2.25
60	75	9.5	4	5.5	30.25
65	64	7	9	-2	4
60	73	9.5	6	3.5	12.25
72	79	2	1.5	0.5	0.25
73	70	1	7	-6	36
70	67	3	8	-5	25
66	79	6	1.5	4.5	20.25
					$\sum d^2$ =134.5

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 134.5}{10(10^2-1)}\right] = 0.185$

There is very low positive relationship between Y and Z

Kendall's rank coefficient

Rx	Ry	C	D
1	7	3	6
2	1.5	7	0
3	8	2	5
4.5	4	3	2
4.5	3	4	1
6	1.5	4	0
7	9	1	2
8	10	0	2
9.5	4	1	0
9.5	6	=25	=18

$\tau = \frac{C-D}{C+D} = \frac{25-18}{25+18} = 0.163$

There is very low relationship positive between X and Y

(iii) Which of the three scales W, X and Y were in good working conditions

W and X are in good working conditions because the show high positive correlation

5. UNEB 1994/2/14

(a) In many government institution, officers complain about typing errors. A test was designed to investigate the relationship between typing speed and errors made. Twelve typist A, B, C …L were picked at random to type a text. The table below shows the rankings of the typist according to speed and errors made. (N.B lowest ranking in error implies the least errors

Typist	A	B	C	D	E	F	G	H	I	J	K	L
Speed	3	4	2	1	8	11	10	6	7	12	5	9
Errors	2	6	5	1	10	9	8	3	4	12	7	11

(i) Calculate the coefficient of rank correlation

Spearman's rank coefficient

Rs	Re	d	d^2
3	2	1	1
4	6	-2	4
2	5	-3	9
1	1	0	0
8	10	-2	4
11	9	2	4
10	8	2	4
6	3	3	9
7	4	3	9
12	12	0	0
5	7	-2	4
9	11	-2	4
			$\sum d^2$ =52

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 52}{12(12^2-1)}\right] = 0.8182$

There is very low positive relationship between Y and Z

Kendall's rank correlation coefficient

Rs	Re	C	D
1	1	11	0
2	5	7	3
3	2	9	0
4	6	6	2
5	7	5	2
6	3	6	0
7	4	5	0
8	10	2	2
9	11	1	2
10	8	2	0
11	9	1	0
12	12	55	11

$\tau = \frac{C-D}{C+D} = \frac{55-11}{55+11} = 0.667$

There is very low relationship positive between X and Y

(ii) Comment on the significance at 1% significance level

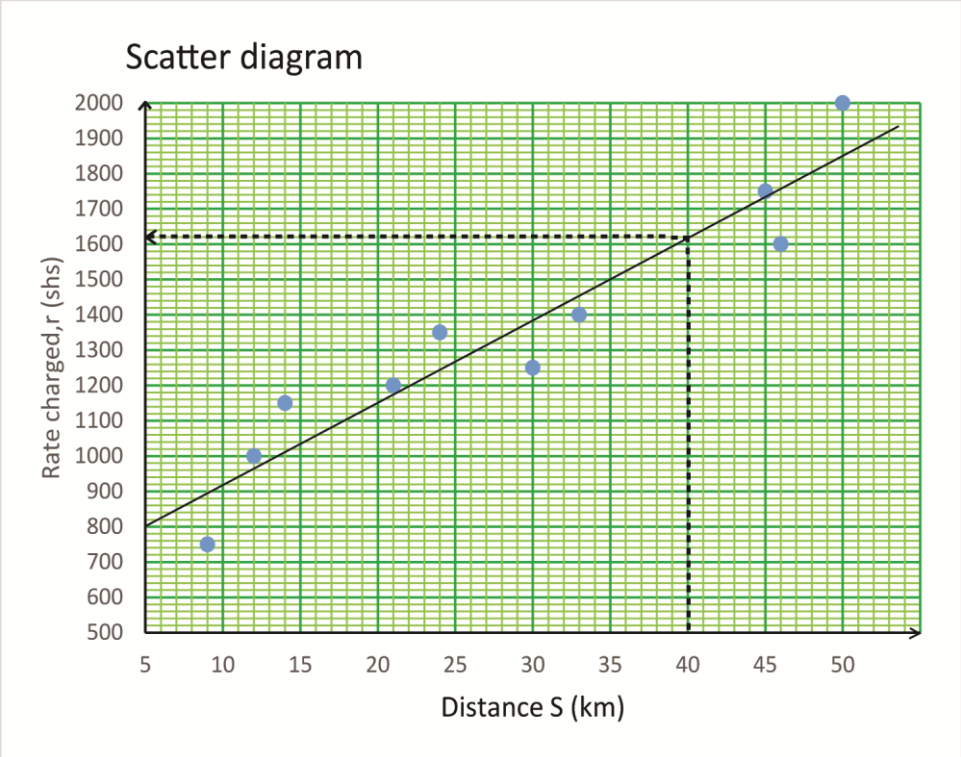
(From the tables of critical values at 1% level of significance based on 12 observations, ρ = 0.727 , τ = 0.545)

Since $|\rho_C(0.8182)| > |\rho_T(0.727)|$ and $|\tau_C(0.667)| > |\tau\rho_T(0.545)|$; there is significant relationship between speed and errors at 1% significance level.

(b) The cost of travelling at a certain distance away from the city centre is found to depend on the route and distance a given place is away from the centre. The table below gives average rates of travel charged for distances to be travelled away from the city centre

Distance (s km)	9	12	14	21	24	30	33	45	46	50
Rate charged (r shs)	750	1000	1150	1200	1350	1250	1400	1750	1600	2000

(i) Plot the above data on a scatter diagram and draw a line of best fit through the points of the scatter diagram



(ii) Estimate the expected value r corresponding to s = 40km

Value of r corresponding to s = 40 is shs 1610

6. UNEB/1995/2/13

In a certain commercial institution, a speed and error typing examination was administered to 12 randomly selected candidates A, B, C, …, L of the institution. The table below shows their speed (y) in seconds and the number of errors in their typed scripts (x)

	A	B	C	D	E	F	G	H	I	J	K	L
No. of errors (x)	12	24	20	10	32	30	28	15	18	40	27	35
Speed (y) in seconds	130	136	124	120	153	160	155	142	145	172	140	157

(i) Calculate the coefficient of rank correlation of the ranking

Spearman's rank correlation

Errors	speed	Re	Rs	d	d^2
12	130	11	10	1	1
24	136	7	9	-2	4
20	124	8	11	-3	9
10	120	12	12	0	0
32	153	3	5	-2	4
30	160	4	2	2	4
28	155	5	4	1	1
15	142	10	7	3	9
18	145	9	6	3	9
40	172	1	1	0	0
27	140	6	8	-2	4
35	157	2	3	-1	1
					46

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 46}{12(12^2-1)}\right] = 0.839$

There is high positive relationship between speed and errors made

Kendall's rank correlation

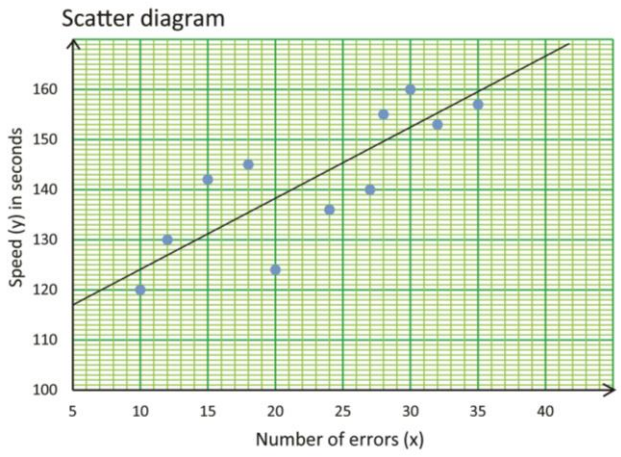
Re	Rs	C	D
1	1	11	0
2	3	9	1
3	5	7	2
4	2	8	0
5	4	7	0
6	8	4	2
7	9	3	2
8	11	2	2
9	6	3	0
10	7	2	0
11	10	1	0
12	12	57	9

$\tau = \frac{C-D}{C+D} = \frac{57-9}{57+9} = 0.72$

There is high positive relationship between speed and errors made

(ii) Comment on your results

(iii) Plot the above data on a scatter diagram and draw a line of the best fit through the points of the scatter diagram



7. UNEB 1996/2/16

The following table gives the marks obtained in calculus, physics and statistics by seven students

Calculus	72	50	60	55	35	48	82
Physics	61	55	70	50	30	50	73
Statistics	50	40	62	70	40	40	60

Determine the rank correlation coefficient for the performance of students in

(i) Calculus and physics

Spearman's correlation coefficient

C	P	Rc	Rp	d	d^2
72	61	2	3	-1	1
50	55	5	4	1	1
60	70	3	2	1	1
55	50	4	5.5	-1.5	2.25
35	30	7	7	0	0
48	50	6	5.5	0.5	0.25
82	73	1	1	0	0
					$\sum d^2 = 5.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 5.5}{7(7^2-1)}\right] = 0.902$

There is high positive relationship between calculus and physics

Kendall' correlation coefficient

Rc	Rp	C	D
1	1	6	0
2	3	4	1
3	2	4	0
4	5.5	1	1
5	4	2	0
6	5.5	1	0
7	7	=18	=2

$\tau = \frac{C-D}{C+D} = \frac{18-2}{18+2} = 0.8$

There is high positive relationship between calculus and physics

(ii) Calculus and statistics ($\rho = 0.64$)

Spearman's correlation

C	S	Rc	Rs	d	d^2
72	50	2	4	-2	4
50	40	5	6	-1	1
60	62	3	2	1	1
55	70	4	1	3	9
35	40	7	6	1	1
48	40	6	6	0	0
82	60	1	3	-2	4
					$\sum d^2 = 20$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6 \times 20}{7(7^2-1)}\right] = 0.643$

Kendall's rank correlation coefficient

Rc	Rs	C	D
1	3	4	2
2	4	3	2
3	2	3	1
4	1	3	0
5	6	0	0
6	6	0	0
7	6	=13	=5

$\tau = \frac{C-D}{C+D} = \frac{13-5}{13+5} = 0.444$

8. UNEB/1999/2/8

Given the table below

x	80	75	86	60	75	92	86	50	64	75
y	62	58	60	45	68	68	81	48	50	70

Determine the rank correlation coefficient between the variable x and y, comment on your results (ρ = 0.715)

Spearman's correlation

x	y	Rx	Ry	d	d^2
80	62	4	5	-1	1
75	58	6	7	-1	1
86	60	2.5	6	-3.5	12.25
60	45	9	10	-1	1
75	68	6	3.5	2.5	6.25
92	68	1	3.5	-2.5	6.25
86	81	2.5	1	1.5	2.25
50	48	10	9	1	1
64	50	8	8	0	0
75	70	6	2	4	16
					47

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 47}{10(10^2-1)}\right] = 0.715$

Kendall's correlation coefficient

Rx	Ry	C	D
1	3.5	6	2
2.5	6	4	4
2.5	1	7	0
4	5	4	2
6	7	3	2
6	3.5	3	2
6	2	3	0
8	8	2	0
9	10	0	1
10	9	=32	=13

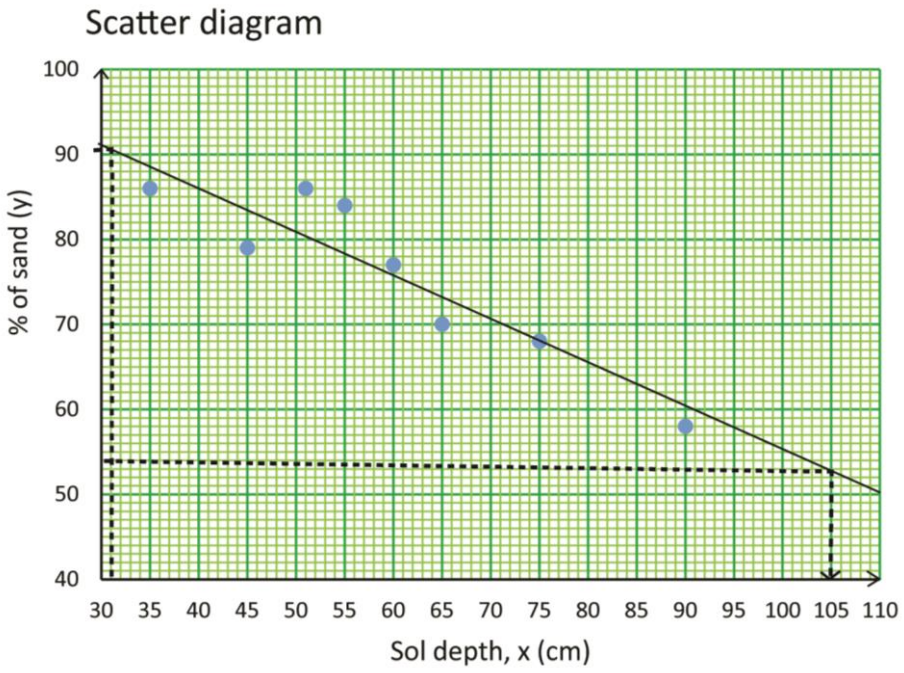
$\tau = \frac{C-D}{C+D} = \frac{32-13}{32+13} = 0.422$

9. UNEB2003/2/15

The table below shows the percentage of sand y in the soils at different depth x (in cm)

Soil depth (x)(cm)	35	65	55	25	45	75	20	90	51	60
% of sand, y	86	70	84	92	79	68	96	58	86	77

(a) Plot the results on a scatter diagram. Comment on the relationship between the depth of the soil and the percentage of sand in the soil



(b) Draw the line of the best fit on you graph and use it to estimate

(i) The percentage of sand in the soil at a depth of 31cm (91%)

(ii) Depth of the soil with 54% sand (105cm)

(iii) Calculate the rank correlation coefficient

Spearman's rank correlation coefficient

soil depth (x)	%of sand (y)	Rx	Ry	d	d^2
35	86	8	3.5	4.5	20.25
65	70	3	8	-5	25
55	84	5	5	0	0
25	92	9	2	7	49
45	79	7	6	1	1
75	68	2	9	-7	49
20	96	10	1	9	81
90	58	1	10	-9	81
51	86	6	3.5	2.5	6.25
60	77	4	7	-3	9
					$\sum d^2$ =321.5

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 321.5}{10(10^2-1)}\right] = -0.948$

Kendall's correlation coefficient

Rx	Ry	C	D
1	10	0	9
2	9	0	8
3	8	0	7
4	7	0	6
5	5	1	4
6	3.5	1	2
7	6	0	3
8	3.5	0	2
9	2	0	1
10	1	2	42

$\tau = \frac{C-D}{C+D} = \frac{2-42}{2+42} =$ -0.91

10. UNEB 2004/2/7

Eight applicants for a certain job obtained the following marks in aptitude and written test

Applicants	A	B	C	D	E	F	G	H
Aptitude test	33	45	16	42	45	35	40	48
Written test	57	60	40	75	68	48	54	68

(i) Calculate the coefficient of rank correlation of applicant's performance in the two tests

Aptitude (A)	Written (W)	R_A	R_W	d	d^2
33	57	7	5	2	4
45	60	2	4	-2	4
16	40	8	8	0	0
42	75	4	1	3	9
45	68	3	2.5	0.5	0.25
35	48	6	7	-1	1
40	54	5	6	-1	1
48	68	1	2.5	-1.5	2.25
					21.5

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 21.5}{8(8^2-1)}\right] = 0.744$

(ii) There is high positive relationship between Aptitude and written

Kendall's correlation coefficient

Rx	Ry	C	D
1	2.5	5	1
2	4	4	2
3	2.5	4	1
4	1	4	0
5	6	2	1
6	7	1	1
7	5	1	0
8	8	=21	=6

$\tau = \frac{C-D}{C+D} = \frac{21-6}{21+6} = 0.555$

(ii) There is moderate positive relationship between Aptitude and written

(ii) Comment on your results

11. UNEB 2005/2/7

The table below shows the marks scored by students in mathematics and fine art tests

students	A	B	C	D	E	F	G	H	I	J
Mathematics,	40	48	79	26	55	35	37	70	60	40
Fine art	59	62	68	47	46	39	63	29	55	67

Calculate the coefficient of rank correlation for the students' performance in the two subjects and comment on your results.

Math	Fine Art	R_M	R_F	d	d^2
40	59	6.5	5	1.5	2.25
48	62	5	4	1	1
79	68	1	1	0	0
26	47	10	7	3	9
55	46	4	8	-4	16
35	39	9	9	0	0
37	63	8	3	5	25
70	29	2	10	-8	64
60	55	3	6	-3	9
40	67	6.5	2	4.5	20.25
					$\sum d^2$ =146.5

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 146.5}{10(10^2-1)}\right]$

$= 0.112$

Kendall's correlation coefficient

RM	RF	C	D
1	1	9	0
2	10	0	8
3	6	3	4
4	8	1	5
5	4	3	2
6.5	5	2	2
6.5	2	3	0
8	3	2	0
9	9	0	1
10	7	23	22

$\tau = \frac{C-D}{C+D} = \frac{23-22}{23+22} = 0.022$

12. UNEB 2007/2/12

Below are marks scored by 8 students A, B, C … H in Mathematics, Economics and geography in the end of term examination.

	A	B	C	D	E	F	G	H
Math	52	75	41	60	81	31	65	52
Economics	50	60	35	65	66	45	69	48
Geography	35	40	60	54	63	40	65	72

Determine the rank correlation coefficient for the performance of students in

(i) Math and economics

Spearman's correlation coefficient

Math, M	Economic C	R_M	R_E	d	d^2
52	50	5.5	5	0.5	0.25
75	60	2	4	-2	4
41	35	7	8	-1	1
60	65	4	3	1	1
81	66	1	2	-1	1
31	45	8	7	1	1
65	69	3	1	2	4
52	48	5.5	6	-0.5	0.25
					$\sum d^2$ =12.5

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 12.5}{10(10^2-1)}\right]$

= 0.851

Kendall's correlation coefficient

RM	RE	C	D
1	2	6	1
2	4	4	2
3	1	5	0
4	3	4	0
5.5	5	3	0
5.5	6	2	0
7	8	1	1
8	7	=25	=4

$\tau = \frac{C-D}{C+D} = \frac{25-4}{25+4}$ = 0.724

(ii) Geography and math

Spearman's rank correlation coefficient

Geog, G	Math, M	R_G	R_M	d	d^2
35	52	8	5.5	2.5	6.25
40	75	6.5	2	4.5	20.25
60	41	4	7	-3	9
54	60	5	4	1	1
63	81	3	1	2	4
40	31	6.5	8	-1.5	2.25
65	65	2	3	-1	1
72	52	1	5.5	-4.5	20.25
					64

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right]$

$= 1 - \left[\frac{6\ x\ 64}{8(8^2-1)}\right]$

= 0.238

Kendall's rank coefficient

RG	RM	C	D
1	5.5	2	4
2	3	4	2
3	1	5	0
4	7	1	3
5	4	2	1
6.5	2	2	0
6.5	8	0	1
8	5.5	16	11

$\tau = \frac{C-D}{C+D} = \frac{16-11}{16+11}$ = 0.185

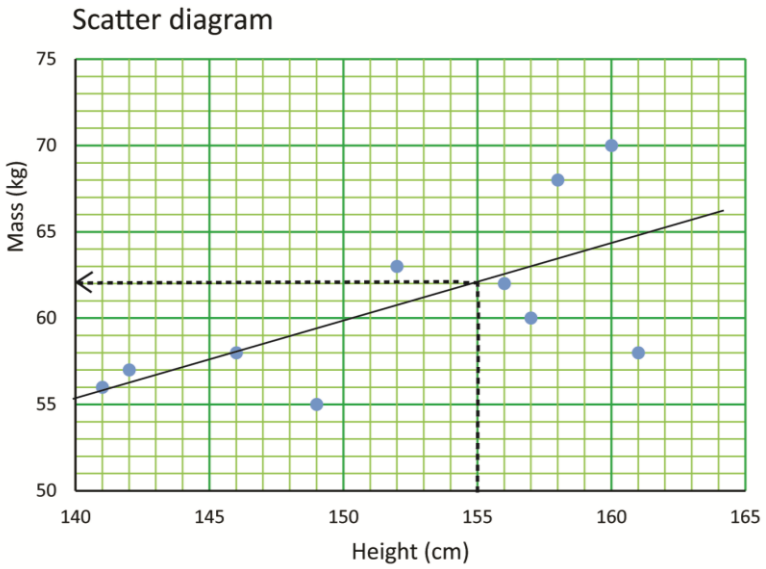
(iii) Comment on the significance of the math in performance of economics and geography. (ρ = 0.86, τ = 0.79 based on 8 observations at a 1% level of significance)
Since $|\rho_C(0.851)| < |\rho_T(0.86)|$ and $|\tau_C(0.724)| < |\tau\rho_T(0.79)|$; there is no significant relationship between math and economics at 1% significance level.

13. UNEB 2011/2/12

The heights and ages of ten students are given in the table below

Height, cm	156	151	152	146	160	157	149	142	158	140
Mass, kg	62	58	63	58	70	60	55	57	68	56

(a) Plot the data on a scatter diagram



(b) Draw the line of best fit on you graph and use it to estimate the mass corresponding to a height of 155cm(63kg)

(c) Calculate the rank correlation coefficient for the data. Comment on the significance of the height on masses of students (ρ = 0.79, τ =0.64 based on 10 observations at 1% level of significance.)

Method I: Using Spearman's rank correlation coefficient

Height (x)	Mass (y)	Rx	Ry	Rx – Ry = d	d^2
156	62	4	4	0	0
151	58	6	6.5	-0.5	0.25
152	63	5	3	2	4
146	58	8	6.5	1.5	2.25
160	70	1	1	0	0
157	60	3	5	-2	4
149	55	7	10	-3	9
142	57	9	8	1	1
158	68	2	2	0	0
140	56	10	9	1	1
					$\sum d^2 = 21.5$

$\rho = 1 - \left[\frac{6\sum d^2}{n(n^2-1)}\right] = 1 - \left[\frac{6\ x\ 21.5}{10(10^2-1)}\right] = 0.87$

Since $|\rho_C(0.0.87)| > |\rho_T(0.79)|$ there is no significant relationship between height and age at 1% significance level.

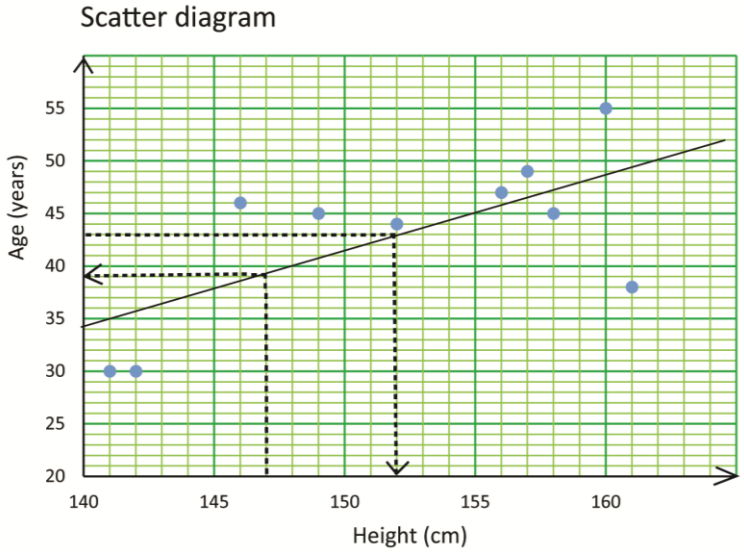
Since $|\tau_C(0.6\ )| < |\tau\rho_T(0.0.64\ )|$;

14. UNEB 2013/2/9

The heights and ages of ten farmers are given in the table below

Height, cm	156	151	152	160	146	157	149	142	158	140
Age, years	47	38	44	55	46	49	45	30	45	20

(a) Plot the data on a scatter diagram



(b) Draw the line of best fit on your diagram and use it to estimate

(i) Age when height = 147 (39)

(ii) Height when the age is 43 (152)

(c) Calculate the rank correlation coefficient for the data. Comment on your results

Method I: Using Spearman's rank correlation coefficient

Height (x)	Age (y)	Rx	Ry	Rx – Ry = d	d^2
156	47	4	3	1	1
151	38	6	8	-2	4
152	44	5	7	-2	4
160	55	1	1	0	0
146	46	8	4	4	16
157	49	3	2	1	1
149	45	7	5.5	1.5	2.25
142	30	9	9.5	-0.5	0.25
158	45	2	5.5	-3.5	12.25
140	30	10	9.5	0.5	0.25
					$\sum d^2 = 41$

p =1 - $\frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6\ x\ 41}{10(10^2-1)} = 0.7515(4D)$

Method II: using Kendall's rank correlation coefficient

Let the farmers be A, B, C, D, E, F, G, H, I, J

Farmers	A	B	C	D	E	F	G	H	I	J
Height	156	151	152	160	146	157	149	142	158	140
Age	47	38	44	55	46	49	45	30	45	30

By re-arranging the findings we have

Farmers	D	I	F	A	C	B	G	E	H	J
Height	1	2	3	4	5	6	7	8	9	10
Age	1	5.5	3	3	7	8	5.5	4	9.5	9.5
agreements	9	4	7	6	3	2	2	2	0	=35
Disagreements	0	3	0	0	2	2	1	0	0	=8

s = total agreements – total disagreements

= 35 – 8 = 27

$\tau = \frac{2s}{n(n-1)} = \frac{2\ x\ 27}{10(10-1)} = \frac{54}{90} = 0.6$ Or τ = $\frac{35-8}{35+8} = 0.63$

Comment

Since$|\rho_C(0.7515)| < |\rho_T(0.79)|$ and $|\tau_C(0.6\ )| < |\tau\rho_T(0.0.64\ )|$; there is no significant relationship between height and age at 1% significance level.

Kendall's rank correlation coefficient

By naming the pairs we have

A(156, 62), B(151, 58), C(152, 63), D(146, 58), E(160, 70), F(157, 60), G149, 55), H(142, 57), I(158, 68), J(141, 56)

	E	I	F	A	C	B	G	D	H	J
x	1	2	3	4	5	6	7	8	9	10
y	1	2	5	4	3	6.5	10	6.5	8	9
C	9	8	5	5	5	3	0	2	1	=38
D	0	0	2	1	0	0	3	0	0	=6

τ (tau) = $\frac{C-D}{C+D}$

τ (tau) = $\frac{38-6}{38+6}$ = 0.73

Since $\tau_C(0.73) > \tau_T$ (0.64), a significant relationship exist between the heights and masses of student.

15. UNEB 2015/2/12

The table gives the points awarded to eight schools by three judges, J_1, J_2 and J_3 during a music competition. J_1 was the chief judge.

J_1	72	50	50	55	35	38	82	72
J_2	60	55	70	50	50	50	73	70
J_3	50	40	62	70	40	48	67	67

(a) Determine the rank correlation coefficients between the judges of

(i) J_1 and J_2

(ii) J_1 and J_3

(b) Who of the two judges had a better correlation with the chief judge? Give a reason.

Solution

(i) J_1 and J_2

J_1	J_2	R_{J_1}	R_{J_2}	D_1	D_1^2
72	60	2.5	4	-1.5	2.25
50	55	5.5	5	0.5	0.25
50	70	5.5	2.5	3	9
55	50	4	7	-3	9
35	50	8	7	1	1
38	50	7	7	0	0
82	73	1	1	0	0
72	70	2.5	2.5	0	0
					$\sum D_1^2$ = 21.5

$$\rho_1 = 1 - \frac{6\sum D_1^2}{n(n^2-1)}$$

$$= 1 - \frac{6\ x\ 21.5}{8(8^2-1)}$$

$$= \frac{125}{168} = 0.7440$$

(ii) J_1 and J_3 (10marks)

J_1	J_3	R_{J_1}	R_{J_3}	D_2	D_2^2
72	50	2.5	5	-2.5	6.25
50	40	5.5	7.5	-4	4
50	62	5.5	4	0.5	2.25
55	70	4	1	3	9
35	40	8	7.5	0.5	0.25
38	48	7	6	1	1
82	67	1	2.5	-1.5	2.25
72	67	2.5	2.5	0	0
					$\sum D_1^2$ = 25

$$\rho_2 = 1 - \frac{6\sum D_2^2}{n(n^2-1)}$$

$$= 1 - \frac{6\ x\ 25}{8(8^2-1)}$$

$$= 0.7023$$

(a) Who of the two other judges had a better correlation with the chief judge? Give a reason. (02marks)

J_2 has a better correlation with the Chief Judge because the coefficient of correlation is smaller showing a stronger mutual relationship

Upper Critical Values for Kendall's Rank Correlation Coefficient $\hat{\tau}$

Note: In the table below, the critical values give significance levels as close as possible to but not exceeding the nominal α.

	Nominal α					
n	0.10	0.05	0.025	0.01	0.005	0.001
4	1.000	1.000	-	-	-	-
5	0.800	0.800	1.000	1.000	-	-
6	0.600	0.733	0.867	0.867	1.000	-
7	0.524	0.619	0.714	0.810	0.905	1.000
8	0.429	0.571	0.643	0.714	0.786	0.857
9	0.389	0.500	0.556	0.667	0.722	0.833
10	0.378	0.467	0.511	0.600	0.644	0.778
11	0.345	0.418	0.491	0.564	0.600	0.709
12	0.303	0.394	0.455	0.545	0.576	0.667
13	0.308	0.359	0.436	0.513	0.564	0.641
14	0.275	0.363	0.407	0.473	0.516	0.604
15	0.276	0.333	0.390	0.467	0.505	0.581
16	0.250	0.317	0.383	0.433	0.483	0.567
17	0.250	0.309	0.368	0.426	0.471	0.544
18	0.242	0.294	0.346	0.412	0.451	0.529
19	0.228	0.287	0.333	0.392	0.439	0.509
20	0.221	0.274	0.326	0.379	0.421	0.495
21	0.210	0.267	0.314	0.371	0.410	0.486
22	0.203	0.264	0.307	0.359	0.394	0.472
23	0.202	0.257	0.296	0.352	0.391	0.455
24	0.196	0.246	0.290	0.341	0.377	0.449
25	0.193	0.240	0.287	0.333	0.367	0.440
26	0.188	0.237	0.280	0.329	0.360	0.428
27	0.179	0.231	0.271	0.322	0.356	0.419
28	0.180	0.228	0.265	0.312	0.344	0.413
29	0.172	0.222	0.261	0.310	0.340	0.404

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Couple of forces

These are equal forces acting in opposite direction

Conditions for forces to form a couple

Forces reduce to a couple if;

- resultant force is zero
- the sum of moments about a point is not zero

Example 1

Forces of (-5i – j)N, -3j and (5i + 4j) act on a body a point with position vectors (i – j)m, (2i + j)m and (4i – 5j)m respectively. Show that these forces reduce to a couple

Solution

$$R = \begin{pmatrix}-5\\-1\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}5\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

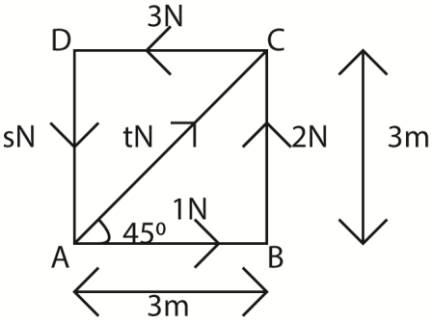
$$G = \begin{vmatrix}1 & -5\\-1 & -1\end{vmatrix} + \begin{vmatrix}2 & 0\\1 & -3\end{vmatrix} + \begin{vmatrix}4 & 5\\-5 & 4\end{vmatrix}$$

= [(1 x -1) – (-5 x -1)]+[(3 x -3) – (1 x 0)] + [(4 x 4) – (5 x -5)] = 29Nm

Since the resultant force is zero an the sum of moment G is not zero, the forces reduce to a couple.

Example 2

ABCD is a square of side 3m. Forces of magnitude 1N, 2N, 3N, sN and tN act along the line AB, BC, CD, DA and AC respectively, in each case the direction of the force being given by the order of letters. Taking AB as horizontal and BC as vertical, find the values of s and t so that the resultant of the forces is a couple.



$$R = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}0\\-s\end{pmatrix} + \begin{pmatrix}tcos45\\tsin45\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

(→) tcos45 = 2; t = $\frac{2}{cos45} = 2\sqrt{2}$

(↑) 2 – s + tsin45 = 0; s = 2 + $2\sqrt{2}$ sin45 = 4N

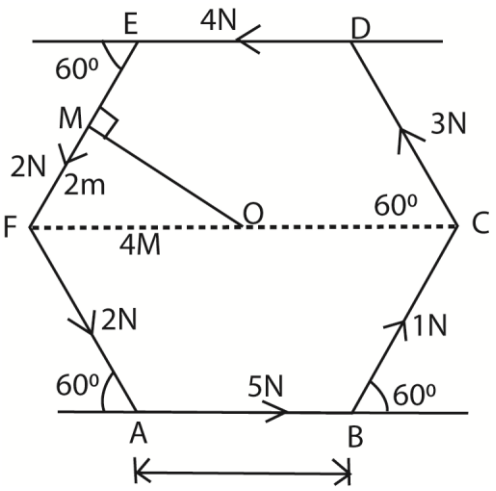
It must also be shown the G≠0

A G = 2 x 3 + 3 x 3 = 15Nm

Example 3

ABCDEF is a regular hexagon of side 4m. Forces of magnitude 5N, 1N, 3N, 4N, 2N and 2N act along the lines AB, BC, CD, DE, EF and FA respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal, show that these forces reduce to a couple.

Solution



OM = $\sqrt{4^2 - 2^2} = 2\sqrt{3}m$

(→) 5 – 4 + (1 – 3 – 2 +2)cos60 = 0

(↑) (1 + 3 – 2 – 2) sin60 = 0

↺ O G = $2\sqrt{3}$ (5 + 1 + 3 + 4 + 2)

= $34\sqrt{3}$N anticlockwise

Since R = 0 and G≠ 0 then is a couple.

Revision exercise

1. Forces of 6N, 8N, 6N and 8N act along sides of a rectangle ABCD where AB =8m and BC = 6m in the direction AB, BC, CD and DA respectively.
   (a) show that the forces reduce to couple
   (b) find the moment of the couple about A. (100Nm)
2. Forces of 5N, 3N, 5N and 3N act along the side of a square ABCD of side 4m in the directions AB, BC, CD and DA respectively.
   (a) show that the forces reduce to couple
   (b) find the moment of the couple about A. (100Nm
3. ABCD is a rectangle with AB = 6m and BC = 2m. A force of 3N acts along each of the four sides AB, BC, CD and DA in the directions indicated by the order of the letters. Show that the forces form a couple and find its moment. [21Nm]
4. ABCD is a rectangle with AB = 6m and BC = 2m. A forces of 5N, 5N, xN, and xN acts along each of the four sides CB, AD, AB and CD in the directions indicated by the order of the letters. If the system is in equilibrium, find the value of x. [15N]
5. ABCD is a square of side 40cm. Forces of magnitude 20N, 15N, 20N and Y act along the line AB, BC, CD and DA respectively in each case the direction of the force being given by the order of letters. If the system is equivalent to a couple, find the magnitude of Y and the moment of the couple. [15N, 14Nm]
6. A force of (3i – 5j)N acts at a point which has position vector (6i + j)m and a force of (-3i + 5j)N acts at the point which has position vector (4i + j)m. Show that the forces reduce to a couple and find the moment of the couple [10Nm]
7. A force of (4i + 3j)N acts at a point which has position vector (6i + 3j)m and a force of (-4i -3j)N acts at the point which has position vector (3i – j)m. Show that the forces reduce to a couple and find the moment of the couple [7Nm]

8. Force of (i + j)N, (-4i + j)N and (3i – 2j)N act at the points having position vectors (2i + 2j)m, (-1 + 4j)m and (4i – 2j)m respectively. Show that the forces reduce to a couple and find the moment of the couple [13Nm]
9. Forces of (ai + bj)N and (6i – 4j)N act at points having position vectors (-2i – 2j)m and (3i – j)m respectively. If these forces reduce to a couple, find a and b and the moment of the couple. [-6, 4 and 26Nm]

Thank You

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Discrete probability distribution

A probability density function (p.d.f) if it takes on specific values

Properties of discrete probability density functions

(i) $\sum P(X = x) = 1 \; or \; \sum f(x) = 1$

(ii) P(X=x)≥0

Examples 1

A discrete random variable has a probability function $P(X = x) = \begin{cases} cx^2 & x = 0, 1, 2, 3, 4 \\ 0 & otherwise \end{cases}$

Find the value of c and draw the graph of P(X = x)

Solution

$\sum P(X = x) = 1$

$c(0^2) + c(1^2) + c(2^2) + c(3^2) + c(4^2) = 1$

$c + 4c + 9c + 16c = 2$

$c = \frac{1}{30}$

Example 2

A discrete random variable has probability function

$f(x) = \begin{cases} kx, & x = 1, 2, 3, 4 \\ 0, & otherwise \end{cases}$, find the value of k and draw the graph of f(x)

Solution

$\sum f(x) = 1$

$k + 2k + 3k + 4k = 1$

$k = \frac{1}{10}$

Example 3

A random variable X of a discrete probability distribution given by

P(X=1) = 0.2, P(X=2) = P(X=3) = 0.1, P(X=4) = P(X=5) = c

Find the value of the constant c and draw the graph of P(X = x)

Solution

$\sum P(X = x) = 1$

0.2 + 0.1 + 0.1 + c + c = 2; c = 0.3

Example 4

A discrete random variable has a probability function

$$P(X = x) = \begin{cases} k\left(\frac{2}{3}\right)^x, & x = 1, 2, 3, \ldots \\ 0, & otherwise \end{cases}$$

Find the value of k

Solution

$k\left(\frac{2}{3}\right)^0 + k\left(\frac{2}{3}\right)^1 + k\left(\frac{2}{3}\right)^2 + k\left(\frac{2}{3}\right)^3 + \cdots . = 1$

$k\left(1 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots .\right) = 1$

Sum to infinity = $S_\infty = \frac{a}{1-r}$

⇨ $k\left(\frac{1}{1-\frac{2}{3}}\right)$ = 1; k = $\frac{1}{3}$

Finding probabilities

Example 5

A discrete random variable has a probability distribution

y	-3	-2	-1	0	1
P(Y=y)	0.1	0.25	0.3	0.15	a

Find

(i) value of a (ii) P(-3≤ $Y < 0$) (iii) P(Y>-1) (iv) P(-1<Y<1) (v) mode

Solution

(i) $\sum P(Y = y) = 1$

0.1 + 0.25 + 0.3 + 0.15 + a = 1; a = 0.2

(ii) P(-3≤ $Y < 0$) = P(Y= -3) + P(Y = -2) + P(Y = -1) = 0.1 + 0.25 + 0.3 = 0.65

(iii) P(Y> -1) = P(Y = 0) + P(Y = 1) = 0.15 + 0.2 = 0.35

(iv) P(-1<Y<1) = P(Y = 0) = 0.15

(v) mode is the value y with the highest probability, mode = -1

Example 6

A discrete random variable X has a probability distribution

X	1	2	3	4	5
P(X = x)	0.15	0.20	0.15	c	0.1

Find

(i) the value of x (ii) P(X<4) (iii) P(X≤4 (iv) P(2≤ $X \leq 4$) (v) P$\left(\frac{X>2}{X\leq 4}\right)$ (vi) mode

Solution

(i) $\sum P(X = x) = 1$

0.15 + 0.20 + 0.15 + c + 0.1 = 1; c = 0.4

(ii) P(X<4) = P(X=1) + P(X= 2) + P(X = 3) = 0.15 + 0.20 + 0.15 = 0.5

(iii) P(X<4) = P(X=1) + P(X= 2) + P(X = 3) + P(X= 4) = 0.15 + 0.20 + 0.15 + 0.4 = 0.9

(iv) P(2≤ $X \leq 4$) = P(X = 2) + P(X = 3) + P(X = 4) = 0.20 + 0.15 + 0.4 = 0.75

(v) P$\left(X > 2/X \leq 4 \right)$ = $\frac{P(X>2,\ X\leq 4)}{P(X\leq 4)}$ = $\frac{P(X=3)+P(X=4)}{\mathrm{P(X=1) + P(X= 2) + P(X = 3) + P(X= 4)}} = \frac{0.15 + 0.4}{0.9}$ = 0.6111

(vi) the mode is a value with highest probability = 4

Example 7

A discrete random variable X has a probability function

$$f(x) = \begin{cases} kx,\ x = 1,2,3,4,5 \\ 0,\ otherwise \end{cases}$$

Find (i) the value of k (ii) P(X = 3) (iii) P(X≥3) (iv) P(X≤ 3) (v) P(1< $X \leq 3$) (vi) P$\left(\frac{X\geq 1}{X<4}\right)$

Solution

(i) $\sum P(X = x) = 1$

k + 2k + 3k + 4k + 5k = 1; k = $\frac{1}{15}$

(ii) P(X=3) = 3k = $\frac{3}{15} = \frac{1}{5}$

(iii) P(X≥3)= P(X=3) + P(X=4) + P(X = 5) = 3k + 4k + 5k = 12k = $\frac{12}{15} = \frac{4}{5}$

(iv) P(X≤ 3)= P(X= 1) + P(X=2) + P(X = 3) = k + 2k + 3k =$\frac{6}{15} = \frac{2}{5}$

(v) P(1< $X \leq 3$) = P(X=2) + P(X = 3) =2k + 3k =$\frac{5}{15} = \frac{1}{3}$

(vi) P$\left(X \geq 2/X < 4 \right) = \frac{P(X\geq 2, X<4)}{P(X<4)} = \frac{P(X=2)+P(X=3)}{P(X=1)+P(X=2)+P(X=3)} = \frac{2k+3k}{k+2k+3k} = \frac{5k}{6k} = \frac{5}{6}$

Revision exercise 1

1. A discrete random variable X has probability distribution

x	1	2	3	4	5
P(X=x)	0.2	0.25	0.4	a	0.05

Find (i) value of a = 0.1 (ii) P($1 \leq x \leq 3$)=0.85 (iii) P(X>2) =0.55 (iv) P(2<X<5) = 0.5 (v) mode =3

2. A random variable x of a discrete pdf is given by P(X x) = kx, x = 12, 13, 14
Write the probability distribution and find the value of k

x	12	13	14
P(X=x)	12k	13k	14k

k = $\frac{1}{39}$

3. A random variable Y of discrete probability distribution is given by
P(Y = 3) = 0.1, P(Y = 5) = 0.05, P(Y=6) = 0.45 P(Y = 8) = 3P(Y = 10). Find P(Y=10) = 0.1

4. A discrete random variable has a distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	k	0.2	0.05

Find
(i) value of k = $\frac{7}{20}$ (ii) P(X≥4) = 0.25 (iii) P(X<1)=0 (iv) P($2 \leq x < 4$) = $\frac{13}{20}$

5. Write out the probability distribution for each of these variables
(a) The number of heads X obtained when two fair coins are tossed

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b) The number of tails, X obtained when three fair coins are tossed.

x	0	1	2	3
P(X=x)	0.125	0.375	0.375	0.125

6. A drawer contains 8 brown socks and 4 blue shocks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. The procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X.

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by P(R=r) = c(3 – r), r = 0, 1, 2, 3
Find (i) value of c = $\frac{1}{6}$ (ii) P(1≤R<3) = 0.5

8. A discrete random variable has probability function

$$P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 1, 2, 3, \ldots \\ 0, & otherwise \end{cases}$$, find the value k = 0.2.

Solutions to revision exercise 1

5 Write out the probability distribution for each of these variables
(a) The number of heads X obtained when two fair coins are tossed
S =(TT, TH, HT, HH)
P(X= 0) = $\frac{1}{4} = 0.25$, P(X= 1) = $\frac{2}{4} = 0.50$, P(X= 2) = $\frac{1}{4} = 0.25$
Probability distribution table

x	0	1	2
P(X=x)	0.25	0.5	0.25

(b) The number of tails, X obtained when three fair coins are tossed.

S = (TTT, TTH, THT, HTH, THH, HTH, HHT, HHH)

Probability distribution table

number of heads, x	0	1	2	3
P(X=x)	$\frac{1}{8}$ = 0.125	$\frac{3}{8}$ = 0.375	$\frac{3}{8}$=0.375	$\frac{1}{8}$ = 0.125

6. A drawer contains 8 brown socks and 4 blue shocks. A sock is taken from the drawer at random, its colour is noted and it is then replaced. The procedure is performed twice more. X is the random variable for the number of brown socks taken. Find the probability distribution for X.
   Let X' represent blue shocks
   P(X = 0) = P(X'∩X'∩X') = $\frac{4}{12} x \frac{4}{12} x \frac{4}{12} = \frac{1}{27}$
   P(X=1) = P(X∩X'∩X')+ P(X'∩X∩X') + P(X'∩X'∩X) = $\frac{8}{12} x \frac{4}{12} x \frac{4}{12} + \frac{4}{12} x \frac{8}{12} x \frac{4}{12} + \frac{4}{12} x \frac{4}{12} x \frac{8}{12} = \frac{2}{9}$
   P(X=2) = P(X∩X∩X')+ P(X'∩X∩X) + P(X∩X'∩X) =$\frac{8}{12} x \frac{8}{12} x \frac{4}{12} + \frac{4}{12} x \frac{8}{12} x \frac{8}{12} + \frac{8}{12} x \frac{4}{12} x \frac{8}{12} = \frac{4}{9}$
   P(X = 3)= P(X∩X∩X)= $\frac{8}{12} x \frac{8}{12} x \frac{8}{12} = \frac{8}{27}$
   Probability distribution table

x	0	1	2	3
P(X=x)	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

7. The discrete random variable R has a p.d.f is given by P(R=r) = c(3 – r), r = 0, 1, 2, 3

   Find (i) value of c
   $\sum P(X = x) = 1$
   3c +2c + c = 3
   c= $\frac{1}{6}$
   (ii) P(1≤R<3) = 2c + c = 3c = 3 x $\frac{1}{6}$ =0.5
8. A discrete random variable has probability function

$$P(X = x) = \begin{cases} k\left(\frac{4}{5}\right)^x, & x = 1, 2, 3, \ldots \\ 0, & otherwise \end{cases}$$, find the value k.

Solution

$$k\left(\frac{4}{5}\right)^0 + k\left(\frac{4}{5}\right)^1 + k\left(\frac{4}{5}\right)^2 + k\left(\frac{4}{5}\right)^3 + \cdots . = 1$$

$$k\left(1 + \left(\frac{4}{5}\right)^1 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{5}\right)^3 + \cdots .\right) = 1$$

Sum to infinity = $S_\infty = \frac{a}{1-r}$

⇨ $k\left(\frac{1}{1-\frac{4}{5}}\right)$ = 1; k = $\frac{1}{5}$ =0.2

Expectation of x, E(x) or mean

The expected value of x is given by E(x) = $\sum xP(X = x)$

Example 8

A discrete random variable has a probability distribution

x	-2	-1	0	1	2
P(X = x)	0.3	0.1	0.15	0.4	0.05

Find expectation, E(x)

Solution

E(X) = (-2 x 0.3) + (-1 x 0.1) + (0 x 0.15) + (1 x 0.4) + (2 x 0.05) = -0.2

Example 9

The discrete random variable Y has a probability distribution is given by

P(Y = y) = cy, y = 1 2,3,

P(Y = y) = c(8-y), y = 4, 5, 6, 7

Find (i) the value of c (ii) mean, μ

Solution

y	1	2	3	4	5	6	7
P(Y = y)	c	2c	3c	4c	3c	2c	c

(i) $\sum P(Y = y) = 1$

c + 2c + 3c + 4c + 3c +2c + c = 1

c = $\frac{1}{16}$

(ii) E(Y) = $\sum y(Y = y)$ = 1 x c + 2 x 2c + 3 x 3c + 4 x 4c + 5 x 3c + 6 x 2c + 7x c = 64c

= 64 x $\frac{1}{16}$ = 4

Example 10

A fair coin is tossed three times write out the probability distribution for the number of heads, X, obtained and hence obtain the expected number of heads

Solution

S = (TTT, TTH, THT, HTH, THH, HTH, HHT, HHH)

Probability distribution table

number of heads, x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

E(X) = $\sum x(X = x)$ = (0 x $\frac{1}{8}$) + 1x $\frac{3}{8}$ + 2 x $\frac{3}{8}$ + 3 x $\frac{1}{8}$ = $\frac{12}{8}$ = 1.5

Example 11

A family plans to have 4children. Given that X is the number of girls in the family. Find the expected number of girls

Solution

S = (BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGGB, BGBG, GGBB, BGGB, GBBG, BGGG, GBGG, GGBG, GGGB GGGG)

Probability distribution table

Number of girls, x	0	1	2	3	4
P(X=x)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

E(X) $= \sum x(X = x) = \left(0\ x\ \frac{1}{16} + 1\ x\ \frac{4}{16} + 2\ x\ \frac{6}{16} + 3\ x\ \frac{4}{16} + 4\ x\ \frac{1}{16}\right) = 2$

Example 12

A box A contains 4 red sweets and 3 green sweets. Box B contains 5 red sweets and 6 green sweets. Box A is twice more likely to be picked as Box B. If a box is chosen at random and two sweets are removed from it, one at a time without replacement.

(a) Find the probability that two sweets removed are of the same colour.

P(same colour) = P(A∩R_1∩R_2) + P(A∩G_1∩G_2) + P(B∩R_1∩R_2) + P(B∩B_1∩B_2)

$= \frac{2}{3}x\frac{4}{7}\ x\frac{3}{6} + \frac{2}{3}x\frac{3}{7}\ x\frac{2}{6} + \frac{1}{3}x\frac{5}{11}\ x\frac{4}{10} + \frac{1}{3}x\frac{6}{11}\ x\frac{5}{10}$

$= \frac{24}{126} + \frac{12}{126} + \frac{20}{330} + \frac{30}{330} = \frac{42}{126} + \frac{50}{330} = 0.4372$

(b) (i) construct a probability distribution table for the number of red sweets removed

Let x = number of red sweets removed

P(X =0) = P(A∩G_1∩G_2) + P(B∩G_1∩G_2) = $\frac{2}{3}x\frac{3}{7}\ x\frac{2}{6} + \frac{1}{3}x\frac{6}{11}\ x\frac{5}{10}$ = 0.1861

P(X=1) = P(A∩R_1∩G_2) + P(A∩G_1∩R_2) + P(B∩R_1∩G_2) + P(B∩G_1∩R_2)

$= \frac{2}{3}x\frac{4}{7}\ x\frac{3}{6} + \frac{2}{3}x\frac{3}{7}\ x\frac{4}{6} + \frac{1}{3}x\frac{5}{11}\ x\frac{6}{10} + \frac{1}{3}x\frac{6}{11}\ x\frac{5}{10}$

$= \frac{24}{126} + \frac{24}{126} + \frac{30}{330} + \frac{30}{330} = \frac{48}{126} + \frac{60}{330}$

= 0.5628

P(X=2) = P(A∩R_1∩R_2)+ P(B∩R_1∩R_2) = $\frac{2}{3}x\frac{4}{7}\ x\frac{3}{6} + \frac{1}{3}x\frac{5}{11}\ x\frac{4}{10} = \frac{24}{126} + \frac{20}{330}$ = 0.2511

Probability distribution table

x	0	1	2
P(X= x)	0.1861	0.5628	0.2511

(ii) find the mean number of red sweets removed

Mean = $\sum x(X = x)$ = 0 x0.1861 + 1 x 0.5628 + 2 x 0.2511 = 1.065

Revision exercise 2

1. A discrete random variable X has a probability distribution.

x	0	1	2	3	4
P(X= x)	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$

Find E(X) = 2.25

2. A discrete random variable X ha probability distribution

x	5	6	7	8	9
P(X=x)	$\frac{3}{11}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{3}{11}$

Find the mean = 7

3. A discrete random variable X has a probability distribution

4. A discrete random variable has a probability distribution

x	0	1	2	3
P(X=x)	c	c^2	c^2+c	$3c^2$ + 2c

Find (i) the value of c= 0.2 (ii) expectation of c=2.08

5. Find the expected number of heads when two fair coins are tossed (E(x) 1)
6. A family plans to have 3 children. Given that x is the number of boys in the family. Find the expected number of boys (=1.5)
7. If X is a random variable for the product of the scores on two tetrahedral dice, where the score is the number on which the die lands, find the expected score for the throw (=6.25)
8. A bag contains 5 black counters and 6 red counters. Two counters are drawn at random, one at a time without replacement. Find the expected number of red counters. (= $\frac{12}{11}$)
9. An unbiased tetrahedral die is tossed once. If it lands on a face marked 1, the player has to pay 10,000/=. If it lands on marked with 2 or 4 the player wins 5000/= and if it lands on a 3, the player wins 3000/=. Find the expected gain in one throw.
10. A discrete random variable X can take on values 10 and 20 only. If E(X) = 16. Write out the probability distribution for X (P(X=x) = 0.4 and P(X=x) = 0.6)
11. A discrete random variable X can take on values 0, 1, 2, and 3 only. If E(X) = 1.4, P(X$\leq$ 2) = 0.9 and P(X$\leq$1) = 0.5. Find (i) P(X=1) = 0.3 (ii) P(X =0) = 0.2
12. The discrete random variable Y has a probability distribution is given by
    P(Y=y) = cy, y = 1, 2, 3, 4
    Find (i) value of c = 0.1 (ii) E(X) = 3
13. A discrete random variable has p.d.f
    $P(X = x) = \begin{cases} k2^x, & x = 0, 1, 2, 3, 4, 5, 6 \\ 0, & otherwise \end{cases}$,
    Find (i) value of k = $\frac{1}{127}$, (ii) mean = 5
14. A discrete random variable X has a probability distribution

x	0	1	2	3	4	5
P(X= x)	0.11	0.17	0.2	0.13	p	0.09

Find (i) the value of p = 0.3 (ii) Expected value of X (= 2.6)

Solutions to revision exercise 2

10. A discrete random variable X can take on values 10 and 20 only. If E(X) = 16. Write out the probability distribution for X
    Let P(X=10) = a and P(X=20) = b
    a+b = 1
    a = (1-b)…… (i)
    10a + 20b = 16 ……. (ii)
    Eqn. (i) and eqn. (ii)
    10(1-b) + 20b = 16
    10 + 20b = 16
    b = 0.6
    a = 1- 0.6 = 0.4

Probability distribution:(P(X=x) = 0.4 and P(X=x) = 0.6)

11. A discrete random variable X can take on values 0, 1, 2, and 3 only. If E(X) = 1.4, P(X$\leq$ 2) = 0.9 and P(X$\leq$1) = 0.5. Find (i) P(X=1) (ii) P(X =0)

Let P(X = 0) = a, P(X=1)=b, P(X=2)= c p(X = 3) = d

a + b + c + d = 1 ……. (i)

P(X$\leq$ 2) = a + b+ c = 0.9 … (ii)

Eqn. (i) and eqn. (ii)

d = 0.1

P(X$\leq$1) = a + b = 0.5 ……. (iii)

Eqn. (i) and eqn. (iii)

0.5 + c + 0.1 = 1

c = 0.4

E(X) = 0 x a + 1 x b + 2 x c + 3 x 0.1 = 1.4

= b + 2c + 0.3 = 1.4

b + 2c = 1.1

b + 2 x 0.4 = 1.1

b = 0.3

a = 0.5 – 0.3 = 0.2

Hence, (i) P(X=1) = 0.3 (ii) P(X =0) = 0.2

12. The discrete random variable Y has a probability distribution is given by

P(Y=y) = cy, y = 1, 2, 3, 4

Find (i) value of c = 0.1 (ii) E(X) = $\frac{11}{3}$

(i) $\sum P(X = x) = 1$

c + 2c + 3c + 4c = 1

10c = 1; c= 0.1

(ii) E(X) =$\sum xP(X = x)$= 1 x 0.1 + 2 x 0.2 + 3 x 0.3 + 4 x 0.4 =3

13. A discrete random variable has p.d.f

$P(X = x) = \begin{cases} k2^x, & x = 0,1,2,3,4,5,6 \\ 0, & otherwise \end{cases}$,

Find

(i) (i) value of k = $\frac{1}{127}$,

$\sum P(X = x) = 1$

k($2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$) = 1

k = $\frac{1}{127}$

(ii) Mean = $\sum Px(X = x) = \frac{1}{127}$(0 x 1 + 2 x 2 + 3 x 8 + 4 x 16 + 4 x 32 + 6 x 64) = 5.01

14. A discrete random variable X has a probability distribution

x	0	1	2	3	4	5
P(X= x)	0.11	0.17	0.2	0.13	p	0.09

Find

(i) the value of p

$\sum P(X = x) = 1$

0.11 + 0.17 + 0.2 + 0.13 + p + 0.09 = 1

p = 0.3

(ii) Expected value of X

E(X) =$\sum Px(X = x)$ = 0.11x0 + 0.17x1 + 0.2x2 + 0.13x3 + 0.3 x 4 + 0.09x5= 2.61

Properties of the mean

(i) E(a) = a
(ii) E(ax) = aE(x)
(iii) E(ax + b) = aE(x) + b
(iv) E(ax - b) = aE(x) - b

Example 13

A random variable X of discrete probability distribution is given by

x	1	2	3	4
P(X=x)	0.1	0.2	0.3	0.4

Find

(i) E(x) = $\sum Px(X = x)$ = 1 x 0.1 + 2 x 0.2 + 3 x 0.3 + 4 x 0.4 = 3
(ii) E(3x) = 3E(x) = 3 x 3 = 9
(iii) E(4x + 6) =4E(x) + 6 = 4 x 3 + 6 = 18

Example 14

A random variable X of discrete probability distribution is given by

x	-1	0	1	2
P(X= x)	0.25	0.10	0.45	0.20

Find

(i) P(-1$\leq$ X <1) = P(X= -1) + p(X = 0) = 0.25 + 0.10 = 0.35
(ii) E(X) =$\sum Px(X = x)$ = -1 x 0.25 + 0 x 0.10 + 1 x 0.45 + 2 x 0.20 = 0.6
(iii) E(6x – 2) = 6E(X) – 2 = 0.6 x 6 – 2 = 1.6

Variance, Var(x)

Var(x) = E(X^2) = [E(x)]^2 where E(X^2) = $\sum x^2 P(X = x)$

Example 15

A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X=x)	0.1	0.3	0.2	0.3	0.1

Find

(i) The mean = $\sum Px(X = x)$ = 1 x 0.1 + 2 x 0.3 + 3 x 0.2 + 4 x 0.3 + 5 x 0.1= 3
(ii) Var(x)
E(X^2) =$\sum Px^2(X = x)$ = 1^2 x 0.1 + 2^2 x 0.3 + 3^2 x 0.3 + 4^2 x 0.3 + 5^2 x 0.1 = 10.4
Var(X) = 10.4 – $(3)^2$ = 1.4

Example 16

The discrete random variable Y has a probability distribution is given by P(Y =y), y= -3, -2, -1, 0, 1, 2,3

Find: (i) value of c (ii) mean (iii) standard deviation

Solution

y	-3	-2	-1	0	1	2	3
P(Y= y)	3c	2c	c	0	c	2c	3c

(i) $\sum P(X = x) = 1$

3c +2c + c+ c + 2c + 3c = 1; c = $\frac{1}{12}$

(ii) Mean = $\sum Px(X = x)$ = -3 x 3c + -2 x 2c + -1 x c + 0 x 0 + 1 x c + 2 x 2c + 3 x 3c = 0

(iii) $E(X^2) = (-3)^2$ x 3c + $(-2)^2$ x 2c + $(-1)^2$ x c + $(0)^2$ x 0 + $(1)^2$ x c + $(2)^2$ x 2c +$(3)^2$ x 3c = 72 x $\frac{1}{12}$ = 6

Var (x) = $E(X^2) - (E(x))^2 = 6 - (0)^2 = 6$

S.D = $\sqrt{Var\ (X)} = \sqrt{6}$ = 2.45

Example 17

Two marbles are drawn without replacement from a box containing 3 red marbles and 4 white marbles. The marbles are randomly drawn. If X is the random variable for the number of red marble drawn find

(i) Expected number of red marbles

P(X= 0) = P(W∩W) = $\frac{4}{7}\ x\ \frac{3}{6} = \frac{2}{7}$

P(X= 1) = P(W∩R) + P(R∩W) = $\frac{4}{7}\ x\ \frac{3}{6} + \frac{3}{7}\ x\ \frac{4}{6} = \frac{4}{7}$

P(X= 2) = P(R∩R) = $\frac{3}{7}\ x\ \frac{2}{6} = \frac{1}{7}$

The probability distribution table

x	0	1	2
P(X=x)	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

E(x) = $\sum Px(X = x)$ = $\frac{2}{7}\ x\ 0 + \frac{4}{7}\ x\ 1 + \frac{1}{7}\ x\ 2 = \frac{6}{7}$

(ii) Standard deviation of X

$E(x^2) = \sum Px^2(X = x) == \frac{2}{7}\ x\ 0 + \frac{4}{7}\ x\ 1^2 + \frac{1}{7}\ x\ 2^2 = \frac{8}{7}$

Var (x) = $E(x^2) - (E(X))^2 = \frac{8}{7} - \left(\frac{6}{7}\right)^2 = \frac{20}{49}$

S.D = $\sqrt{Var\ (X)} = \sqrt{\frac{20}{49}}$ = 0.6389

Example 19

A vendor stocks 12 copies of a magazine each week and the probability for each possible total number of copies sold is shown below

Number of copies	9	10	11	12
probability	0.2	0.35	0.30	0.15

(a) Estimate the mean and variance of the number of copies

Mean = $\sum Px(X = x)$ = 9 x 0.2 + 10 x 0.35 + 11 x 0.3 + 12 x 0.15 = 10.4

$E(X^2) = 9^2$ x 0.2 + 10^2 x 0.35 + 11^2 x 0.3 + 12^2 x 0.15 =109.1

Var(x) = 109.1 – $(10.4)^2$ = 0.94

(b) The vendor buys the magazine at 8,500/= and sells at 14,500/=. Any copies not sold are destroyed. Construct a probability distribution table for vendor's weekly profit and hence find the expected weekly profit

Profit = S.P – C.P

Profit for 9 copies = 9 x 14,500 – 12 x 8500 = 28500
Profit for 10 copies= 10 x 14,500 – 12 x 8500 =43000
Profit for 11 copies= 11 x 14,500 – 12 x 8500 = 57500
Profit for 12 copies= 12 x 14,500 – 12 x 8500 = 72000

y	28500	43000	57500	72000
P(Y=y)	0.2	0.35	0.30	0.15

E(Y) = 0.2 x 28500 + 0.35 x 43000 + 0.30 x 57500 + 0.15 x 72000 = 48000/=

Example 20

The table below shows the number of red and green balls put in three identical boxes A, B and C.

Boxes	A	B	C
Red balls	4	6	3
Green balls	2	7	5

A box is chosen at random and two balls are then drawn from it successively without replacement. If the random variable X is "the number of green balls drawn".

(a) Draw a probability distribution table for X (06marks)

Using combination

$P(X = 0) = \frac{1}{3}\left[\frac{{}^4C_2}{{}^6C_2} + \frac{{}^6C_2}{{}^{13}C_2} + \frac{{}^3C_2}{{}^8C_2}\right]$

$= \frac{1}{3}\left[\frac{2}{5} + \frac{5}{26} + \frac{2}{28}\right] = \frac{1273}{5460}$

$P(X = 1) = \frac{1}{3}\left[\frac{{}^2C_1 \, x \, {}^4C_1}{{}^6C_2} + \frac{{}^7C_1 \, x \, {}^6C_1}{{}^{13}C_2} + \frac{{}^3C_1 \, x \, {}^3C_1}{{}^8C_2}\right]$

$= \frac{1}{3}\left[\frac{8}{15} + \frac{7}{13} + \frac{15}{28}\right] = \frac{8777}{16380}$

$P(X = 2) = \frac{1}{3}\left[\frac{{}^2C_2}{{}^6C_2} + \frac{{}^7C_2}{{}^{13}C_2} + \frac{{}^5C_2}{{}^8C_2}\right]$

$= \frac{1}{3}\left[\frac{1}{15} + \frac{7}{26} + \frac{5}{14}\right] = \frac{946}{4095}$

x	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$

(b) Calculate the mean and variance of X (06marks)

1	0	1	2
P(X =x)	$\frac{1273}{5460}$	$\frac{8777}{16380}$	$\frac{946}{4095}$
xP(X =x)	0	$\frac{8777}{16380}$	$\frac{1892}{4095}$
$x^2P(X =x)$	0	$\frac{8777}{16380}$	$\frac{3784}{4095}$

$E(X) = \frac{8777}{16380} + \frac{1892}{4095} = 0.9979$

$E(X^2) = \frac{8777}{16380} + \frac{3784}{4095} = 1.4599$

Var(X) = 1.4599 – 0.9979

= 0.4642

Properties of the variance

(i) Var(a) = 0
(ii) $Var(aX) = a^2Var(X)$
(iii) $Var(aX + b) = a^2Var(X)$
(iv) $Var(aX – b) = a^2Var(X)$

Example 21

A discrete random variable X has a probability distribution

x	1	2	3	4	5
P(X = x)	0.2	0.25	0.4	0.1	0.05

Find

(i) Mean = $\sum Px(X = x)$ = 1 x 0.2 + 2 x 0.25 + 3 x 0.4 + 4 x 0.1 + 5 x 0.05 = 2.55
(ii) The variance
$E(X^2) = 1^2 \times 0.2 + 2^2 \times 0.25 + 3^2 \times 0.4 + 4^2 \times 0.1 + 5^2 \times 0.05 = 7.65$
$Var(x) = E(X^2) – (E(X))^2 = 7.65 – (2.55)^2 = 1.148$
(iii) $Var (3x -2) = 3^2Var(x) = 9 \times 1.148 = 10.332$

Example 22

A random variable X of a discrete probability distribution given by

x	10	20	30
P(X = x)	0.2	0.3	0.5

Find

(i) E(X) = 10 x 0.2 + 20 x 0.3 + 30 x 0.5 =22
(ii) $Var (X) = E(X^2) – (E(X))^2$
$E(X^2) = 10^2 \times 0.2 + 20^2 \times 0.3 + 30^2 \times 0.5 = 520$
$Var(x) = 520 – 22^2 = 36$
(iii) $Var(4X + 3) = 4^2Var(x) = 16 \times 36 = 576$

Revision exercise 3

1. A random variable X of discrete probability distribution is given by

x	1	2	3
P(X = x)	0.2	0.3	0.5

Find (i) E(X) = 2.3 (ii) $E(X^2) = 5.9$ (iii) Var (X) = 0.61

2. A random variable X of discrete probability distribution is given by

x	-1	0	1	2
P(X = x)	0.25	0.1	0.45	0.2

Find: (i) $P(-1 \leq X < 2) = 0.8$ (ii) E(X) = 0.6 (iii) E(2x + 3) = 4.2

3. A random variable X of a discrete probability distribution
P(X = 0) = 0.05, P(X = 1) = 0.45 P(X = 2) = 0.5

Find: (i) E(X) = 1.45, (i) E(X^2) = 2.45 (iii) Var (X) = 0.348

4. A random variable X of discrete probability distribution is given by
   P(X = 1) 0.1, P(X= 2) = 0.2, P(X = 3) = 0.3, P(X =4)= 0.4
   Find (i) E(X) = 3 (ii) Var (X) = 1 (iii) P(X = 2/X≥2) = $\frac{2}{9}$
5. The discrete random variable Y has a probability distribution P(Y = y) = k =1, 2, 3, 4, 5,6
   Find (i) mean, μ = 3.5 (ii) E(3X +4) = $15\frac{1}{6}$ (iii) E(X^2)=14.5 (iv) standard deviation = 1.708
6. The discrete random variable R has a probability distribution is given by
   P(R = r) = $\frac{3r+1}{22}$; r = 0, 1, 2, 3
   Find (i) mean, μ= $\frac{24}{11}$, E(R2) = $\frac{61}{11}$ (iii) E(3R-2) = $\frac{50}{11}$
7. The discrete random variable R has a probability distribution given by
   $$P(R=r)=\begin{cases}\frac{2r+1}{20}; r=0,1,2,3\\ \frac{11-r}{20}, \quad r=4,5\end{cases}$$
   Find (i) E(R) = 2.55, (ii) Var(R) = 1.45
8. The discrete random variable X has a probability distribution given by
   $$P(X=x)=\begin{cases}kx, & x=1,2,3,4,5\\ k(10-x), & x=6,7,8,9\end{cases}$$
   Find (i) constant, k = 0.04, (i) E(X) = 5 (iii) Var(X) = 4
9. The discrete random variable X has a probability distribution is given by
   P(X =x) = kx, x = 1, 2, 3,…..n ; where k is a constant
   Show that k = $\frac{2}{n(n+1)}$, hence find in terms of n the mean X = $\frac{1}{3}(2n+1)$
10. A random variable X of a discrete probability distribution given by
    P(X = 0) = P(X=1) = 0.1, P(X = 2) = 0.2, P(X= 3) = P(X = 4) = 0.3. Find Var (X) = 1.64
11. A random variable X of a discrete probability distribution given by
    P(X = 2) = 0.1; P(X= 4) = 0.3; P(X = 6) = 0.5; P(X = 8) = 0.1. Find Var (X) 2.56

Cumulative distribution function F(X)

F(X) is given by FX = $\sum P(X=x)$

Note F(+∞) = 1 where +∞ is the upper limit.

Example23

A discrete random variable has a probability distribution

x	1	2	3	4	5
P(X = x)	0.2	0.25	0.4	0.1	0.05

Find the cumulative distribution function

Solution

x	1	2	3	4	5
F(X))	0.2	0.45	0.85	0.95	1

Example 24

The random variable X has a cumulative function below

X	-1	0	1	2
F(X)	0.25	0.35	0.80	1

Find the probability distribution function

X	-1	0	1	2
F(X)	0.25	0.1	0.45	0.2

Example 25

A discrete random variable has a cumulative distribution

x	1	2	3	4	5
F(X)	0.2	0.32	0.67	0.91	1

Find (i) probability distribution function

x	1	2	3	4	5
F(X)	0.2	0.12	0.35	0.24	0.09

(ii) P(X = 3) =0.35

(iii) P(X>2) = P(X= 3) + P(X =4) + P(X=5) = 1- 0.12 = 0.68

Example 26

The random variable X has a cumulative function

X	1	2	3	4
F(X)	0.1	0.5	0.8	1

Find (i) mean (ii) Var(X) (iii) mode

Solution

X	1	2	3	4
P(X = x)	0.1	0.4	0.3	0.2

(i) Mean = $\sum xP(X = x)$ = 1 x 0.1 + 2 x 0.4 + 3 x 0.3 + 4 x 0.2 = 2.6

(ii) Var(X)

$E(X^2) = 1^2 \times 0.1 + 2^2 \times 0.4 + 3^2 \times 0.3 + 4^2 \times 0.2 = 5.92$

$Var(X) = E(X^2) - (E(X))^2 = 5.92 - (2.6)^2 = 0.84$

(iii) Mode 2

Revision Exercise 4

1. A discrete random variable has a cumulative distribution

x	0	1	2	3	4
F(X)	01	0.3	0.6	0.8	1

Find (i) E(X) = 15.2 (ii) Var(X)=1.56 (iii) Var(6X + 2) = 56.16

2. The random variable X has a cumulative function below

x	1	2	3	4
F(X)	0.13	0.54	0.75	1

Find (i) P(X=2) = 0.41 (ii) P(X>1) = 0.87 (iii) P(X ≥ 3) = 0.46 (iv) P(X<2) = 0.13 (V)E(X) = 2.58

3. A discrete random variable X has a cumulative distribution

x	3	4	5	6	7
F(X)	0.01	0.23	0.64	0.85	1

Find (i) probability distribution function (ii) Var (X) = 0.9724)

4. A discrete random variable has a cumulative probability function F(X) = $\frac{x^2}{9}$, x = 1, 2, 3.
   Find (1) F(2) = $\frac{4}{9}$ (ii) P(X = 2)= $\frac{1}{3}$ (iii) E(2X -3) = $\frac{17}{9}$
5. A discrete random variable has a cumulative probability function. F(X) = k, x = 1, 2,3
   Find the
   (i) constant k = $\frac{1}{3}$
   (ii) P(X<3) = $\frac{2}{3}$
   (iii) Standard deviation, σ = 0.816
6. A discrete random variable has a cumulative probability function
   F(X) = $1 - \left(1 - \frac{x}{4}\right)^x$ x = 1, 2, 3, 4
   Find the
   (i) F(3) = $\frac{63}{64}$
   (ii) F(2) =$\frac{3}{4}$
   (iii) Var (X) = 0.547

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Equilibrium of forces

several forces acting on a particle are said to be in equilibrium when the resultant force is equal to zero

i.e. $F_R = \begin{pmatrix}0\\0\end{pmatrix}$

Example 1

For the following set of forces in equilibrium find the values of a and b in each case

(i) (6i + 4j)N, (-2i -5j)N, (ai + bj)N

$\begin{pmatrix}6\\4\end{pmatrix} + \begin{pmatrix}-2\\-5\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

6 – 2 + a = 0 => a = -4

4 – 5 + b = 0 => b = 1

(ii) (5i + aj + ck)N, (bi – 6j – k)N, and (-3i + 2j + ck)

$\begin{pmatrix}5\\a\\c\end{pmatrix} + \begin{pmatrix}b\\-6\\-1\end{pmatrix} + \begin{pmatrix}-3\\2\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

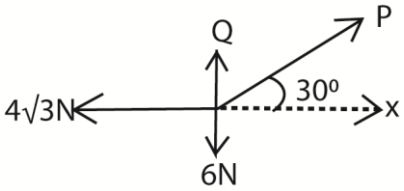
5 + b – 3 = 0 => b = -2

a – 6 +2 = 0 => a = 4

2c – 1 = 0 => c = 0.5

Example 2

In the diagram below, the particle is in equilibrium, find the values of P and Q.



Solution

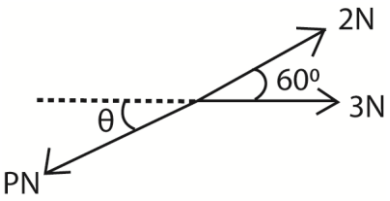
$\begin{pmatrix}0\\Q\end{pmatrix} + \begin{pmatrix}Pcos30\\Qcos30\end{pmatrix} + \begin{pmatrix}0\\-6\end{pmatrix} + \begin{pmatrix}-4\sqrt{3}\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

0 + Pcos30 + 0 -$4\sqrt{3}$ = 0; => P = 8N

Q + Qcos 30 – 6 + 0 = 0; => Q = 2N

Example 3

Diagram below shows three coplanar forces of magnitude 2N, 3N and PN all acting at point O in the direction shown. Given that the forces are in equilibrium, Find the value of P



Solution

-Pcos θ + 2cos 60 + 3 = 0 …. (i)

cos θ = $\frac{2cos60+3}{P}$

-Psinθ + 2sin60 = 0

sin θ = $\frac{2sin60}{P}$ …….. (ii)

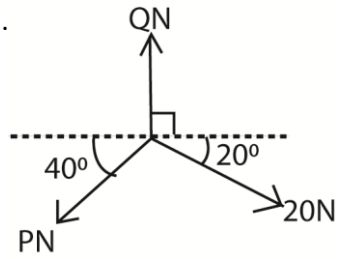
Eqn. (i) and (ii)

θ = $\tan^{-1}\left(\frac{2sin60}{2cos60+3}\right)$= 23.413

from (ii) P = $\frac{2\sin 60}{\sin 23.413}$ =4.3589N

Revision Exercise

1 .



(i) The diagram above shows three coplanar forces in equilibrium. Find the value of P and Q.
(ii) If the direction of Q s now reversed, find the magnitude and direction of the resultant
[(i) 24.5N, 22.6N; (ii) 45.2N]

2. Forces F_1 = (-3i + 7j)N, F_2= (i – j)N and F_3 =(pi +qj) act on a particle.

(i) If the particle is in equilibrium, find the vlues of p and q. [p = 2, q= -6]

(ii) Find the magnitude and direction of the resultant of F1 and F2 . [6.3246N, 71.57^0]

3. Forces of 6N, 5N, 8N, 5N and 9N act on a particles in the direction $N30^0E$, $N30^0W$, $S50^0E$, $N60^0W$, $N80^0E$ and $S40^0W$ respectively. Find the additional force that will keep the system of force in equilibrium. [5.358N at 68.920 above the positive axis]

4. Forces of 7N, 2N, 4N and 5N act on a particle in directions of 0600, 1600 2000 and 3150 respectively. Find the additional force that will keep the system of forces in equilibrium. [2.3125N at 37.18^0 below the negative axis]

5. Forces of 2N, 1N, 3N and 4N act on a particle in the direction 0^0, 90^0, 270^0 and 330^0 respectively. Find the additional force that will keep the system of forces in equilibrium. [6.8N at 36^0 above the negative axis]

6. Forces of 6N, 5N, 7N, 4N, $3\sqrt{2}\,N$ and $7\sqrt{2}$N act in direction AB, CB, CD, DA, CA and DB respectively on a square ABCD. Find the additional force that will keep the system of forces in equilibrium. [19.2N at 81^0 above the negative axis]

7. Forces 8N, 7N, 6N, 4N, 7N and 6N act along the sides of a regular hexagon ABCDEF in direction AB, CB, CD, DE, EF and FA respectively. Find the additional force that will keep the system of forces in equilibrium. [12.49N at 76^0 above AB]

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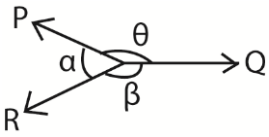
Equilibrium of three forces L ami's theorem

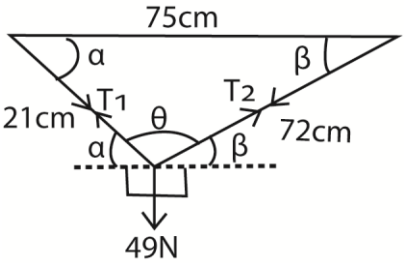
For any three forces acting on a particle in equilibrium where none of them is parallel to each other, Lami's theorem is applicable



$$\frac{P}{sin\beta} = \frac{Q}{sin\alpha} = \frac{R}{sin\theta}$$

Example 1

A weight of 49N is suspended by two strings of length 21 cm and 72cm attached to 2 points in a horizontal line a distance of 75cm apart. Find the tension in the strings so that the particle remain in equilibrium



By cosine rule:

$75^2 = 21^2 + 72^2 - 2 \times 21 \times 72\cos\theta$

$\theta = 90^0$

Similarly, $\beta = 16.26^0$ and $\alpha = 73.74^0$

$$\frac{T_1}{\sin(16.26+90)} = \frac{49}{\sin 90};$$

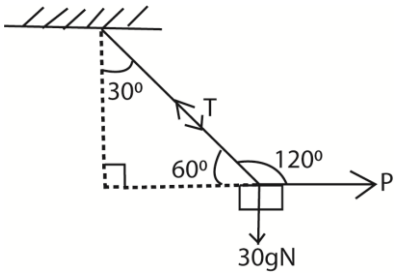
$$\therefore T_1 = 47N$$

$$\frac{T_2}{\sin(73.74+90)} = \frac{49}{\sin 90};$$

$$\therefore T_2 = 13.72N$$

Example 2

Mass of 30kg hangs vertically at the end of a light string. If the mass is pulled by a horizontal force P so that the string makes 30^0 with the vertical. Find the magnitude of the force and the tension in the string so that the particle remain in equilibrium.

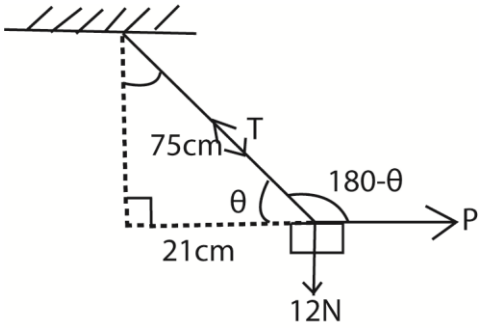


$$\frac{T}{\sin 90} = \frac{30\ x\ 9.8}{\sin 120};\ T = 339.48N$$

$$\frac{P}{\sin(60+90)} = \frac{30\ x\ 9.8}{\sin 120};\ P = 16974N$$

Example 3

One end of a light inextensible string of length 75cm is fixed to a point on a rigid pole. The particle of weight 12N is attached to the other end of the string. The particle is held 21cm away from the pole by a horizontal force, P. Find the magnitude of the force, P and the tension of the string so that the particle remain in equilibrium



$\theta = \cos^{-1}\left(\frac{21}{75}\right) = 73.74^0$

$\frac{T}{\sin 90} = \frac{12}{\sin(180-73.74)}$

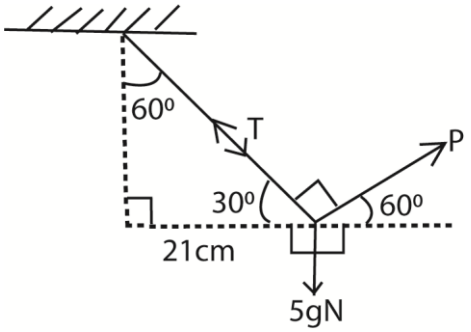
T = 12.5N

$\frac{P}{\sin(90+73.4)} = \frac{12}{\sin(180-73.74)}$

P = 3.5N

Example 4

A light inextensible string AB whose end A is fixed has end B attached to a particle of mass 5kg. A force P acting perpendicular to the string is applied on the particle keeping it in equilibrium with the string inclined at 60^0 to the vertical. Find the value of P and the tension in the string



$\frac{T}{\sin(90+60)} = \frac{5\ x\ 98}{\sin 90}$

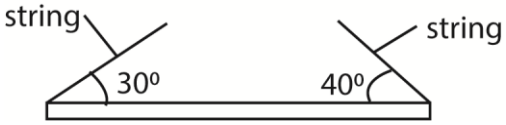
T = 24.5N

$\frac{5\ x\ 98}{\sin 90} = \frac{P}{\sin\ (90+30)}$

P = 42.44N

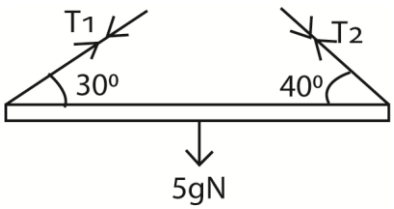
Example 5

A non-uniform beam of mass 5kg rests horizontally in equilibrium supported by two strings attached to the ends of the beam.



The strings makes 300 and 400 with the horizontal beam as shown above. Find the tension in the strings.

Solution



(→) $T_1\cos30 = T_2\cos40$; $T_1 = 0.8846T_2$

(↑)$T_1\sin30 + T_2\sin40 = 5g$

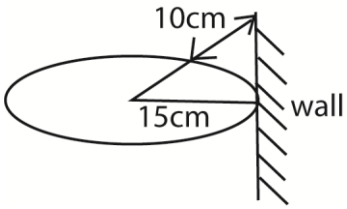
$0.8846T_2\sin30 + T_2\sin40 = 5 \times 9.8$

$T_2 = 45.159N$

$T_1 = 0.8846 \times 45.159 = 39.94N$

Example 6

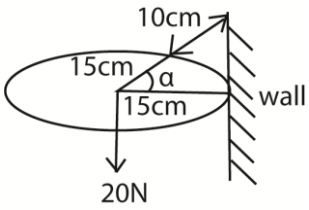
A sphere of weight 20N and radius 15cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 10cm attached to a point on the sphere and to a point on the wall as shown.



(i) calculate the reaction on the sphere due to the wall

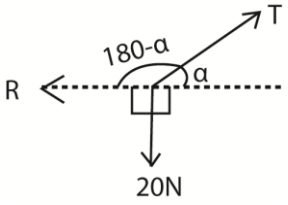
(ii) Find the tension in the string

Solution



$\alpha = \cos^{-1}\left(\frac{15}{25}\right) = 53.13^0$

Using Lami's theory



$\frac{T}{sin90} = \frac{20}{\sin(180-53.13)}$; T =25N

$\frac{R}{\sin(90+53.13)} = \frac{20}{\sin(180-53.13)}$; R = 15N

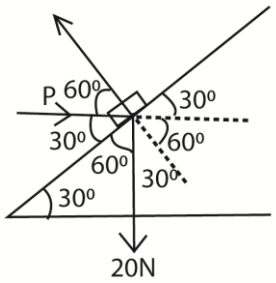
Example 7

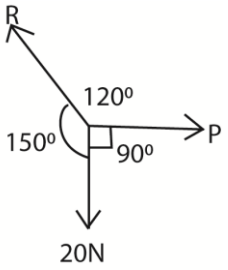
A particle of weight 20N is held at equilibrium on a smooth plane inclined at 30^0 to the horizontal by a horizontal force P.

(i) Find the value of P and the reaction between the particle and the plane.

(ii) If the force P is removed and a string parallel to the plane is used to hold the particle, find the tension in the string and the new value of the reaction.

Solution

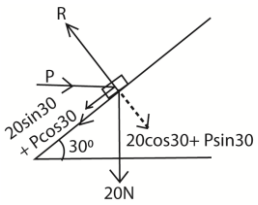




$\frac{P}{sin150} = \frac{R}{sin90} = \frac{20}{sin120}$

R = 23.09N and P = 11.55N

Alternatively: by resolving forces



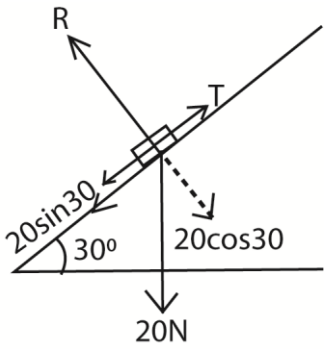
At equilibrium parallel to plane forces = 0

Pcos30 + 20sin30 = 0; P = 11.55N

R = 20cos30 + Psin30

R = 20cos30 + 11.55sin30 = 23.09N

(ii)



Parallel to the plane T = 20sin30 = 10N

Perpendicular to the plane R = 20cos30 = 13N

Alternatively by Lami's theory

$$\frac{T}{sin150} = \frac{R}{sin120} = \frac{20}{sin90}$$

T = 10N

R = 1.3N

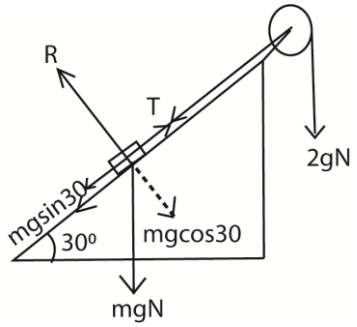
Example 8

A light inextensible string passes over a smooth fixed pulley at the top of a smooth plane inclined at 300 to the horizontal. A particle of mass 2kg is attached to one end of the string and rests vertically in equilibrium when the particle of ma resting on the surface of the plane is attached to the other end of the string. Find

(i) the normal reaction between m and the plane
(ii) tension in the string and the value of m.

Solution

By resolving forces



For 2kg mass: T – 2 x 9.8 = 0; T = 19.62N

Parallel to the plane

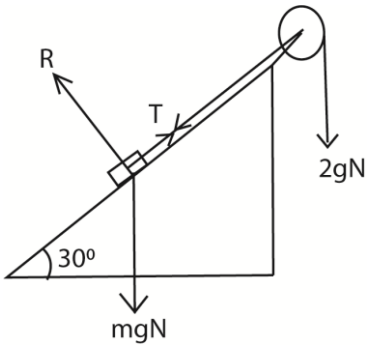
T – mgsin30 = 0; m = 4kg

Perpendicular to the plane

R = mgco30

R = 4 x 98cos30 = 33.98

Alternatively by using Lami's theorem



For 2kg mass: T – 2 x 9.8 = 0; T = 19.62N

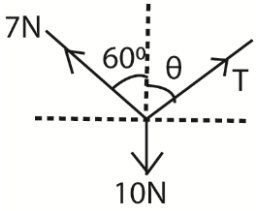
$$\frac{T}{sin150} = \frac{mg}{sin90} = \frac{R}{sin120}$$

$$\frac{19.62}{sin150} = \frac{mg}{sin90} = \frac{R}{sin120}$$

m = 4kg and R = 33.98N

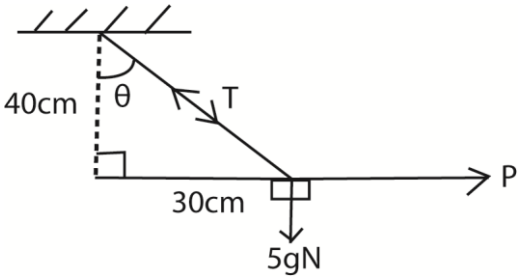
Revision exercise

1. A particle P of mass 2kg is suspended from a fixed point O by means of a light inextensible string. The string is taut and makes an angle of 300 with the downward vertical through O and a particle is held in equilibrium by means of a horizontal force of magnitude F acting on the particle. Find the value of F and the tension in the string [F = 11.3161, T = 22.6321N]
2. A particle of mass 3kg lies on a smooth plane inclined at angle θ to the horizontal, where tanθ = $\frac{3}{4}$. The particle is held in equilibrium by horizontal force of magnitude FN. The line of action of this force is the same vertical plane as a line of greatest slope of inclined plane. Find the value of F. [22.05N]
3. The diagram below shows a body of weight 10N supported in equilibrium by two light inextensible strings. The tension in the strings are 7N and T and the angle the string makes with the upward vertical are 60^0 and θ respectively.



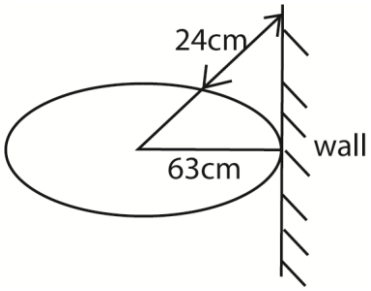
Find T and θ. [T = 8.9N, θ = 43^0]

4. A particle of weight 8N is attached to a point B by a light inextensible string AB. It hangs in equilibrium with point A fixed and AB at an angle of 30^0 to the downward vertical. A force F at B acting at right angles to AB, keeps the particle in equilibrium. Find the magnitude of force F and the tension in the string. [4N, $4\sqrt{3}N$]
5. The diagram shows a light inextensible string with one end fixed at A and a mass of 5kg suspended at the other end.



The mass is held in equilibrium at an angle θ to the downward vertical by a horizontal force P. Find the value of θ, P and the tension in the string [θ = 36.9^0, P = 36.75N, T = 61.25N)

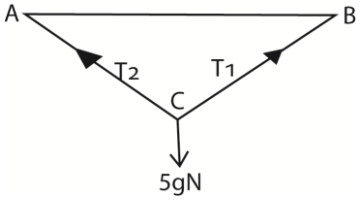
6. A sphere of mass 5kg and radius 63cm rests against a smooth vertical wall. A sphere is supported in its position by a string of length 24cm attached to a point on the sphere an to a point on the wall as shown.



Find the tension in the string. [71.05N]

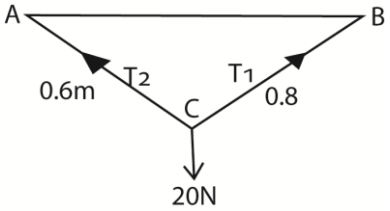
7. A particle whose weight is 50N is suspended by a light string which is 35^0 to the vertical under the action of a horizontal force F. Find the force F and the tension in the string. [35.0N, 61.0N]

8. A particle of weight w rests on a smooth plane which inclined at 40^0 to horizontal. The particle is prevented from slipping by a force of 50.0N acting parallel to the plane and up a line of greatest slope. Calculate w and reaction due to the plane. [77.8N, 59.6N]
9. A mass of 2kg is suspended by two light inextensible strings. One making an angle of 600 with the upward vertical and the other 300with the upward vertical. Find the tension in each string. [9.8N, 17.0N]
10. A heavy uniform rod of weight W is hung from a point by two equal strings, one attached to each end of the rod. A body of weight w is hang half-way between A and the center of the rod. Prove that the ratio of tension in the string is $\frac{2W+3w}{2W+w}$.
11. A non-uniform beam AB of length 8m and its weight 10N acts from a point G between A and B such that AG = 6m. The beam is supported horizontally by strings attached to A and B. The string attached to A makes an angle of 30^0 with AB. Find the angle that the string attached to B makes with AB and find the tension in the strings. [60^0, 5N, 8.66N]
12. A light inextensible string of length 40cm has its upper end fixed to a point A and carries a mass of 2kg at its lower end. A horizontal force applied to the mass keeps it in equilibrium, 20cm from the vertical through A. Find the magnitude of this horizontal force and the tension in the string. [11.3N, 22.6N]
13. The diagram shows a body of mass 5kg supported by two light inextensible strings, the other ends of which are attached to two points A and B on same level as each other end 7m apart.



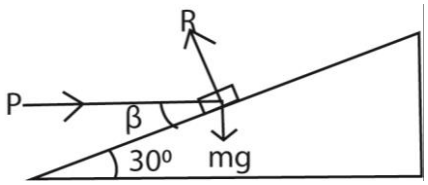
The body rests in equilibrium at 3m vertically below AB. If angle CBA = 45^0, find T_1 and T_2 the tensions in the strings. [35N, $28\sqrt{2}N$]

14. The diagram shows a body of weight 20N supported by two light inextensible strings of length 0.6m and 0.8m from two points 1m apart on a horizontal beam.



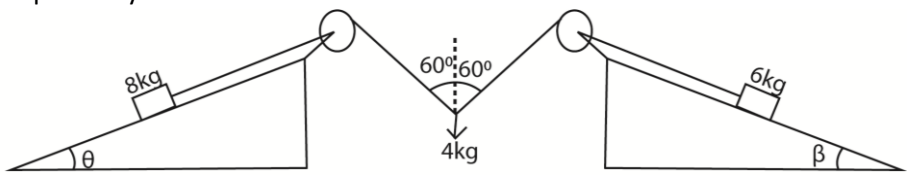
The body rests in equilibrium, find T_1 and T_2 the tensions in the strings. [16N, 12N]

15. A light inextensible string of length 50cm has its upper end fixed at point A and carries a particle of 8kg at its lower end. A horizontal force P applied to the particle in equilibrium 30cm from the vertical through A, find the magnitude of P and the tension in the string. [58.8N, 98N}
16. A article is in equilibrium under the action of forces 4N due north, 8N due west, $5\sqrt{2}$N south east and P, find the magnitude and direction of P. [3.16N, N71.6^{0}E]
17. A force P holds a particle of mass mkg in equilibrium on a smooth plane which is inclined at 300 to the horizontal.



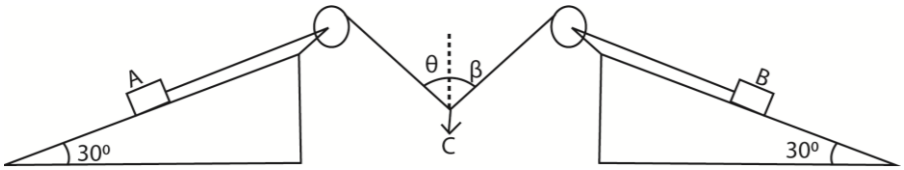
If P makes an angle β with the plane, find β when R the normal reaction between the particle and the plane is 15mg [51.7^0]

18. The diagram below shows masses of 8kg and 6kg lying on smooth planes of inclination θ and β respectively



Light inextensible strings attached to these masses pass along the line of greatest slopes over smooth pulleys and are connected to 4kg mass hanging freely. The strings both make an angle of 600 with the upward vertical as shown above. If the system rest in equilibrium find θ and β. [$\theta = 30^0$ and β 41.8^0]

19. The diagram below shows masses A and B each lying on smooth planes of inclination30^0.



Light inextensible strings attached to A and B pass along the lines of greatest slopes, over smooth pulleys and are connected to a third mass C hanging freely. The strings make angles of θ and β with the upward vertical as shown above. If A, B and C have masses 2m, m, and m respectively and the system rests in equilibrium show that $\sin\theta = 2\sin\beta$ and $\cos\beta + 2\cos\theta = 2$. Hence find θ and β. [$29.0^0$, 75.5^0]

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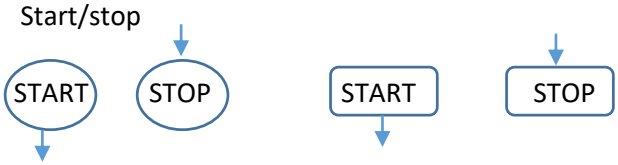
Flowcharts in mathematics

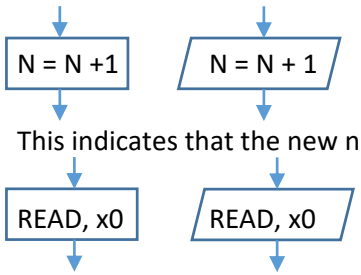
A flow chart is a diagram comprising of systematic steps followed in order to solve a problem.

Shapes used

1. Start/stop

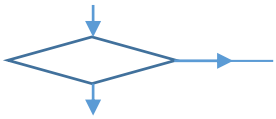


2. OPERATIONASSIGNMENT



This indicates that the new number is obtained by adding one to the previous N

3. Decision box



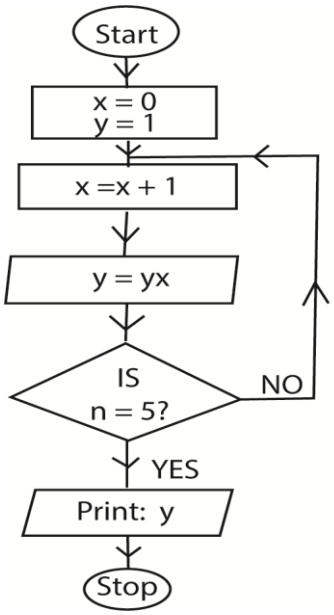
Note: all other shapes can be interchanged except for the decision box

Dry run or trace

This is the method of predicting the outcome of a given flow chart using a table

Example 1

Perform a dry run and state the purpose of the flowchart



Solution

Dry run

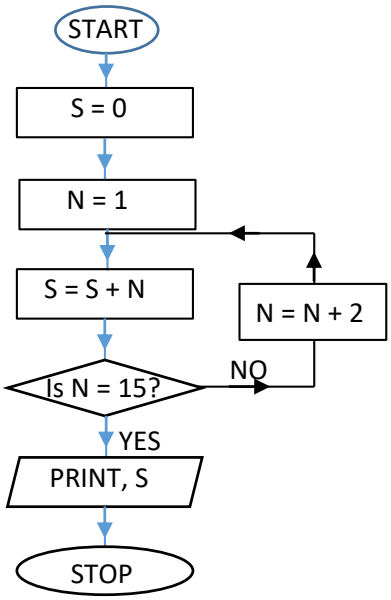
x	y
0	1
1	1
2	2
3	6
4	24
5	120
6	720

Purpose is to compute and print 6!

Relationship is y = x!

Example 2

Study the flow chart below and perform dry run of the flowchart



Solution

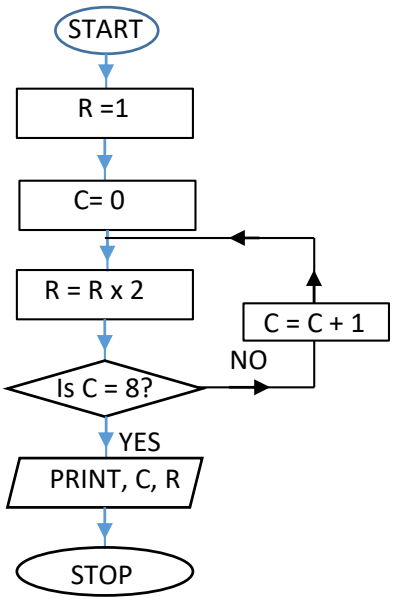
Dry run

N	S	Is N = 15?
1	1	NO
3	4	NO
5	9	NO
7	16	NO
9	25	NO
11	36	NO
13	49	NO
15	64	YES

Purpose is to compute and print the first 8 square numbers

Example 3

Perform a dry run and state the purpose of the flowchart



Solution

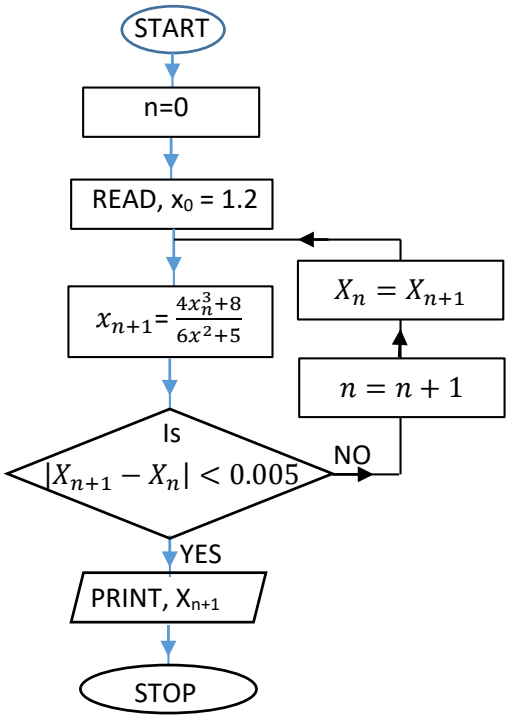
Dry run

C	R	Is C = 8?
0	1	NO
1	2	NO
2	4	NO
3	8	NO
5	32	NO
6	64	NO
7	128	NO
8	256	YES

Purpose is to compute and print 2^8

Example 4

The flowchart below is used to read the root of the equation $2x^3 + 5x - 8 = 0$



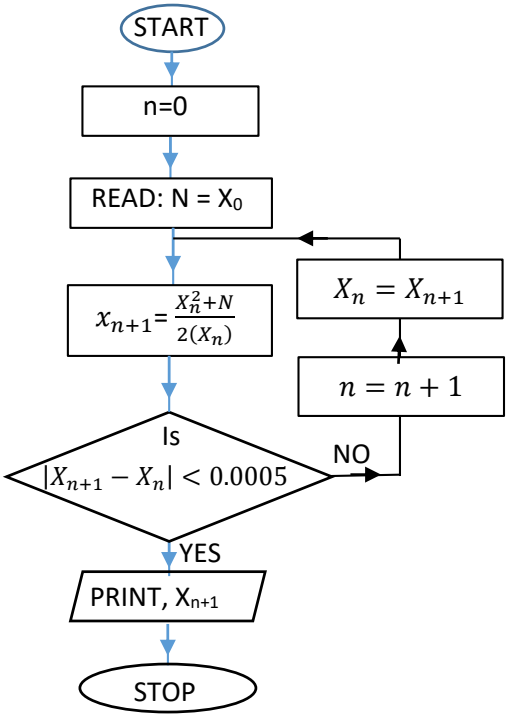
Carry out a dry run of the flow chart and obtain the root of $2x^3 + 5x - 8 = 0$ with an error less than 0.005

N	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	1.2	1.0933	0.1067
1	1.0933	1.0867	0.0066
2	1.0867	1.0866	0.001

Root is 1.087

Example 5

Study the flowchart below



(i) Carry out a dry run of the flowchart, taking N = 20, X0 = 4 and obtain the root of correct to 3dp.

(ii) State its purpose

Solution

N	X_n	X_{n+1}	$\lvert X_{n+1} - X_n \rvert$
0	4.0	4.5	0.5
1	4.5	4.4722	0.0278
2	4.4722	4.4721	0.0001

Root is 4.472

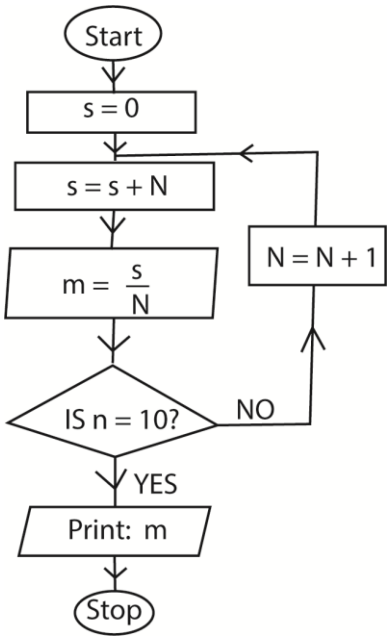
(ii) to print the square root of a number N

Constructing flowcharts

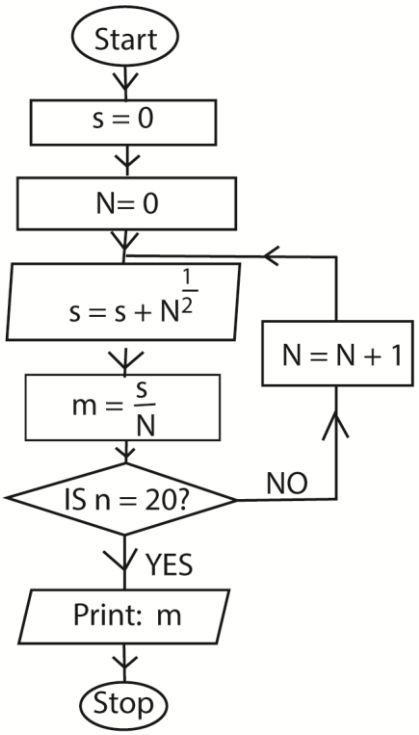
1. Draw a flow chart that reads and prints the mean of the first ten counting numbers

Solution

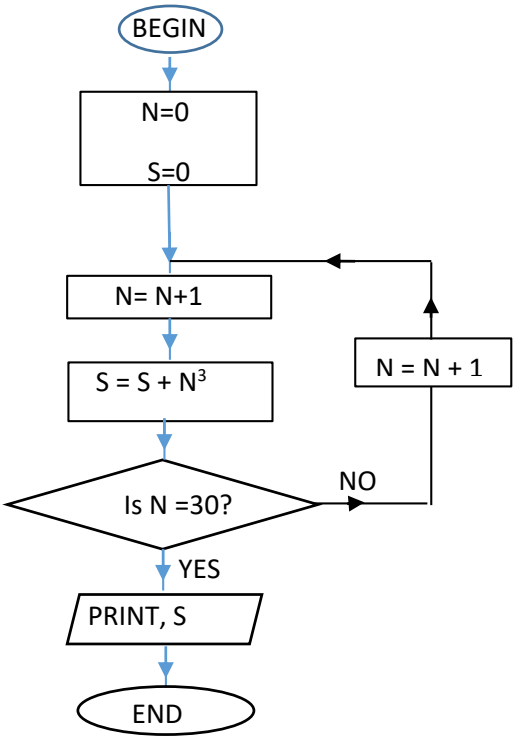
Let S be sum and m the mean



2. Draw a flowchart for computing and printing the mean of the square roots of the first 20 natural numbers

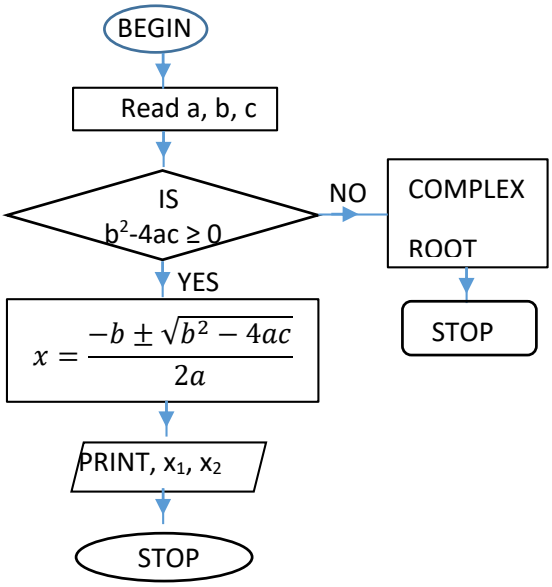


3. Draw a flowchart that computes and prints the sum of the cubes of the first 30 natural numbers



4. Draw a flowchart that computes the root of the equation ax2 + bx + c = 0

Solution



Newton Raphson's method and Flowcharts

Example 6

(a) Show that the iterative formula based on Newton Raphson's method for approximating the root of the equation 2lnx – x + 1 = 0 is given by

$x_{n+1} = x_n\left(\frac{2Inx_n-1}{x_n-2}\right)$, n = 0, 1,2 …….. (03marks)

f(x) = 2lnx – x + 1

$f'(x) = \frac{2}{x} - 1$

also

$f(x_n) = 2Inx_n - x_n + 1$

$f'(x_n) = \frac{2}{x_n} - 1$

Using $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$

By substitution, we get

$$x_{n+1} = x_n - \frac{2Inx_n - x_n + 1}{\left(\frac{2}{x_n} - 1\right)}$$

$$= \frac{x_n\left(\frac{2}{x_n} - 1\right) - 2Inx_n - x_n + 1}{\left(\frac{2}{x_n} - 1\right)}$$

$$= \frac{x_n(2 - x_n) - x_n(2Inx_n - x_n + 1)}{(2 - x_n)}$$

$$= \frac{x_n(2 - x_n - 2Inx_n + x_n - 1)}{(2 - x_n)}$$

$$= \frac{x_n(1 - 2Inx_n)}{(2 - x_n)}$$

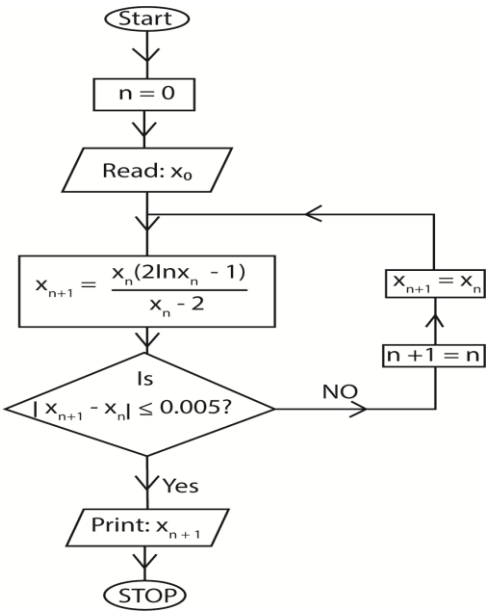
$$= \frac{-x_n(2Inx_n - 1)}{-(x_n - 2)}$$

$$= \frac{x_n(2Inx_n - 1)}{(x_n - 2)}$$

(b) Draw a flow chart that:

(i) reads the initial approximation x_0 of the root

(ii) computes and prints the root correct to two decimal places, using the formula in (a) (05marks)



(ii) Taking x_0 = 3.4, perform a dry run to find the root of the equation (04marks)

Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	3.4	3.51548	0.11548
1	3.51548	3.51286	0.00262
2	3.51286	3.51286	0.0000

Example 7

(a) Show that the Newton-Raphson formula for finding the root of the equation x = $N^{\frac{1}{5}}$ is given by $X_{n+1} = \frac{4X_n^5+N}{5X_n^4}$, n = 0, 1, 2, … (04marks)

$x = N^{\frac{1}{5}}$

$x^5 = N$

$x^5 - N = 0$

Let f(x) = $x^5 - N$

$f(x_n) = x_n^5 - N$

$f'(x_n) = 5x_n^4$

Using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5-N}{5x_n^4} = \frac{5x_n^5 - x_n^5-N}{5x_n^4}$$

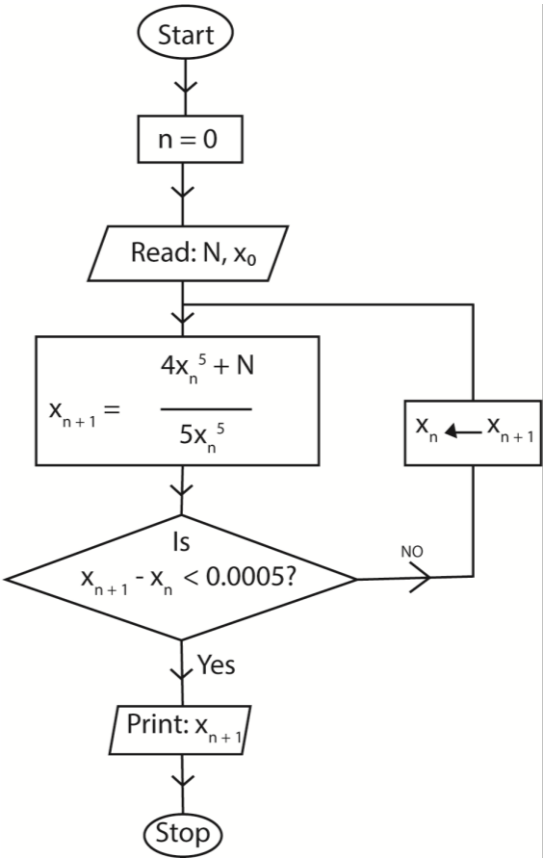
$x_{n+1} = \frac{4x_n^5+N}{5x_n^4}$, n = 0, 1, 2 …

(b) Construct a flow chart that

(i) reads N and the first approximation x_0.

(ii) computes the root to three decimal places

(iii) Prints the root (x_n) and the number of iteration (n) (05marks)



(c) Taking N = 50, x_0 = 2.2, perform a dry run for the flow chart. Give your root correct to three decimal places.(03marks)

N = 50, x_o = 2.2

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.2	2.18688	0.01312
1	218688	2.18672	0.00016

Root = 2.187(3D)

Example 8

(a) Show that the iterative formula based on Newton Raphson's method for solving the equation

Inx + x – 2 = 0 is given by

$X_{n+1} = \frac{x_n(3-Inx_n)}{1+x_n}, n = 0, 1, 2 \ldots$ (04marks)

$let\ f(x) = Inx + x - 2$

$f(x_n) = Inx_n + x_n - 2$

$f'(x) = \frac{1}{x_n} + 1 = \frac{1+x_n}{x_n}$

Using N.R.M

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \left(\frac{Inx_n + x_n - 2}{\frac{1+x_n}{x_n}}\right)$

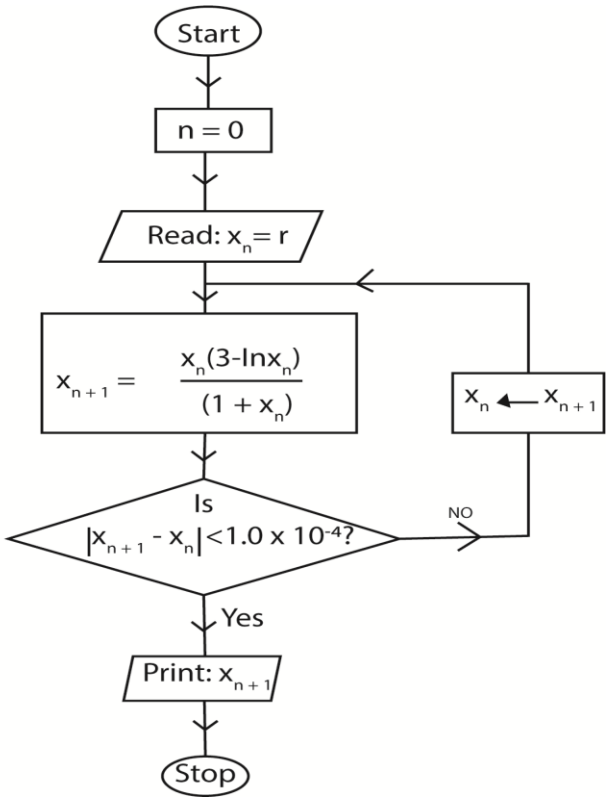
$= \frac{x_n}{1} - \frac{x_n(Inx_n + x_n - 2)}{1+x_n}$

$= \frac{x_n(1+x_n - Inx_n - x_n + 2)}{1+x_n}$

(b)(i) Construct a flow chart that ;

- reads the initial approximation as r

- computes using the interactive formula in (a), and prints the root of equation Inx + x -2 = 0, when the error is less than 1.0 x 10^{-4}.



(ii) Perform a dry run of the flow chart when r = 1.6. (08marks)

n	x_n	X_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	1.6	1.5569	0.0431
1	1.5569	1.5571	0.0002
2	1.5571	1.5571	0.0000

Hence the root = 1.557(3D)

Example 9

(a) Show that iterative formula based on Newton Raphson's method for approximating the sixth root of a number N is given by $x_{n+1} = \frac{1}{6}\left(5x_n + \frac{N}{x_n{}^5}\right)$

(b) Draw a flowchart that
   (i) Reads N and the initial approximation x_0 of the root
   (ii) computes and prints the root to three decimal places

(c) Taking N = 60, x0 = 1.9, perform a dry run for the flow chart, give your root correct to three decimal places.

Solution

$x = N^{\frac{1}{6}}$
$x^6 = N$
$x^6 - N = 0$
Let f(x) = $x^6 - N$
$f(x_n) = x_n^6 - N$
$f'(x_n) = 6x_n^5$

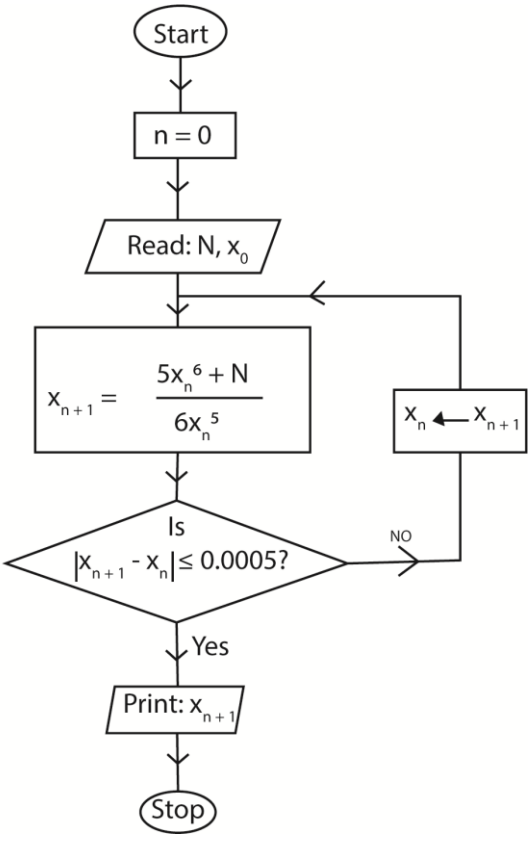
Using NRM

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - N}{6x_n^5}$$

$$= \frac{x_n(6x_n^5) - (x_n^6 - N)}{6x_n^5}$$

$$x_{n+1} = \frac{5x_n^6 + N}{6x_n^5}, \text{ n = 0, 1, 2 ...}$$

(b) Flowchart



(c) Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	1.9	1.9872	0.082
1	1.9872	1.9787	0.0085
2	1.9787	1.9786	0.0001

Example 10

(a) Show that the iterative formula based on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = \frac{3}{4}\left(x_n + \frac{N}{3x_n{}^3}\right)$.

(b) Draw a flowchart that
   (i) Records N and initial approximation x_0 of the root
   (ii) computes and prints the root after four iterations.

(c) Taking N = 39.0, x0 = 2.0, perform a dry run for the flowchart, give your root correct to three decimal places

Solution

$x = N^{\frac{1}{4}}$
$x^4 = N$
$x^4 - N = 0$
Let f(x) = $x^4 - N$
$f(x_n) = x_n^4 - N$
$f'(x_n) = 4x_n^3$

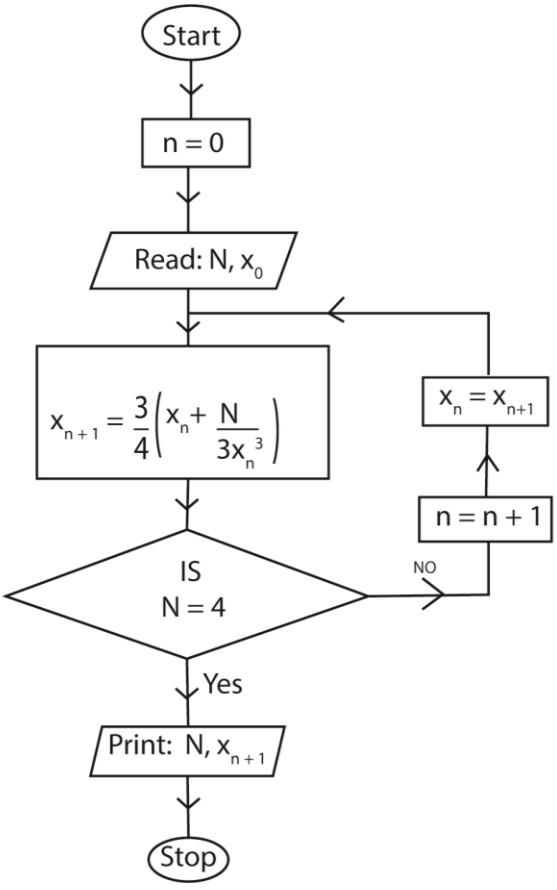
Using NRM

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - N}{4x_n^3}$$

$$= \frac{x_n(4x_n^3) - (x_n^3 - N)}{4x_n^3}$$

$$x_{n+1} = \frac{3x_n^3 + N}{4x_n^4} = \frac{3}{4}\left(x_n + \frac{N}{3x_n{}^3}\right), \text{ n = 0, 1, 2 ...}$$

(b) Flowchart



(c) Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.0	2.7188	0.7188
1	2.7188	2.5242	0.1945
2	2.5242	2.4994	0.0249
3	2.4994	4.4990	0.0004

Example 11

(a) Show that the iterative formula based on Newton's Raphson's method for finding the natural logarithm of a number N is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+N}{e^{x_n}}$, n = 0, 1, 2, …

(b) Draw a flowchart that

(i) Records N and initial approximation x_0 of the root

(ii) computes and prints the natural logarithm after four iteration and gives natural logarithm to 3 decimal places.

(c) Taking N = 10, x_0 = 2, perform a dry run for the flowchart, give your root correct to three decimal places

Solution

(a) $x = \ln N$; $e^x = N \Rightarrow e^x - N = 0$

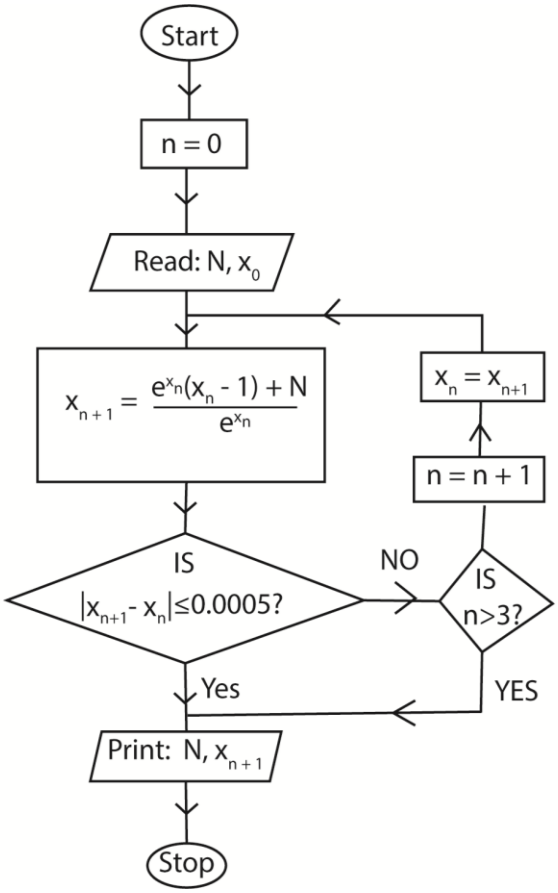
$f(x) = e^x - N$; $f^1(x) = e^x$

$x_{n+1} = x_n - \left(\frac{e^{x_n}-N}{e^x}\right)$

$= \frac{x_n e^x - (e^{x_n}-N)}{e^x}$

$= \frac{e^x(x_n-1)+N}{e^x}$

(b) Flowchart



(c) Dry run

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2.0	2.3533	0.3533
1	2.3533	2.3039	0.0494
2	2.3039	2.3026	0.0013
3	2.3026	2.3026	0.0000

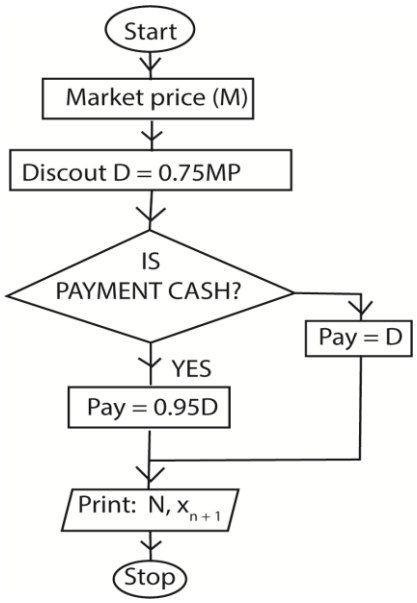
Example 12

A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.

(a) Construct a flowchart for the above information

(b) perform a dry run for (i) a shoe of 75,000/= cash and (ii) a shirt of 45,000/= credit

(a) Flowchart



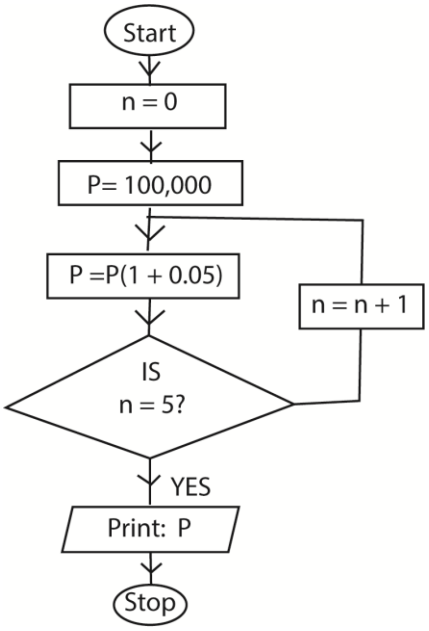
(c) dry run

MP	D= 0.75MP	Pay	Cash = 0.95D	Credit = D
75,000	56,250	cash	53437.50	--------
45,000	33,750	credit	----------	33750

Example 13

Given that a man deposited 100,000/= to a bank which gives a compound interest of 5%. Draw a flowchart to compute the amount of money accumulated after 5 years and perform a dry run for the flowchart.

Flowchart



Dry run

n	P	A
0	100,000	100,000
1	100,000	105,000
2	105,000	110,250
3	110,250	115,762.0
4	115,762,5	121,550.625
5	121,550.625	127,628.1563

Revision Exercise

1. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of a number N is given by $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root to three decimal places.
   (c) Taking N = 54, x_0 = 3.7, perform a dry run for the flowchart, give your root to three decimal places [3.780]
2. (a) show that the iterative formula based on Newton Raphson's method for approximating the fourth root of a number N is given by $x_{n+1} = \frac{3}{4}\left(x_n + \frac{N}{3x_n^3}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root to two decimal places.
   (c) Taking N = 35, x_0 = 2.3, perform a dry run for the flowchart, give your root to two decimal places. [2.43]
3. (a) show that the iterative formula based on Newton Raphson's method for finding the root of a number $N^{\frac{1}{5}}$ is given by $x_{n+1} = \left(\frac{4x_n^5+N}{5N_n^4}\right), n = 0, 1, 2, \ldots$
   (c) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root to three decimal places.
   (d) Taking N = 50, x_0 = 2.2, perform a dry run for the flowchart, give your root to three decimal places [2.187]
4. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of 2Inx – x + 1 = 0 is given by $x_{n+1} = x_n\left(\frac{2Inx_n-1}{x_n-2}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads N and the initial approximation x_0 of the root
      (ii) computes and prints the root
   (c) Taking x_0 = 3.4, perform a dry run for the flowchart, give your root to three decimal places
5. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of Inx + x – 2 = is given by $x_{n+1} = x_n\left(\frac{3-Inx_n}{1+x_n}\right), n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (iii) reads N and the initial approximation r of the root
      (iv) computes and prints the root of the equation, when the error is less than 10. x 10^{-4}.
   (c) Taking r = 1.6, perform a dry run for the flowchart, give your root to three decimal places
6. (a) show that the iterative formula based on Newton Raphson's method for approximating the cube root of x = In(x +2) is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+2}{e^{x_n}-1} n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that
      (i) reads the initial approximation x_0 of the root
      (ii) computes and prints the root to three decimal places
   (c) Taking x_0 = 1.2, perform a dry run for the flowchart, give your root to three decimal places
7. (a) show that the iterative formula based on Newton Raphson's method for finding the natural logarithm of number N is given by $x_{n+1} = \frac{e^{x_n}(x_n-1)+N}{e^{x_n}}, n = 0, 1, 2, \ldots$
   (b) Draw a flowchart that

(i) reads N and the initial approximation x_0 of the root

(ii) computes and prints the root to two decimal places.

(c) Taking N = 45, x_0 = 3.5, perform a dry run for the flowchart, give your root to two decimal places [3.81]

8. (a) show that the iterative formula based on Newton Raphson's method for finding the root of the $2x^3 + 5x - 8$ is given by $x_{n+1} = \left(\frac{4x_n^3+8}{6x_n^2+5}\right), n = 0, 1, 2, \ldots$

(b) Draw a flowchart that

(i) reads N and the initial approximation α of the root

(ii) computes and prints the root when the error is less than 0.001.

(c) Taking α= 1.1, perform a dry run for the flowchart, give your root to three decimal places [1.087]

9. A shop offers a 25% discount on all items in their store and a second discount of 5% for paying cash.

(a) Construct a flowchart for the above information

(b) perform a dry run for (i) a radio of 125,000/= cash and (ii) a T.V of 340,000/= credit [89.062.50, 255,0000]

10. Given that a man deposited 120,000/= to a bank which gives a compound interest of 15%. Draw a flowchart to compute the amount of money accumulated after 4 years and perform a dry run for the flowchart. [209,880.75/=]

Thank you

Dr. Bbosa Science

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Force and Newton's laws of motion

Law I: A body continues in its state of rest or uniform motion in a straight line unless acted upon by an external force

Law II: The rate of change of momentum of a body is directly proportional to the applied force and takes place in the direction of the force.

$F = m\left(\frac{v-u}{t}\right)$ but $\left(\frac{v-u}{t}\right)$ =a

= ma

NB: F must be the resultant force

Example 1

Find the acceleration produced when a body of mass 5kg experiences a resultant force of 10N

F = ma | 10 = 5a | $a = 2ms^{-2}$

Example 2

A car of mass 600kg travels a distance of 24m while uniformly accelerated from rest to $12ms^{-1}$

(i) Find the acceleration of the car
$v^2 = u^2 + 2as$
$12^2 = 0^2 + 2a \times 24$
$a = 3ms^{-2}$

(ii) determine the accelerating force
F = ma = 600 x 3 = 1800N

Example 3

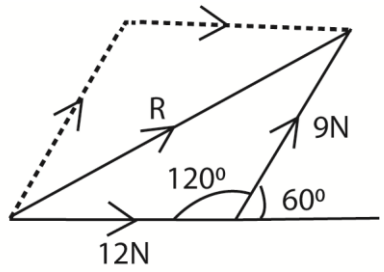
A body of mass 500g experiences a resultant force 3N. Find

(i) Acceleration produced
F = ma | 3 = 0.5 x a | $a = 6ms^{-2}$

(ii) Distance travelled by the body while increasing speed from $1ms^{-1}$ to $7ms^{-1}$
$v^2 = u^2 + 2as$
$7^2 = 1^2 + 2x\ 6 \times s$
s = 4m

Example 4

Two forces of magnitude 12N and 9N act on a particle producing an acceleration of $3.65ms^{-2}$. The two forces act at an angle of 600 to each other. Find the mass of the particle.



$R^2 = 12^2 + 9^2 - 2 \times 12 \times 9 \cos 120^0$

R = 18.25N

F = ma

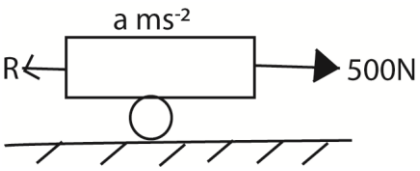
18.25N = 3.65m

m = 5kg

When resistance or friction is involved

Example 5

A car moves along a level road at constant velocity of $22ms^{-2}$. If its engine is exerting a forward force of 500N, what resistance is the car experiencing.



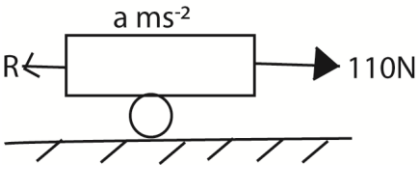
F = ma

500 – R = m x 0

R = 500N

Example 6

A car of mass 500kg moves along a level road with acceleration of $2ms^{-2}$. Its Engine is exerting a forward force of 110N. What is the resistance a car is experiencing?



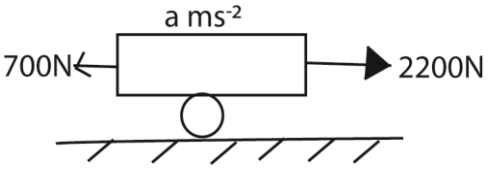
F = ma

500 – R = 500 x 2

R = 100N

Example 7

A van of mass 2tonnes moves along a level road against resistance of 700N. If its engine is exerting a forward force of 2200N. Find the acceleration of the van



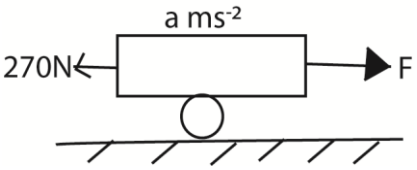
F = ma

2200 – 700 = 2000 x a

$a = 0.75ms^{-2}$

Example 8

Find the constant force necessary to accelerate a car of mass 1000kg from $15ms^{-1}$ to $20ms^{-1}$ in 10s against a resistance of 270N



v = u + at

20 = 15 + 10a

a = $0.5ms^{-2}$

F = ma

F – 270 = 1000 x 0.5

F = 770N

Calculations involving vector form

Find the resultant force required to make a body of mass 2kg at $(5i + 2j)ms^{-2}$.

F = ma | F = $2\left(\frac{5}{2}\right) = \begin{pmatrix}10\\4\end{pmatrix} N$

Example 9

F = ma $\quad \begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = 0.5a$ | a = $\begin{pmatrix}6\\6\end{pmatrix} ms^2$

Find the acceleration produced in a body of mass 500N is subjected to forces of (4i + 2j)N and (-i + j)N

F = ma | $\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = 0.5a$ | a = $\begin{pmatrix}6\\6\end{pmatrix} ms^2$

Example 10

Find the magnitude of the acceleration produced in a body of mass 2kg subjected to forces of (2i – 3j +4k)N and (i + 5j + 2k)N

F = ma | $\begin{pmatrix}2\\-3\\4\end{pmatrix} + \begin{pmatrix}1\\5\\2\end{pmatrix} = 2a$ | a = $\begin{pmatrix}1.5\\1\\3\end{pmatrix} ms^2$ | $|a| = \sqrt{1.5^2 + 1^2 + 3^2} = 2.3ms^{-2}$

Example 10

A particle of mass 2.5kg is acted on by a resultant force of 15N acting in the direction (2i – j – 2k). find the magnitude of the acceleration

F = 15 x $\frac{2i-j-2k}{\sqrt{2^2+(-1)^2+(-2)^2}}$

= 15 x $\frac{2i-j-2k}{3}$

F = 10i – 5j -10k

F = ma

$\begin{pmatrix}10\\-3\\-10\end{pmatrix} = 2a$

a = $\begin{pmatrix}4\\-2\\-4\end{pmatrix}$

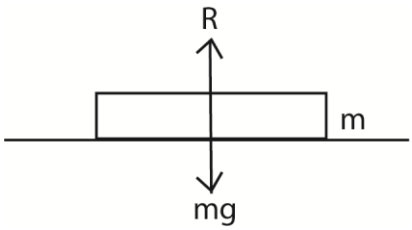
$|a| = \sqrt{10^2 + (-3)^2 + (-10)^2}$

=6ms-2

Law III: To every action there is an equal but opposite reaction

Consider

1. a body of mass m place on a smooth horizontal surface

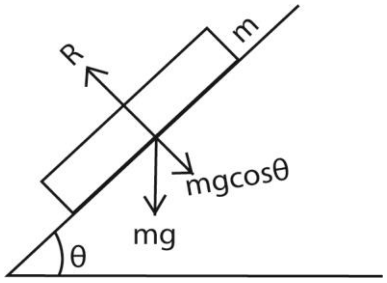


R = mg

R = normal reaction

mg gravitational pull (weight)

2. Mass m laced on a smooth inclined lane of angle of inclination θ



R = mgcosθ

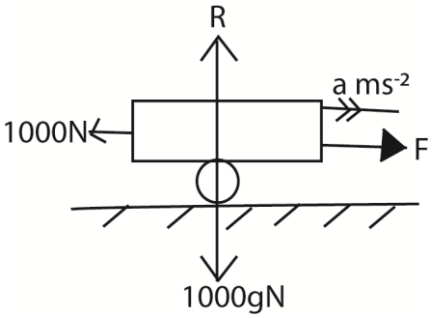
- All objects placed on or moving on an inclined plane experience a force mgsinθ **down** the plane no matter the direction of movement.
- If the plane is rough, the body experience a frictional force whose direction is opposite of the direction of motion

Motion on horizontal plane

Example 11

A car of 1000kg is accelerating at $2ms^{-2}$. If the resistance to motion is 100N

(i) Find the normal reaction of the car on the road surface



R = 1000gN

= 1000 x 9.8 = 9800N

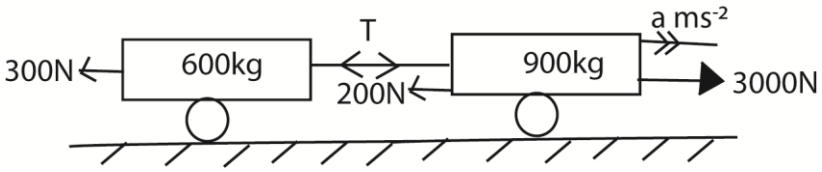
(ii) What accelerating force acts on the car?

F = ma

F – 1000 = 1000 x 2; F = 3000N

Example 12

A car of mass 900kg tows a trailer of mass 600kg along a level road by means of a rigid bar. The car experiences a resistance of 200N and the trailer a resistance of 300N, if the car engine exerts a force of 3kN, find the acceleration produced and the tension in the tow bar



For 900kg: 3000 – (T + 200) = 900a … (i)

For 600kg: T – 300 = 600a ……. (ii)

(i) and (ii) a= 1.6667ms^{-2}

Alternatively

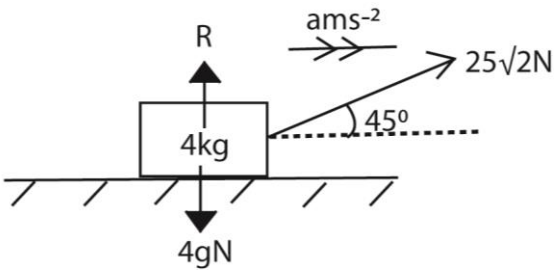
3000 – (200 + 300) = (900 + 600)a

a = 1.6667ms^{-2}

Force inclined at an angle to the horizontal

Example 13

A body of mass 4kg is acted on by force of $25\sqrt{2}$N which is inclined at 450 to a smooth horizontal surface. Find the acceleration of the body and the normal reaction between the body and the surface.



(→) $25\sqrt{2}$Ncos45 = 4a

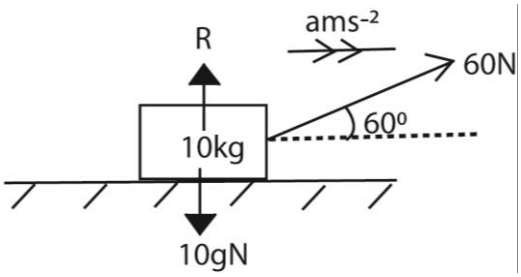
a = 6.25ms^{-2}

(↑) R + $25\sqrt{2}$Nsin 45 - = 0

R = 14.2N

Example 14

A body of mass 10kg is initially at rest on a rough horizontal surface. It is pulled along the surface by constant force of 60N inclined at 600 above the horizontal. If the resistance to motion totals 10n, find the acceleration of the body and the distance travelled in the first 3s.



(→) 60sos60 - 10 = 10a

a = 2ms^{-2}

s = ut + $\frac{1}{2}at^2$

s = 0 x 3 + $\frac{1}{2}$ x 2 x 3^2 =9m

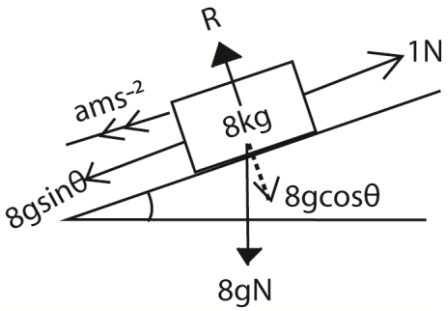
Revision exercise 1

1. A railway engine of mass 100tonnes is attached to a line of trucks of total mass 80 tonnes. Assuming there is no resistance to motion, find the tension in the coupling between the engine and the leading truck when the train has acceleration of 0.020ms^{-2} [25.6kN]
2. A body of mass 5kg, initially at rest on a smooth horizontal surface is pulled along the surface by a constant force P inclined at 450 above the horizontal. In the first 5 seconds of motion, the body moves a distance of 10m along the surface. Find the
   (i) acceleration of the body [0.8ms^{-2}]
   (ii) magnitude of P [$4\sqrt{2}$N]
   (iii) normal reaction between the body and the surface. [45N]
3. A body of mass m kg, initially at rest on a smooth horizontal surface is pulled along the surface by a constant force P inclined at θ above the horizontal. Show that the body moves a distance in time t along the surface given by $\frac{Pt^2cos\theta}{2m}$.
4. A body of mass m kg, initially at rest on a rough horizontal surface is pulled along the surface by a constant force P inclined at θ above the horizontal. If the mass acquire velocity v in a distance d. Show that the resistance to motion is given by Pcosθ = $\frac{mv^2}{2d}$

Motion on an inclined plane

Example 15

A body of mass 8kg is released from on the surface of a plane at 1 in 40. If the resistance to motion is 1N, find the acceleration of the body and the speed it acquired after 6s.



sinθ= $\frac{1}{40}$

F = ma

8gsinθ – 1 = 8a

a = 0.12ms^{-2}

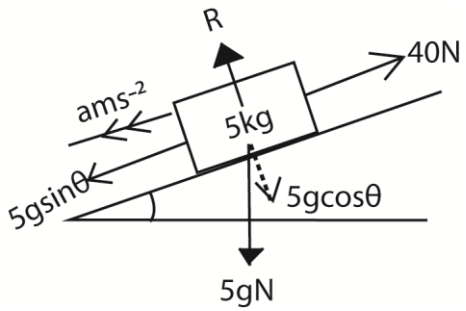
v = u + at

= 0 + 0.12 x 6 =0.72ms^{-2}

Example 16

A body of mass 5kg is pulled u a smooth plane inclined at 30^0 to the horizontal by a force of 40N acting parallel to the plane. Find

(i) acceleration of the body
(ii) force exerted on the body by the plane R



F = ma

40 -5gsin30^0 = 5a

a = 3.095ms^{-2}

(ii) R = 5gcos30

= 5 x 9.8cos30 = 42.4N

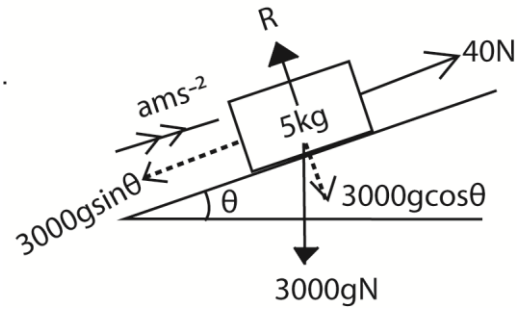
Example 17

A lorry of mass 3tonnes travelling at 90k/h starts to climb an incline of 1 in 5. Assuming the attractive pull between its tyres and the road remains constant and that its velocity reduces to 54kmh in a distance of 500m. Find the attractive pull.

u = 90kmh = $\frac{90\ x\ 1000}{3600}$ = 25ms^{-1}

V = 54kmh = $\frac{54\ x\ 1000}{3600}$ = 15ms^{-1}

a = $\frac{v^2 - u^2}{2s} = \frac{15^2 - 25^2}{2\ x\ 500}$ = -0.4ms^{-2}



sinθ = $\frac{1}{40}$

F = ma

F - 3000gsinθ = 3000a

F – 3000 x 9.8 x $\frac{1}{5}$ = 3000 x 0.4

F = 4686N

Revision exercise 2

1. A particle of mass 5kg resting on smooth plane inclined at $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$to the horizontal. Find the magnitude of the horizontal force required to keep the particle in equilibrium and the normal reaction. [28.29N, 56.58N]
2. The engine of a train exerts a force of 35,000N on a train of mass 240 tonnes and draws up a slope of 1 in 120 against resistance totalling to 60N per tonne. Find the acceleration of the train. [0.004167ms^{-2}]
3. A car of mass 2.5 metric tonnes is drawn up a slope of 1 in 10 from rest with acceleration of 1.2ms^{-2} against a constant frictional force of $\frac{1}{100}$ of the weight of the vehicle using a cable. Find the tension in the cable. [5695N]
4. A mass 5kg is initially at the bottom of a smooth slope which is inclined at $\sin^{-1}\left(\frac{3}{5}\right)$ to the horizontal. The mass is pushed up the slope by horizontal force 50N, find
   (i) the normal reaction between the mass and the plane [69.2N]
   (ii) calculate the acceleration up the slope [2.12ms^{-2}]
   (iii) how far up the slope the mass travels in the first 4s [16.96m]
5. A body of mass 100kg is released from rest at the top of a smooth slope which is inclined at 30^0 to the horizontal. Find
   (i) velocity of the body when it has travelled 20m down the slope. [14ms^{-1}]
   (ii) velocity, if the mass of the body was 50kg. [14ms^{-1}]
6. A body of mass 20kg is released from rest at the top of a smooth slope which is inclined at 30^0 to the horizontal. If the body accelerates down the slope at 3ms^{-2}, find the constant resistance to motion experienced by the body. [38N]
7. A body of mass 20kg is released from rest at the top of a rough slope which is inclined at 30^0 to the horizontal. 6s later the body has a velocity of 21ms^{-1} down the slope, find the constant resistance to motion experienced by the body. [28N]

8. A car of 1 tonne accelerated from 36kmh^{-1} to 72kmh^{-1} while moving 0.5km up a road inclined at an angle of α to the horizontal where sinα = $\frac{1}{20}$. If the total resistive force to its motion is 0,3kN, find the driving force of the car engine. [1090N]
9. A railway truck of mass 6.0 tonnes moves with an acceleration of 0.050ms^{-2} down a track which is inclined to the horizontal at an angle α where sinα = $\frac{1}{120}$. find the resistance to motion. [190N]
10. A body of mass 5.0kg is pulled along a smooth horizontal ground by means of 40N acting at 60^0 above the horizontal. find
    (i) Accelerating force [4ms^{-2}]
    (ii) Force the body exerts on the ground [14.4N]
11. A body of mass 3.0kgslides down a plane which is inclined at 300 to the horizontal. Find the acceleration of the body if
    (i) the plane is smooth [4.9ms^{-2}]
    (ii) there is frictional resistance of 9.0N [1.9ms^{-2}]
12. A car of mass 1000kg tows a caravan of mass 600kg u a road which rises 1m vertically for every 20m of its length. There is a constant frictional resistance of 200N and 100N to the motion of the car and caravan respectively. The combination has an acceleration of 1.2ms^{-2} with the engine exerting a constant driving force. (Take g = 10ms^{-2}). Find
    (a) driving force[3020N]
    (b) Tension in the tow-bar [1120N]

Thank You

Dr. Bbosa Science

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Friction

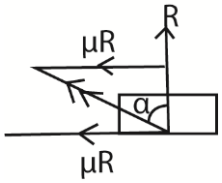
Friction is a force that opposes relative motion or attempted motion between two bodies in contact.

Friction force F = μR where R = normal reaction and μ = coefficient of friction

At limiting equilibrium, the body is at the point of moving (slip or slide) and friction force is maximum.

Angle of friction

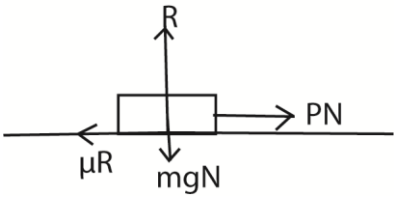
This is the angle between the resultant force and the normal reaction



$\tan\alpha = \frac{\mu R}{R} = \mu$

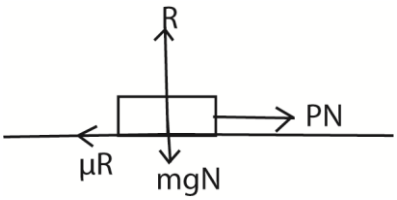
A horizontal plane

(i) at limiting equilibrium (about to slip or slid)



$P = \mu R$

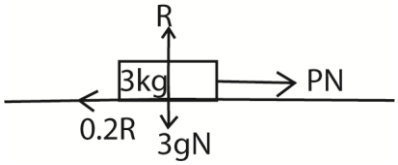
(ii) In motion



$P - \mu R = ma$

Example 1

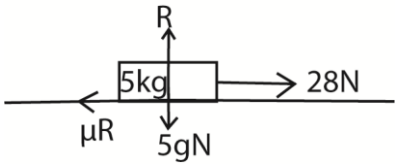
Calculate the maximum frictional force which can act when a block of mass 3kg rests on a rough horizontal surface, the coefficient of friction between the surface being 0.2



F = μR = 0.2 x 3 x 9.8 = 5.88N

Example 2

When a horizontal force of 28N is applied to a body of mass 5kg which is resting on a rough horizontal surface, the body is found to in limiting equilibrium. Find the coefficient of friction between the body and the plane



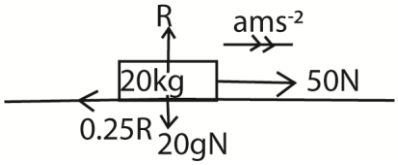
28 = μR

28 = μ x 5 x 9.8

$\mu = 0.57$

Example 3

A block of mass 20kg rests on a rough horizontal plane. The coefficient of friction between the block and the plane is 0.25. If a horizontal force of 50N acts no the body, find the acceleration of the body.



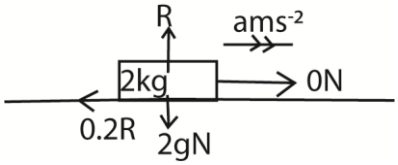
50 - μR = 20a

50 - $(0.25\ x\ 20\ x\ 9.8)$ = 20a

a = 0.05ms^{-2}

Example 4

A block of mass 2kg sliding along a smooth surface at a constant speed of 2ms-1. When the mass encounter a rough surface of coefficient of friction 0.2, it comes to rest. Find the distance the body will move across the rough surface before it comes to rest.



F = ma

0 – μR = 20a

-0.2 x 2 x 9.8 = 20a

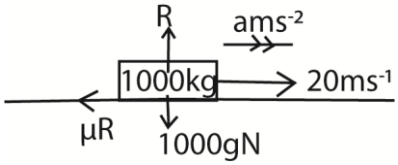
a = -1.96ms^{-2}

s = $\frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2x\ (-1.96)}$ = 1.02m

Example 5

A car of mass 1000kg moving along a straight road with speed of 72kmh^{-1} is brought to rest by a speedy application of brakes in a distance of 5m. Find the coefficient of kinetic friction between the tyres and the road.

u = $\frac{72\ x\ 1000}{2600}$ = 20ms^{-1}



a = $\frac{v^2 - u^2}{2s} = \frac{0^2 - 20^2}{2x\ 5}$ = -4ms^{-2}

ma = μR

4 x 1000 = 1000 x 9.8 x μ

μ = 0.41

Alternatively

Work done against friction = loss in kinetic energy

$\mu(mg)x\ s = \frac{1}{2}mv^2$

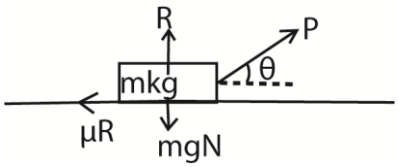
$\mu \times 9.8 \times 50 = \frac{1}{2}x\ 20^2$

$\mu = 0.408$

Revision exercise 1

1. When a horizontal force of 0.245N is applied to a body of mass 250g which is resting on a rough horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane. [0.1]
2. A body of mass 40kg is resting on a rough horizontal plane and can just move by a force of 98N acting horizontally. Find the coefficient of friction. [0.25]
3. A block of mass 0.5kg rests on a rough horizontal plane. The coefficient of friction between the block and the table is 0.1. When a horizontal force of 1N acts on the block, find
   (i) friction force experienced by the block. [0.49N]
   (ii) acceleration with which the block will move. [1.02ms^{-2}]
4. When a horizontal force of 37N is applied to the body of mass 10kg which is resting on a rough horizontal surface, the body moves along the surface with acceleration 1.25ms^{-2}. Find the coefficient of friction between the body and the surface. [0.25]

5. A force inclined at an angle θ to the horizontal



At limiting equilibrium

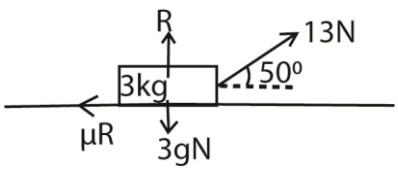
(→): Pcosθ = μR

(↑): R + Psinθ = mg

Example 6

A particle of mass 3kg resting on a rough horizontal plane is pulled by a force of magnitude 13N inclined at an angle 50^0 to the horizontal, if the particle does not move find the

(i) Normal reaction (ii) coefficient of friction

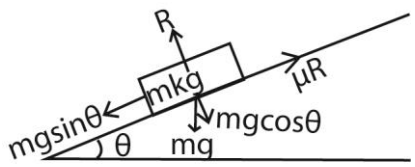


(↑):R = 3 x 9.8 – 13sin 50^0 = 19.4414N

(→): 13cos50^0 = μ x 19.4414

μ = 0.4298

Friction and inclined planes



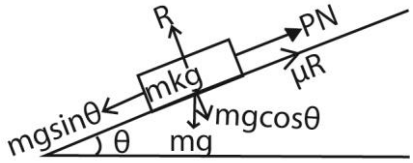
At limiting equilibrium

mgsinθ = μR

mgsinθ = μmgcosθ

μ = tanθ

(ii) A force P applied parallel to and up the plane to just move the particle upwards



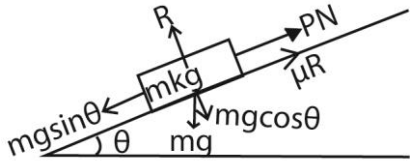
Normal to the plane: $mg\cos\theta = R$

Parallel to the plane; $mg\sin\theta + \mu R = P$

$P = mg\sin\theta + \mu mg\cos\theta$

(iii)

(iv) A force P applied parallel to and up the plane so that the particle is on the point of moving downwards (prevent moving downwards)



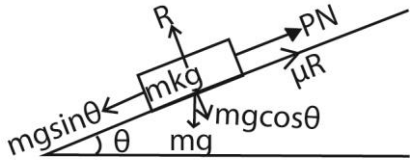
Normal to the plane: $mg\cos\theta = R$

Parallel to the plane; $mg\sin\theta = P + \mu R$

$P = mg\sin\theta - \mu mg\cos\theta$

(v)

(vi) A force P applied parallel to and up the plane to move the particle upwards



Normal to the plane: $mg\cos\theta = R$

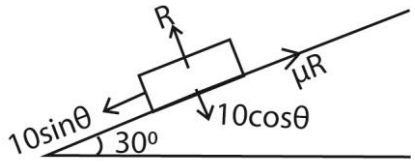
Parallel to the plane; $P - (mg\sin\theta + \mu R) = ma$

$P - (mg\sin\theta + \mu mg\cos\theta) = ma$

(vii)

Example 7

A particle of weight 10N rests on a rough plane inclined at $30^0$to the horizontal and is just about to slip. Find the value of coefficient of friction between the plane and the particle.



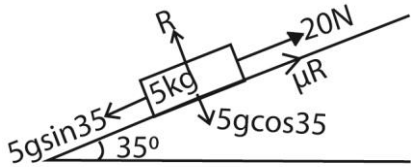
$R = 10\cos 30$ and $\mu R = 10\sin 30$

$\mu(10\cos 30) = 10\sin 30$

$\mu = 0.5774$

Example 8

A body of mass 5kg lies on a rough plane which is inclined at 35^0 to the horizontal. When a force of 20N is applied to the body parallel to and up the plane, the body is on the point of moving down the plane. Find the coefficient of friction between the body and the plane.



At limiting equilibrium

$R = 5g\cos 35$

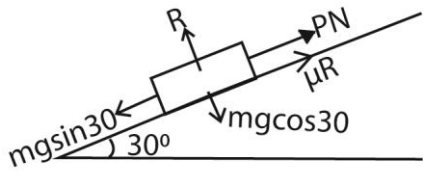
$20 + \mu R = 5g\sin 35$

$20 + \mu(5g\cos 35) = 5g\sin 35$

$\mu = 0.2$

Example 9

A block of wood of mass 150g rest on an inclined plane. If the coefficient of friction between the surface of contact is 0.3. Find the force parallel to the plane necessary to prevent slipping when the angle of the plane to the horizontal is 30^0.



At limiting equilibrium

R =0.15 x 9.8cos 30

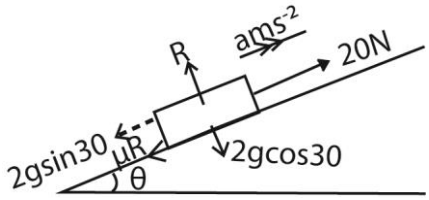
P +μR = 0.15 x 9.8sin30

P +0.3(0.15 x 9.8cos 30) =5gsin30;

P= 0.353N

Example 10

A body of mass 2kg lies on a rough plane which is inclined at $\sin^{-1}\left(\frac{5}{13}\right)$ to the horizontal. A force of 20N is applied to the body, parallel to and up the plane. If the body accelerates up the plane at $1.5ms^{-2}$, find the coefficient of friction between the body and the plane.



Sin θ =$\frac{5}{13}$, cos θ = $\frac{12}{13}$

R = 2 x 9.8 cosθ = 2 x 9.8 x $\frac{12}{13}$ =18.09N

F = ma

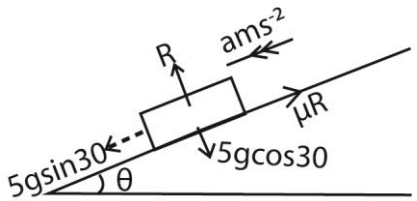
20 – (2gsinθ + μR) = 2a

20 –(2 x 9.8 x $\frac{5}{13}$ + 2 x 9.8 x $\frac{12}{13}$ x μ) = 2 x 1.5

μ =0.523

Example 11

A body of mass 5kg is released from rest on a rough surface of a plane inclined at 30^0 to the horizontal. If the body takes2.5s to acquire a speed of 4ms-2 from rest, find the frictional force and coefficient of friction.



v = u + at

4 = 0 + 2.5a

a = $1.6ms^{-2}$

F = ma

5 x 9.8sin30 – μR = 5 x 1.6;

Frictional force, μR = 16.5N

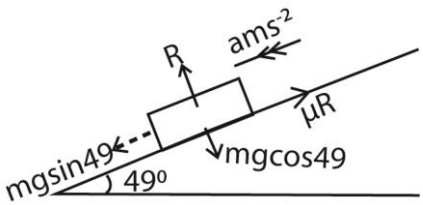
μR = 16.5N

μ = $\frac{16.6}{5\ x\ 9.8cos30}$ = 0.243

Example 12

A car of mass 500kg moves from rest with engine switched off down a road which is inclined at an angle 490 to the horizontal.

(a) calculate the normal reaction
(b) if the coefficient of friction between the tyres and the surface of the road is 0.32. Find the acceleration of the car.



(a) R =mgcos49 = 500 x 9.8cos49 = 3217.97N

(b) F = ma

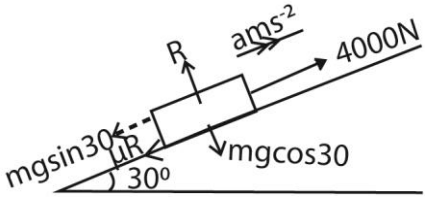
mgsin49 – μR = 500a

500 x 9.8sin49 – 0.32 x 3217.97 = 500a

a = 5.34ms^{-2}

Example 13

A car of mass 1000kg climbs a plane which is inclined at 30^0 to the horizontal. The speed of the car at the bottom of the incline is 36kmh^{-1}. If the coefficient of friction between the plane and the car tyres is 0.3 and the engine exerts a force of 4000N, how far up the incline does the car move in 5s



u = 36kh^{-1} = $\frac{36\ x\ 1000}{3600}$ = 10ms^{-1}

F = ma

4000 – (mgsin30 + μR) = ma

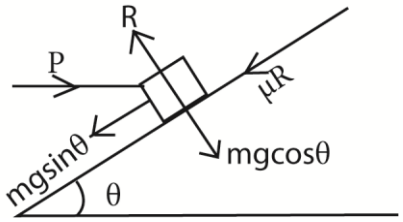
4000 – (1000 x 9.8sin30 + 0.3 x 1000 x 9.8cos30) = 1000a

a = -3.45ms^{-2}

s = ut + $\frac{1}{2}at^2$ = 10 x 5 + $\frac{1}{2}\ x\ (-3.45)x\ 5^2$ = 6.9m

Horizontal force on inclined planes

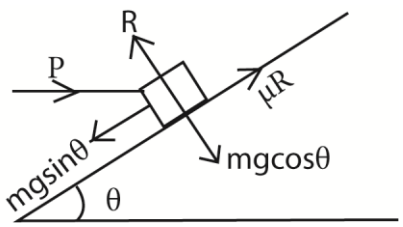
(i) A horizontal force P required to just move the particle upwards



Normal to the plane: mgcosθ + Psinθ = R

Parallel to the plane: mgsinθ + μR = Pcosθ

(ii) a horizontal force O required to prevent the particle from moving downwards

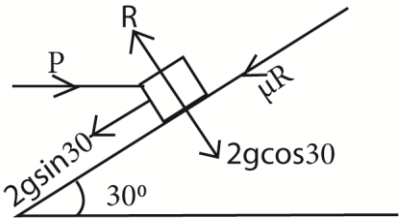


Normal to the plane: mgcosθ + Psinθ = R

Parallel to the plane: mgsinθ - μR = Pcosθ

Example 14

A body of mass 2kg lies on a rough plane inclined at 300 to the horizontal. When a horizontal force of 20N is applied to the body in an attempt to push it up the plane, the body is found to be on the point of moving up the plane. Find the coefficient of friction between the body and the plane.



At limiting

Normal to the plane: 2gcos30 + 20sin30 = R……(i)

Parallel to the plane: 2gsin30 + μR = 20cos30 …… (ii)

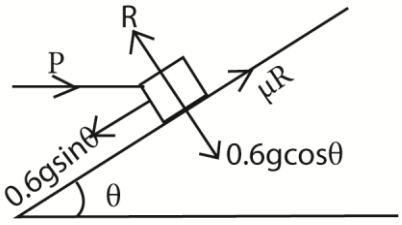
(i) and (ii)

2gsin30 +μ(2gcos30 + 20sin30) = 20cos30

μ =0.279

Example 15

A horizontal force of 1N is just sufficient to prevent a brick of mass 600g sliding down a rough plane which is inclined at $\sin^{-1}\left(\frac{5}{13}\right)$to horizontal. Find the coefficient of friction between the brick and the plane.



sinθ = $\frac{5}{13}$, cosθ =$\frac{12}{13}$

At limiting

Normal to the plane: 0.6gcosθ + 1xsinθ = R……(i)

Parallel to the plane: 0.6gsinθ - μR = 1 x cosθ …… (ii)

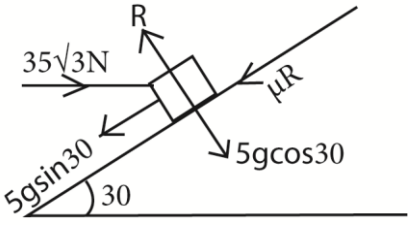
(i) and (ii)

0.6x 9.8 x $\frac{5}{13}$ - μ(0.6 x 9.8 x $\frac{12}{13}$ + 1 x $\frac{5}{13}$) = 1 x $\frac{12}{13}$

μ =0.23

Example 16

A body of mass 5kg is initially at the bottom of a rough inclined plane of length 6.3m. The plane is inclined to the horizontal and the coefficient of friction between the body and the plane is $\frac{1}{2}\sqrt{3}$. A constant horizontal force of $35\sqrt{3}N$is applied to the body causing it to accelerate up the plane. Find the time taken for the body to reach the top and its speed on arrival.



R = 5gcos 30 + $35\sqrt{3}$sin30

$35\sqrt{3}$cos 30 – (μR + 5gsin30) = 5a

a = 1.39ms^{-2}

s = ut + $\frac{1}{2}at^2$

6.3 = 0 x t + $\frac{1}{2}x1.39\ x\ t^2$; t = 3s

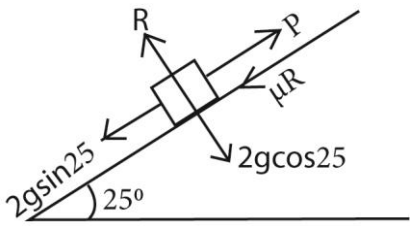
v = u + at

v = 0 + 1.39 x 3 =4.17ms^{-1}

Example 17

A box of mass 2kg at rest on a plane inclined at 25^0 to the horizontal. The coefficient of friction between the box and the plane is 0.4. What minimum force applied parallel to the plane would move the box up the plane?

Let the minimum force required be P



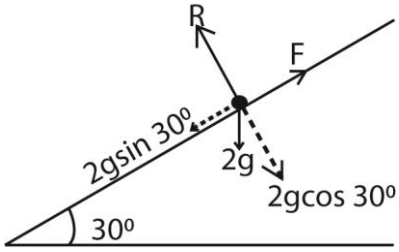
Perpendicular to the plane: R = 2gcos25 …. (i)
Along the plane: P = 2gsin25 + μR ……(ii)

(i) and (i)

P = 2gsin25 + 0.4(2gcos25) = 8.3 + 7.1 = 15.388N

Example 18

A particle of mass 2kg rests in limiting equilibrium on a rough plane inclined at 30^0 to the horizontal. Find the value of coefficient of friction.

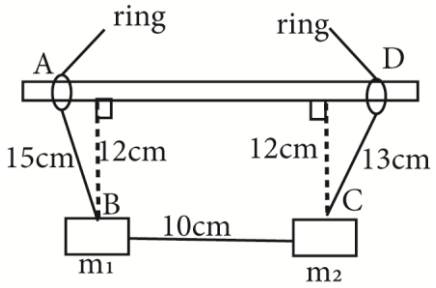


R = 2gcos 30^0
F = 2gsin 3^0
μR = 2gsin30^0
$\mu[2gcos\ 30^0]$ = 2gsin30^0
$\mu = \frac{2gsin\ 30^0}{2gcos\ 30^0}$ = tan 30^0 = 0.57735

Revision exercise

1. The resistance to motion of a lorry of mass mkg is 1/200 of its weight. When travelling at 108kmh^{-1} on a level road and ascends a hill inclined at 1 in 100. Its engine fails to work. Find how far up the hill (in km) the lorry moves before it comes to rest. [36.12m]
2. A vehicle of mass 2.5metric tonnes is drawn up on a slope of 1 in 10 from rest with an acceleration of 1.2 ms-2 against a constant frictional resistance of $\frac{1}{100}$ of the weight of the vehicle, using a cable. Find the tension in the cable. [T = 5695N]
3. (a) A particle of mass, m kg is projected with a velocity of 10ms-1 up a rough plane of inclination 300 to the horizontal. If the coefficient of friction between the particle and the plane is $\frac{1}{4}$. Calculate how far up the plane the particle travels.[s = 7.121m]
   (b) A car is working at 5kW and is travelling at a constant speed of 72knh^{-1}. Find the resistance to motion. [250N]

4. A body of mass 8kg rests on a rough plane inclined at θ to horizontal. If the coefficient of friction is μ, find the least horizontal force in terms of μ, θ and g which will hold the body in equilibrium. [$\frac{8g(sin\theta-\mu cos\theta)}{(cos\theta+\mu sin\theta)}$]
5. A carton of 3kg rests on a rough plane inclined at an angle 30^0 to the horizontal. The coefficient of friction between the carton and the plane is $\frac{1}{3}$ > find a horizontal force that should be applied to make the carton just about to move up the plane. [33.155N]
6. A particle of weight 20N is placed on a rough plane inclined at an angle of 400 to the horizontal. the coefficient of friction between the plane and the particle is 0.25. When a horizontal force P is applied on the particle it rests in equilibrium. Calculate the value of P. [9.739N]
7. The diagram below shows the three strings AB = 15cm, BC = 10cm and CD = 13cm, A and D are fixed to small rings each of mass 2kg which can slide on a rough horizontal rail AD. Masses m_1 and m_2 are attached at B and C respectively. The system rests in equilibrium with BC at a distance 12cm below AD.



   (a) Show that $9m_1 = 5m_2$.
   (b) If the coefficient of friction between each ring and the rail is 0.25 and the ring A is on the pint of slipping, determine the value of m_1. [m_1 = 1kg]
8. A 2kg body lies on a plane of inclination 600. The coefficient of friction between the body and the plane is 0.25. Find the least horizontal force which prevents the body from sliding down the plane. [20.27N]
9. A particle od mass 12kg slides from rest down a plane inclined at 50^0 to the horizontal. If the coefficient of friction between the particle and the plane is 0.4, calculate the acceleration of the particle. [4.99ms^{-2}]
10. A body of mass 3kg is released from a rough surface which is inclined at $\sin^{-1}\left(\frac{3}{5}\right)$ to the horizontal. If after 2.5s the body has acquired a velocity of 4.9ms-1 down the surface. Find the coefficient of friction between the body and the surface.

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Hooke's law, elastic strings and Simple harmonic motion

Hooke's law states that the tension in a stretched string is proportional to the extension,e from its natural (unstretched) length, L.

$T = \lambda\frac{e}{L}$ where λ is modulus of elasticity.

Example 1

An elastic string is of natural length 4m and modulus 25N. Find

(i) The extension in the string when the extension is 20cm

$T = \lambda\frac{e}{L}$ = 25 $x\frac{0.2}{4}$= 1.25N

(ii) The extension of the string when the tension is 6N

$e = T\frac{L}{\lambda} = 6\ x\frac{4}{25} = 0.96m$

Example 2

A spring is of natural length 1.6 and modulus 20N. Find the thrust in the spring when it is compresses to a length of 1m

$T = \lambda\frac{e}{L}$ = 20 $x\frac{0.6}{1.6}$= 7.5N

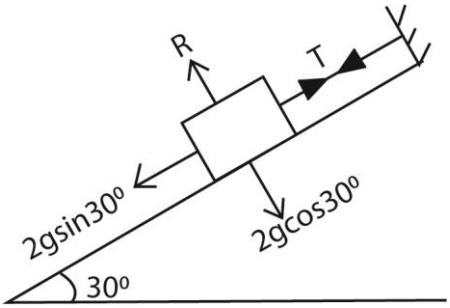
Example 3

A body of mass mkg lies on a smooth horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 60cm and modulus 90N. When the body moves in a horizontal circular path about O with constant speed of $3ms^{-1}$, the extension in the string is 30cm. find the mass of the body.

$T = \lambda\frac{e}{L}$ = 90 $x\frac{0.3}{0.6}$= 45N | T = $m\frac{v^2}{r}$ | 45 = $m\frac{3^2}{(0.6+0.3)}$ | m = 4.5kg

Example 4

A smooth surface inclined at 30^0 to the horizontal has a body A of mass 2kg that is held at rest on the surface by a light elastic string which has one end attached to A and the other to a fixed point on the surface 1.5m away from A up to the line of greatest slope. If the modulus of elasticity of the string is 2gN, find the natural length



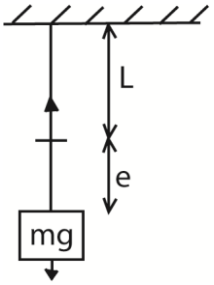
$T = \lambda\frac{e}{L}$

2gsin30 = 2g x $\frac{1.5-L}{L}$

L = 1m

Equilibrium of a suspended body

When an elastic string one end fixed and a mass attached to its other end so that the mass is suspended in equilibrium, the string is stretched due to the mass.



T = mg

mg = $\lambda\frac{e}{L}$ at equilibrium

Example 5

A light elastic string of natural length 65 has one end fixed and a mass of 500g freely suspended from the other end. Find the modulus of elasticity of the string if the total length of the string in equilibrium position is 85cm

mg = $\lambda\frac{e}{L}$ | 0.5 x 9.8 = $\lambda\ x\ \frac{0.85-0.65}{0.65}$ | λ = 15.93N

Example 6

A light spring of natural length 1.5m has one end fixed and a mass of 400g freely suspended from the other. The modulus of the spring is 44.1N

(a) Find the extension of the spring when the body hangs in equilibrium

mg = $\lambda\frac{e}{L}$ | 0.4 x 9.8 = $44.1\ x\ \frac{e}{1.5}$ | e= 0.13m

(b) The mass is pulled vertically downwards a distance of 10cm and released, find the acceleration of the body when released

$T_1 = \lambda\frac{e+x}{L}$

$T_1 = 44.1\ x\frac{0.13\ +0.1}{1.5} = 6.762N$

F =ma

$T_1 - T = ma$

6.762- 3.92 = 0.4a

A = 7.11ms^{-2}

Potential energy stored in an elastic string

Work done = average force x extension = $\lambda\frac{e^2}{2L}$

Example 7

An elastic string is of natural length 6.4m and modulus 55N. Find the work done in stretching it from 6.4m to 6.8m

Work done = $\lambda\frac{e^2}{2L} = 55\ x\ \frac{0.4^2}{2\ x\ 6.4}$ = 0.688J

Example 8

An elastic string of natural length 4m is fixed at one end and stretched to 5.6m by a force of 8N. Find the modulus of elasticity and the work done

$T = \lambda\frac{e}{L}$

8 = $\lambda\frac{5.6-4}{4}$

λ = 20N

Work done = $\lambda\frac{e^2}{2L} = 20\ x\ \frac{1.6^2}{2\ x\ 4}$ = 6.4J

Example 9

An elastic string of natural length 1.2m and modulus of elasticity 8N is stretched until the extending force is 6N. Find the extension and work done

$T = \lambda\frac{e}{L}$

6 = $8\ x\ \frac{e}{1.2}$

e = 0.9m

Work done = $\lambda\frac{e^2}{2L} = 8\ x\ \frac{0.9^2}{2\ x1.2}$ = 2.7J

Example 10

A light elastic string of natural length 1.2m has one end fixed at A and a mass of 5kg freely suspended from the other. The modulus of the string is such that a 5kg mass hanging vertically would stretch the string by 15cm. the mass is held at A and allowed to fall vertically, how far from A it comes to rest.

$mg = \lambda\frac{e}{L}$

$5\ x\ 9.8 = \lambda\ x\ \frac{0.15}{1.2}$

λ = 392N

At A; u = 0ms^{-1}, s = 1.2m, g = 9.8ms^{-2}

v^2 =u^2 + 2gs = 0^2 + 2 x 9.8 x 1.2 = 23.52m^{2}s^{-2}

w = k.e + p.e

$\lambda\frac{e^2}{2L} = \frac{mv^2}{2} + mge$

$392\ x\ \frac{e^2}{2\ x\ 1.2} = \frac{5\ x\ 23.52}{2} + 5\ x\ 9.8\ x\ e$

e = 0.769m

Depth = 1.2 + 0.769 = 1.969m

Revision exercise

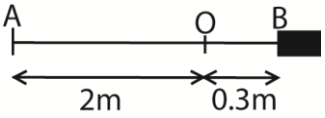
1. An elastic string of natural length 1m and modulus 20N. find the tension in the string when the extension is 20cm [4N]
2. A spring of natural length 50cm and modulus 10N. Find the thrust in the spring when it is compressed to a length 40cm [2N]
3. When the length of a spring is 60% of its original length, the thrust in the spring is 10N. find the modulus of the spring [25N]

4. An elastic string of natural length 60cm and modulus 18N. Find the extension of the string when the tension in the string is 6N [20cm]
5. The tension in an elastic string is 8N when the extension in the string is 25cm. If the modulus of the string is 8N. Find the un-stretched length. [25cm]
6. A light elastic string of natural length 20cm and modulus 2gN has one end fixed and a mass of 500g freely suspended from the other. Find the total length of the string [25cm]
7. When a mass of 5kg is freely suspended from one end of a light elastic string, the other end of it fixed, the string extends to twice its natural length. Find the modulus of the string [49N]
8. A body of mass 4kg lies on a smooth horizontal surface and is connected to point O of the surface by a light elastic string of natural length 64cm and modulus 25N. when the body moves in a horizontal circular path about O with constant speed of vms^{-1}, the extension in the string is 36cm. Find v [1.875ms^{-1}]
9. A body of mass 5kg lies on a horizontal surface and is connected to a point O on the surface by a light elastic string of natural length 2m and modulus 30N. When the body moves in a horizontal circular path about O with a constant speed 3ms^{-1}, find the extension in the string[1m]
10. An elastic string of natural length 2m and modulus 10N. Find the energy stored when it is extended to a length of 3m [2.5J]
11. An elastic string of natural length 1m and modulus 20N. Find the energy stored when it is extended by a length of 30cm [0.9J]

Simple harmonic motion in strings

Example 11

One end of a light elastic string of natural length 2m and modulus 10N is fixed to point A on a smooth horizontal surface. A body of mass 200g is attached to the other end of the string and is held at rest at point B on the surface causing the spring to extend by 30cm. Show that when released, the body will move with S.H.M. State its amplitude and find the maximum speed.



$T = \lambda\frac{x}{L} = 10\,x\frac{x}{2} = 5x$

F = ma

5x = -0.2a

a = -25x

Since a = -ω^{2}x

ω^2 = 25

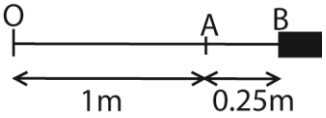
ω = 5rads^{-1}

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2}s$ and r = 0.3m

v_{max} = ωr = 5 x 0.3 = 1.5ms^{-1}

Example 12

One end of a light elastic string of natural length 1m and modulus 5N is fixed to a point O on a smooth horizontal surface. A body of mass 1kg is attached to the other end, A of the string and is held at rest at point B where OB = 1.25m. Show that when released, the body will move with S.H.M. Find the maximum speed and the total time taken from B to O



$T = \lambda\frac{x}{L} = 5\,x\frac{x}{1} = 5x$

F = ma

5x = -1a

a = 5x

Since a = $-\omega^2 x$

$\omega^2 = 5$

$\omega = 2.236 rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.236} s = 2.81s$ and r = 0.25m

$v_{max} = \omega r = 2.236 \times 0.25 = 0.559 ms^{-1}$

$t_{OA} = \frac{distance}{speed} = \frac{1}{0.559} = 1.788s$

$\frac{distance}{time} = \frac{1}{T} = \frac{0.25}{t_{AB}}$

$t_{AB} = \frac{0.25\ x\ 2.81}{1} = 0.7025s$

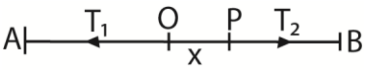
$t_{OB} = t_{OA} + t_{AB}$

$t_{OB} = 1.788 + 0.7025 = 2.491s$

Example 13

A light elastic string of natural length 2.4m and modulus 15 is stretched between two points A and B, 3m apart on a smooth horizontal surface. A body of mass 4kgis attached to the midpoint of the string is pulled 10 cm towards B and released

(i) Show that the subsequent motion is simple harmonic
(ii) Find the speed of the body when it is 158cmfrom A



At equilibrium $T_1 = T_2$

$\lambda \frac{e_1}{L} = \lambda \frac{e_2}{L}$

$e_1 = e_2$ ……. (i)

$e_1 + e_2 + 2.4 = 3$

$e_1 = 0.3m$

F = ma

$T_2 - T_1 = ma$

$\lambda\left(\frac{0.3-x}{1.2}\right) - \lambda\left(\frac{0.3+x}{1.2}\right) = 4a$

$4.8a = -2\lambda x$

$a = -\frac{2\ x\ 15x}{4.8} = -6.25x$

It is in form of $a = \omega^2 x$ hence S.H.M

$\omega^2 = 6.25$

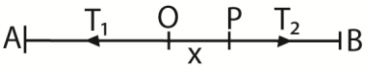
$\omega = 2.5 rads^{-1}$

$v^2 = \omega^2(r^2 - x^2)$ when 158 from A, x = 8cm

$v = \sqrt{0.1^2 - 0.08^2} = 0.15 ms^{-1}$

Example 14

A particle of mass m is attached by means of a light string AP and BP of the same natural length a m and modulus of elasticity mgN and 2mgN respectively, to point A and B on a smooth horizontal surface. The particle is released from the midpoint AB where AB = 3a. Show that the subsequent motion is simple harmonic with period $T = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$.



At equilibrium $T_1 = T_2$

$mg \frac{e_1}{a} = 2mg \frac{e_2}{a}$

$e_1 = 2e_2$ ……. (i)

$e_1 + e_2 + 2a = 3a$

$2e_2 + e_2 + 2a = 3a$

$e_2 = \frac{a}{3}$ and $e_1 = \frac{2a}{3}$

F = ma

$T_2 - T_1 = ma$

$2mg\left(\frac{a/3-x}{a}\right) - mg\left(\frac{2a/3+x}{a}\right) = ma$

$a = -\frac{3g}{a}x$

It is in form of $a = \omega^2 x$ hence S.H.M

$\omega^2 = \frac{3g}{a}$

$\omega = \left(\frac{3g}{a}\right)^{1/2} rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{3g}{a}\right)^{1/2}} = \left(\frac{4\pi^2 a}{3g}\right)^{\frac{1}{2}}$

Example 15

A particle of mass 1.5kglies on a smooth horizontal table and attached to two light elastic string fixed at points P and Q 12m apart. The strings are of natural length 4m and 5m and their modulus are λ and 2.5λ respectively.

(a) Show that the particles stays in equilibrium at a point R midway between P and Q

At equilibrium $T_1 = T_2$

$\lambda\frac{e_1}{4} = 2.5\lambda\frac{e_2}{5}$

$e_1 = 2e_2$ ……. (i)

$e_1 + e_2 + 4 + 5 = 12$

$2e_2 + e_2 + 9 = 12$

$e_2 = 1$

$e_1 = 2$

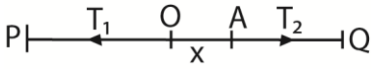
At R (midpoint) $4 + e_1 = 5 + e_2$

4 + 2 = 5 + 1 = 6

(b) If the particle is held at some point S in the line PQ with PS = 4.8m and then released. Show that the particle performs S.H.M and find the

(i) Period of oscillation

(ii) Velocity when the particle is 5.5m from P



F= ma

$T_2 - T_1 = ma$

$2.5\lambda\left(\frac{1-x}{5}\right) - \lambda\left(\frac{2+x}{4}\right) = 15a$

$a = -\frac{\lambda}{2}x$

It is in form of $a = \omega^2 x$ hence S.H.M

$\omega^2 = \frac{\lambda}{2}$

$\omega = \left(\frac{\lambda}{2}\right)^{1/2} rads^{-1}$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{2}\right)^{1/2}} = \left(\frac{8\pi^2}{\lambda}\right)^{\frac{1}{2}}$

$v^2 = \omega^2(r^2 - x^2)$ when 5.5m from P, x = 0.5m

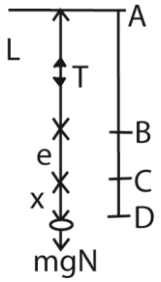
$v = \sqrt{\frac{\lambda}{2}((6-4.8)^2 - 0.5^2)}$

$v = \sqrt{\frac{\lambda}{2}((1.2)^2 - 0.5^2)} = \sqrt{0.595\lambda}\, ms^{-1}$

Elastic strings or springs hanging vertically

Example 16

A particle of mass m is suspended by a string from a fixed point A and has a natural length L. If the spring is extended from B to C where BC = e and this extension is due to weight of the body (mg), CD = x is the length a particle is pulled vertically downwards.



At equilibrium T = mg

T = $\lambda\frac{e}{L}$

When pulled a distance x:

T – T_1 = ma

$\lambda\frac{e}{L} - \lambda\frac{e+x}{L} = ma$

$a = -\frac{\lambda}{mL}x$

It is in form of a =ω^2x hence S.H.M

$\omega^2 = \frac{\lambda}{mL}$

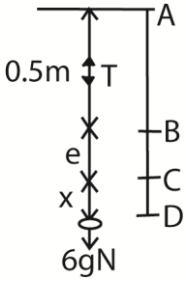
$\omega = \left(\frac{\lambda}{mL}\right)^{1/2}$ rads^{-1}

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{\lambda}{mL}\right)^{1/2}} = 2\pi\sqrt{\frac{mL}{\lambda}}$

Example 17

A light elastic spring of natural length 50cm and modulus 20gN, hangs vertically with its upper and fixed and the body of mass 6kg attached to its lower end. The body initially rests in equilibrium and then pulled down a distance of 25cm and released.

(a) Show that the ensuing motion will simple harmonic and
(b) Find the period of motion and maximum speed of the body



At equilibrium T = mg

6g = $20g\frac{e}{0.5}$

e = 0.15m

When pulled a distance x:

T – T_1 = ma

$6g - \lambda\frac{e+x}{0.5} = 6a$

$a = -\frac{196}{3}x$

It is in form of a =ω^2x hence S.H.M

$\omega^2 = \frac{196}{3}$

$\omega = \left(\frac{196}{3}\right)^{1/2} = 8.083$ rads^{-1}

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{196}{3}\right)^{1/2}} = 0.773s$

$v_{max} = \omega r = 8.083\ x\ 0.25 = 2.021ms^{-1}$

Thank you

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A-level math paper 2: Index numbers

This the percentage ratio of one quantity to the other, e.g. price index

Index number is a technique of measuring changes in a variable or group of variables with respect to time, geographical location or other characteristics. It is a statistical measure of change in a representative group of individual data points.

For example, if the price of a certain commodity rises from shs. 100 in the year 2007 to shs. 150 in the year 2017, the price index number will be 150 showing that there is a 50% increase in the prices over this period

Simple index numbers

The simple index numbers include;

(i) Price index or price relative

Price relative = $\frac{P_1}{P_o}\ x\ 100$ where P_1= price the current year and P_0 – price in the base year

(ii) Wage index

Wage index = $\frac{W_1}{W_o}\ x\ 100$

(iii) Quantity index (quantum) index

Quantum index = $\frac{Q_1}{Q_o}\ x\ 100$

Example 1

A loaf of bread costed shs. 1200/= in 2008 and shs. 1800/= in 2014. Taking 2008 as the base year, find the price relative in 2014

Solution

Price relative = $\frac{P_1}{P_o}\ x\ 100 = \frac{1800}{1200}\ x\ 100 = 150$

Example 2

In 2020, the price index of a commodity using 2019 as the base was 180. In 2021, the price index using 2020 as the base year was 150. What is the price index in 2021 using 2019 as the base year?

Solution

$\frac{P_{2020}}{P_{2019}}\ x\ 100 = 180$

$\frac{P_{2020}}{P_{2019}} = 1.80$ ..... (i)

$\frac{P_{2021}}{P_{2020}}\ x\ 100 = 150$

$\frac{P_{2021}}{P_{2020}} = 1.50$ ..... (ii)

Eqn. (i) x Eqn. (ii)

$\frac{P_{2020}}{P_{2019}} . \frac{P_{2021}}{P_{2020}}$ = 1.8 x 1.5

$\frac{P_{2021}}{P_{2019}} = 2.7$

$\frac{P_{2021}}{P_{2019}} x\ 100 = 2.7x\ 100 = 270$

∴ the price index in 2021 using 2019 as the base year = 270

Example 3

The wage of a nurse in Uganda in 2010 was 350,000/=, the wage of the nurse in 2015 was increase by shs. 150,000/=. Using 2010 as the a base year calculate the nurses wage index in 2015.

Wage index = $\frac{W_1}{W_o}\ x\ 100$ = $\frac{500,000}{350,000}\ x\ 100 = 142.9$

Price indices

Price indices are divided into

(a) Simple price index
(b) Simple Aggregate price index
(c) Weighed price index

(a) Simple price index
This is the average of the price relative
It is given by simple price index = $\frac{\sum \frac{P_1}{P_0}}{n}$ x100
Where n = number of items.

(b) Simple aggregate price index
It is given by simple aggregate price index = $\left(\frac{\sum P_1}{\sum P_0}\ x\ 100\right)$

Example 4

The table shows the prices 0f bread and meat per kg in 2000 and 2008

	Year	
item	2000	2008
Beans	700	1200
Meat	2500	4500

Using 2000 as the base year, find

(a) Price relatives of each commodity
(b) Simple price index
(c) Simple aggregate price index

Solution

(a) Price index = $\frac{P_1}{P_o}\ x\ 100$
For beans: P.R = $\frac{1200}{700}\ x\ 100 =\ 171.43$
For meat: P.R = $\frac{4500}{2500}\ x\ 100 =\ 180$

(b) Simple price index (S.P.I) = $\frac{171.43+180}{2} = 175.72$

(c) Simple aggregate price index (S.A.P.I) = $\frac{1200+450}{700+2500}\ x\ 100 = 178.13$

Example 5

In 2014 the price of a shirt, a dress and a pair of shoes were shs. 20,000; shs. 35,000 and shs. 45, 000 respectively. Given that in 2017 the prices were shs. 25,000, shs. 50,000 and shs. y respectively. Find the value of y if the aggregate price index was 130 while taking 2014 as the base year.

Solution

Simple aggregate price index = $\left(\frac{\sum P_1}{\sum P_0}\ x\ 100\right)$

$\left(\frac{25{,}000+50{,}000+y}{20{,}000+35{,}000+45{,}000}\right)\ x\ 100 = 130$

$\left(\frac{75.000+y}{1000}\right) = 130$

$y = shs.55{,}000$

Weighted price index (composite index)

If the weight or quantity in the base year and current year are the same, we use

(i) Weighted aggregate price index

Weighted aggregate price index = $\left(\frac{\sum P_1 w}{\sum P_0 w}\right) x\ 100$

Example 6

The table below shows the prices (shs.) and amounts of items bought for assembling a phone in 2012 and 2015

Items	Prices (shs		Quantity
	2012	2015	
Transistor	12,000	18,000	8
Resistor	16,500	21,000	22
Capacitor	15,000	17,000	9
Diode	16,000	18,000	2
Circuit	20,000	25,000	1

Calculate the composite price index for a phone taking 2012 as the base year

W.A.P.I = $\left(\frac{18{,}000\ x\ 8\ +21{,}000\ x\ 22+17{,}000\ x\ 9\ +18{,}000\ x\ 2+25{,}000\ x\ 1}{12{,}000\ x\ 8+16{,}500\ x\ 22+15{,}000\ x\ 9+16{,}000\ x\ 2+20{,}000\ x\ 1}\right)\ x100 = 126.94$

Example 7

The table below shows the prices (shs) and amount of items bought for making a cake in 2008 and 2009.

Items	Prices (shs		Quantity
	2008	2009	
Flour per kg	6,000	7,800	3
Sugar per kg	5,000	4,000	1
Milk per litre	1,000	1,500	2
Eggs per egg	200	300	8

(a) Calculate the weighted aggregate price index taking 2008 as the base year

(b) In 2009, the cost of making a cake was shs. 80,000/=. Using the weighted aggregate price index above, find the cost of the coke in 2008.

Solution

(a) W.A.P.I = $\left(\frac{7800\ x\ 3+4000\ x\ 1+1500\ x\ 2+3\ x\ 8}{6000\ x\ 3+5000\ x\ 1+1000\ x\ 2+200\ x\ 8}\right) x\ 100 = 123.3083$

(b) $\frac{P_1}{P_o}\ x\ 100 = 123.3083$

$\frac{80{,}000}{P_o}\ x\ 100 = 123.3083$

$P_0 = shs.64{,}878.0333$

(ii) Average weighted price index

Average weighted price index = $\frac{\sum \frac{P_1}{P_0} w}{\sum w}\ x\ 100$

When the price relative (P.R) is given then

Average weighted price index = $\frac{\sum (P.R\ x\ w)}{\sum w}$

Example 8

The table shows the expenditure (Ug. Shs.) of a student during the first and second terms

Items	Expenditure (shs.)		Amount
	1st term	2nd term	
clothing	46,500	49,350	5
Pocket money	55,200	37,500	3
Books	80,000	97,500	8

Using the first term expenditure as the base, find the average weighted price index

Average weighted price index = $\frac{\sum \frac{P_1}{P_0} w}{\sum w}\ x\ 100$

$= \left(\frac{\frac{49{,}350}{46{,}500} x\ 5 + \frac{37{,}500}{55{,}200} x\ 3 + \frac{97{,}500}{80{,}000} x\ 8}{5+3+8}\right) x\ 100$

= 106.841

Example 9

The table below shows the price relatives together with their weights for a certain family

Item	Weight	Price relative
Food	172	120
Water	160	124
Housing	170	125
Electricity	210	135
Clothing	140	104

Find the:

(i) simple price index

(ii) cost of living

Solution

(i) S.P.I = $\frac{\sum\left(\frac{P_1}{P_0}\right)}{n} = \frac{120+124+125+135+104}{5} = 121.6$

(ii) Cost of living = $\frac{\sum (P.R\ x\ w)}{\sum w} = \frac{120\ x\ 172\ +124x\ 160\ +125\ x\ 170\ +135\ x\ 210+104\ x\ 140}{172+160+170+210+140}$ = 122.82

Weighted aggregate price indices/ Paache's theory/value index

If the weight or quantity in the base year and current year are different, we use

Weighted aggregate price index = $\frac{\sum P_1W_1}{\sum P_0W_0}\ x\ 100$

Example 10

The table below shows the prices of items per kg in the year 2001 and 2002

Item	2001 = 100		2002	
	Price (shs.)	Quantity (kg)	Price (shs.)	Quantity (kg)
Rice	2800	20	3200	30
Millet	1500	10	1900	10
Beans	2000	5	2500	70

Calculate for 2002

(i) Price index
(ii) Simple aggregate price index
(iii) Simple aggregate quantity index
(iv) Weighted aggregate price index

Solution

(i) Price index = $\frac{P_{2002}}{P_{2001}}x\ 100$

For rice, price index = $\frac{3200}{2800}x\ 100 = 114.29$

For millet, price index = $\frac{1900}{1500}x\ 100 = 126.67$

For beans, price index = $\frac{2500}{2000}x\ 100 = 125$

(ii) S.A.P.I = $\left[\frac{3200+1900+250}{2800+1500+2000}\right]\ x\ 100 = 120.63$

(iii) S.A.Q.I = $\frac{\sum Q_{2002}}{\sum Q_{2001}}\ x\ 100$

$= \left[\frac{30+10+70}{20+20+5}\right]x\ 100 = 314.29$

(iv) W.A.P.I = $\frac{\sum P_1W_1}{\sum P_0W_0}\ x\ 100$

$= \left[\frac{3200\ x\ 30\ +1900\ x\ 10+2500\ x\ 70}{2800\ x\ 20+1500\ x\ 10+2000\ x\ 5}\right]x\ 100$

= 358.02

Revision exercise

1. UNEB 2020/2/5

The table below shows the price indices of beans, maize, rice and meat with corresponding weights

Item	Price index 2008 (2007 = 100%)	Weight
Beans	105	4
Maize	x	7
Rice	104	2
Meat	113	5

Calculate the;

(a) Value of x given that the price indices of maize in 2007 and 2008 using 2006 as the base year are 112 and 130 respectively. (**116.0714**)

(b) Weighted price index for 2008 using 2007 as the base year (**111.4167**)

2. UNEB 2018/2/5

The price index of an article in 2000 based on 1998 was 130. The price index for the article in 2005 based on 2000 was 80. Calculate

(a) Price index of the article in 2005 based on 1998. (104)

(b) Price of the article in 1998 if the price of the article was 45,000 in 2005.

UNEB 2017/2/7

The table below shows the price (shs.) and amount of items bought weekly by a restaurant in 2002 and 2003.

Items	Price (shs.)		Amount
	2002	2003	
Mil k per litre	400	500	200
Eggs per tray	2500	3000	18
Cooking oil per litre	2400	2100	2
Flour per packet	2000	2200	15

Calculate

(a) the weighted aggregate price index taking 2002 as the base year. (**119.63**)

(b) In 2003, the restaurant spent shs. 450,000/=. Using the weighted aggregate price index, find how the restaurant could have spent in 2002. (**376,096.95**)

3. The table below shows the prices (shs.) an amount of items bought in 2006 and 2007

Items	Price (shs.)		Amount
	2006	2007	
Mil k per litre	300	400	1
Eggs per tray	2500	3000	3
Cooking oil per litre	3000	8000	4
Flour per packet	1500	1800	15

Taking 2006 as the base year

(a) Calculate the simple aggregate price index (**180.92**)

(b) Calculate the weighted aggregate price index(**161.702**)

4. The table below shows the prices in US dollars and weights of five components of an engine, in 1998 and 2005

components	weight	Prices (US D)	
		1998	2005
A	6	35	60
B	5	70	135
C	3	43	105
D	2	180	290
E	1	480	800

(a) Taking 1998 as the base year, calculate the

(i) Simple aggregate price index (**172.03**)

(ii) Price relative of each component (**171.4, 192.9, 244.2, 161.1, 166.7**)

(iii) Weighted price index (**178.55**)

(b) Using the price index in (a)(i) estimate the cost of the engine in 1998 if the cost of the engine in 2005 was 1600 USD (**896.11**)

5. The table below shows the prices and amounts of items bought in 2004 and 2005

Item	Prices (shs)		Amount
	2004	2005	
A	635	887.5	6
B	720	815	4
C	730	1045	3
D	362	503	7

(a) Calculate the simple aggregate price index (**132.836**)
(b) Calculate the weighted aggregate price index (**133.52**)
(c) Calculate the price of an item costing 500 in 2004 using weighted aggregate price index above. (**667.64**)

6. The table below shows the prices of items per kg in the year 2005 and 2007

Item	Posho	Beans	Rice	Beef	Chicken
Price in 2005	1200	2000	1200	4000	8000
Price in 2007	1600	2500	1600	6000	9500

Calculate for 2007 using 2005a as the base year
(a) Simple price index (**132.0833**)
(b) Simple aggregate price index (**129.2683**)

7. The table below shows the prices in the year 2010 and 2018

Item	Price in 2010	Price in 2018
Flour per kg	3000	5400
Eggs per dozen	5,000	7800

Calculate for 2018 using 2010 as the base year
(i) Simple price index (**168**)
(ii) Simple aggregate price index (**165**)

8. The table below shows the prices and quantities of four items in the year 2020 and 2021

Item	Price per unit		Quantities	
	2020	2021	2020	2021
A	100	120	36	42
B	110	100	96	88
C	50	65	10	12
D	80	85	11	10

(a) Calculate the price index (120, 90.91, 130, 106.25)
(b) Simple aggregate price index (108.82)
(c) Weighted aggregate price index (99.55)
(d) Cost in 2021, A, B, C and Dare ingredients to make chapatti and in 2020a price of chapatti costed shilling 600 using index in (iii) above. (**shs. 597.3**)

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Line of action of the resultant force

The equation of line of action is given by

$$G = \begin{vmatrix} x & F_x \\ y & F_y \end{vmatrix} = xF_y - yF_x$$

$G - xF_y + yF_x = 0$

Note

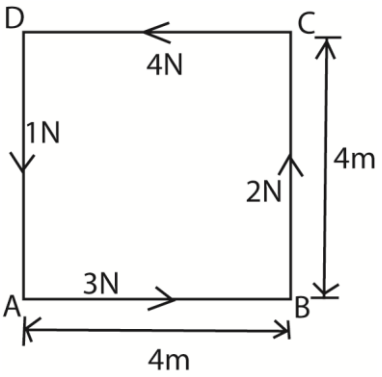
The line of action cuts the horizontal when y = o and the cuts the vertical axis when x = 0

Example1

Forces of 3N, 2N, 4N and 1N act along the sides of a square ABCD of side 4m in the direction AB, BC, CD and DA respectively, the direction of force in each case being the order of the letters. Find the;

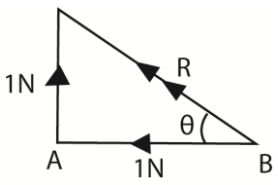
(i) Magnitude and direction of the resultant force
(ii) equation of the line of action
(iii) point where the line of action of the resultant of forces cuts AB

Solution



$$R = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$|R| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}N$



Direction, θ =$\tan^{-1}\left(\frac{1}{1}\right)$ = 45^0 to AB

A: G = 4 x 4 + 2 x 4 = 24Nm anticlockwise

Equation is 24 – x + y = 0

(ii) Line cuts AB when y = 0; 24 – x + 0 = 0

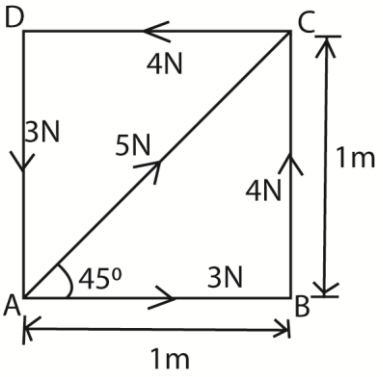
x = 24m from A

Example 2

Five forces of magnitude 3N, 4N, 4N, 3N and 5N act along AB, BC, CD, DA and AC respectively of squares of side 1m. The directions of the forces being given in the order of the letters Taking AB and AD as horizontal and vertical respectively. Find the

(a) magnitude and direction of the resultant force
(b) equation of the line of action

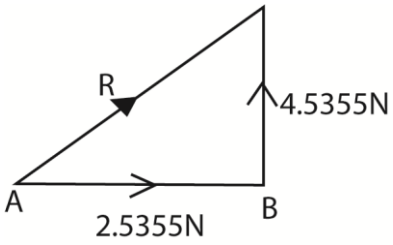
Solution



$R = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}4\\0\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} = \begin{pmatrix}0\\-3\end{pmatrix} + \begin{pmatrix}5cos45\\5sin45\end{pmatrix}$

$= \begin{pmatrix}2.5355\\4.5355\end{pmatrix}$

$|R| = \sqrt{(2.5355)^2 + (4.5355)^2} = 5.1961N$



Direction, θ =$\tan^{-1}\left(\frac{4.5355}{2.5355}\right)$ = 60.8^0 to AB

A: G = 4 x 1 + 4 x 1 = 8Nm anticlockwise

Equation is 8 – 4.5355x + 2.5355y = 0

(ii) Line cuts AB when y = 0; 8 – 4.5355x + 0 = 0

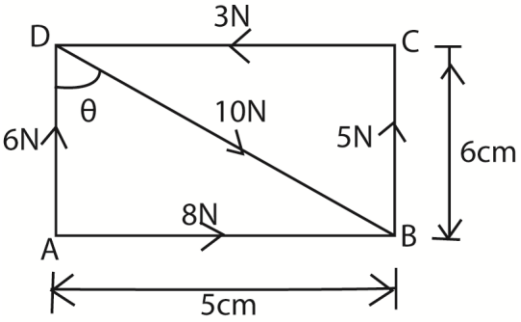
x = $\frac{8}{4.5355} = 1.769$m from A

Example 3

ABCD is a rectangle with AB = DC = 5m and AD = BC = 6m. Forces of magnitude 8N, 5N, 3N, 6N and 10N act along AB, BC, CD, AD and DB of the rectangle respectively. The directions of the forces being given by the order of the letters. Find

(i) the magnitude and the direction of a single force that could replace this system of force
(ii) equation of the line of action
(iii) where the line of action cuts AB

Solution



(i) $\theta = \tan^{-1}\left(\frac{5}{6}\right) = 398^0$

$R = \begin{pmatrix}8\\0\end{pmatrix} + \begin{pmatrix}0\\5\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}0\\6\end{pmatrix} + \begin{pmatrix}cos39.8\\-sin39.8\end{pmatrix}$

$= \begin{pmatrix}11.4011\\3.3172\end{pmatrix}$

$|R| = \sqrt{11.4011^2 + 3.3172^2}$ =11.8739N

Direction, α =$\tan^{-1}\left(\frac{3.3172}{11.4011}\right)$ = 16.2^0 to AB

(ii) A G= (5 x 0.05) + (3 x 0.06) + (10 sin39.8 x 0.06) = 0.046Nm anticlockwise

0.046 - 0.3172x + 11.4011y = 0

(iii) line cuts AB where y = 0

0.046 - 0.3172x =0

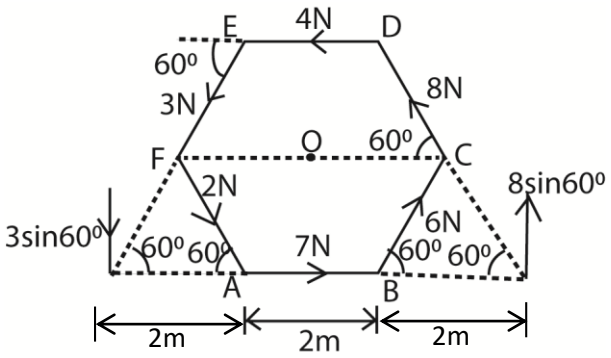
x = 0.0139m from A

Example 4

ABCDEF is a regular hexagon 2m. Forces of magnitude 7N, 6N, 8N, 4N, 3N and 2N act along AB, BC, CD, DE, EF and FA respectively in each case the direction of force being given by the order of letters. Given that AB is horizontal, find the

(a) the magnitude and direction of resultant force
(b) where its line of action of the resultant cuts AB

Solution



$R = \begin{pmatrix}7\\0\end{pmatrix} + \begin{pmatrix}6cos60\\6sin60\end{pmatrix} + \begin{pmatrix}-8cos60\\8sin60\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} + \begin{pmatrix}-3cos60\\-3sin60\end{pmatrix} + \begin{pmatrix}2cos20\\-2sin60\end{pmatrix} = \begin{pmatrix}1.5\\7.794\end{pmatrix}$

$|R| = \sqrt{1.5^5 + 7.794^2} = 7.94N$

Direction, α = $\tan^{-1}\left(\frac{7.794}{1.5}\right)$= 79.11⁰ to AB

A: G =(6sin60⁰ x 2) + (8sin60⁰ x 4) + (4 x 4sin60⁰) + (3sin60⁰ x 2) = $33\sqrt{3}$ = 57.158Nm

$G = xF_y - yF_x$

57.158 = 7.794x - 1.5y

Line cuts AB when y = 0

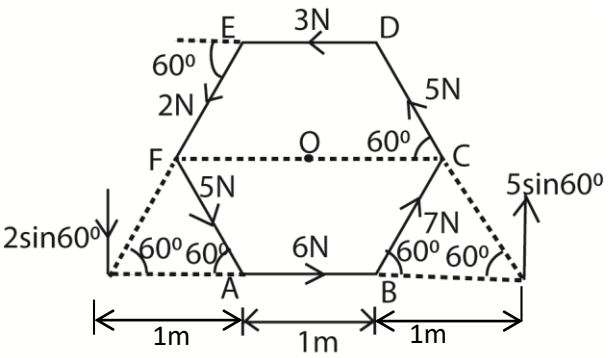
x = $\frac{57.158}{7.794}$ = 7.334m from A

Example 5

ABCDEF is a regular hexagon 1m. Forces of magnitude 6N, 7N, 5N, 3N, 2N, and 52N act along AB, BC, CD, DE, EF and FA respectively in each case the direction of force being given by the order of letters. Given that AB is horizontal, find the

(a) the magnitude and direction of resultant force
(b) where its line of action of the resultant cuts AB

Solution



$R = \begin{pmatrix}6\\0\end{pmatrix} + \begin{pmatrix}7cos60\\7sin60\end{pmatrix} + \begin{pmatrix}-5cos60\\5sin60\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}-2cos60\\-2sin60\end{pmatrix} + \begin{pmatrix}5cos20\\-5sin60\end{pmatrix} = \begin{pmatrix}5.5\\4.33\end{pmatrix}$

$|R| = \sqrt{5.5^5 + 433^2} = 7.0N$

Direction, α = $\tan^{-1}\left(\frac{4.33}{5.5}\right)$= 38.21^0 to AB

A: G = (7sin60^0 x 1) + (5sin60^0 x 2) + (4 x 4sin60^0) + (3 x 2sin60^0)+ (2sin60 x 1= 12.5$\sqrt{3}$

= 21.65Nm anticlockwise

$G = xF_y - yF_x$

21.65 = 4.33x - 5.5y

Line cuts AB when y = 0

x = $\frac{21.65}{4.33}$ = 5m from A

Example 6

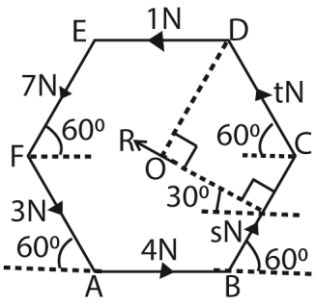
The center of a regular hexagon ABCDEF of side 2a is O. Forces of magnitude 4N, sN, tN, 1N, 7N and 3N act along the sides AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters.

(i) Given that the resultant of these six forces is of magnitude $2\sqrt{3}N$ acting in the direction perpendicular to BC, determine the value of s and t.

(ii) show that the sum of moments of forces about O is 27a$\sqrt{3}Nm$

(iii) If the midpoint of BC is M, find the equation f the line of action of the resultant, refer to OM as x-axis and OD as y-axis

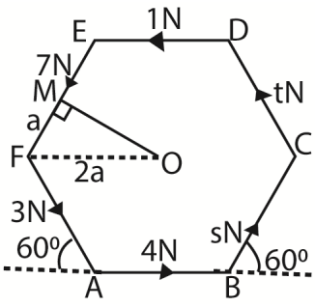
Solution



$$\begin{pmatrix} -2\sqrt{3}cos30 \\ 2\sqrt{3}sn30 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} s\ cos60 \\ s\ sin60 \end{pmatrix} + \begin{pmatrix} -tcos60 \\ tsin60 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -7cos60 \\ -7sin60 \end{pmatrix} + \begin{pmatrix} 3cos60 \\ -3sin60 \end{pmatrix}$$

$s = 2\ and\ t = 10$

(ii)



OM = $\sqrt{(2a)^2 - a^2}$ =a$\sqrt{3}$

O: G =(4 + 2 + 10 + 1 + 7 + 3) a$\sqrt{3}$ = 27 a$\sqrt{3}$Nm

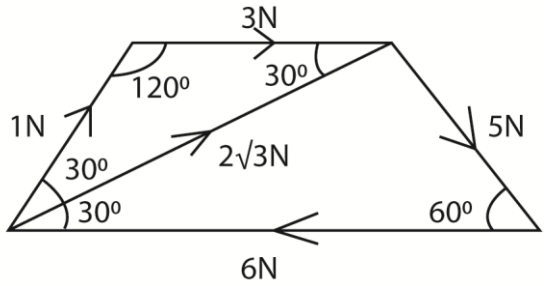
Since the resultant is in direction OM, => x = 2$\sqrt{3}$, y =0

G = xY + yX; resultant cuts the x-axis when Y = 0

27 a$\sqrt{3}$ = -2$\sqrt{3}$y; y = -13.5a

Example 7

The diagram below shows a trapezium rule ABCD, AD = DC = CB = 1 and AB = 2meters. Forces of magnitude 1N, 3N, 5N, 6N and $2\sqrt{3}$ N respectively.



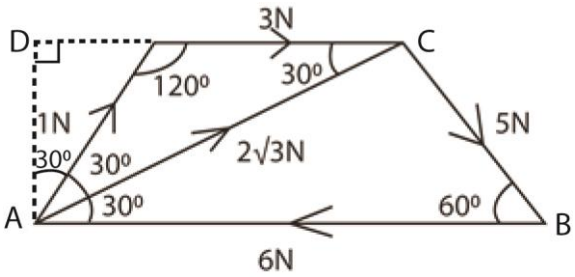
(a) Calculate the magnitude of the resultant force and the angle it makes with side AB (09marks)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\sqrt{3}\cos 30^0 \\ 2\sqrt{3}\sin 30^0 \end{pmatrix} + \begin{pmatrix} \cos 60^0 \\ \sin 60^0 \end{pmatrix} + \begin{pmatrix} 5\cos 60^0 \\ -5\sin 60^0 \end{pmatrix} = \begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix}$$

$Resultant\ force, R = \sqrt{(3)^2 + \left(-\sqrt{3}\right)^2}$ = 3.464N

$Direction, \alpha = tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ = 30⁰

(b) Given that the line of action of the resultant force meets AB at X, find AX. (03marks)



Equation of the line action of the resultant is given by G-xY + yX = 0

Taking moments about A

G= -3 x 1cos 30^0 – 5 x 2 cos 30^0

= -3 x $\frac{\sqrt{3}}{2}$ -10 x $\frac{\sqrt{3}}{2}$ = $\frac{-13\sqrt{3}}{2}$

By substitution

$\frac{-13\sqrt{3}}{2} + \sqrt{3}x + 3y = 0$

The line of action of the resultant cuts AB when y = 0

$\frac{-13\sqrt{3}}{2} + \sqrt{3}x + 3\ x\ 0 = 0$

$x = 6.5m$

Hence $\overline{AX}$= 6.5m

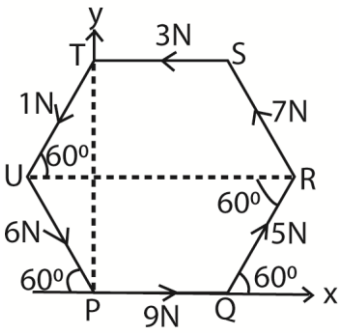
Example 8

Six forces, 9N, 5N, 7N, 3N, 1N and 4N act along the sides PQ, QR, RS, ST, TU and UP of a regular hexagon of side 2m, their direction being indicated by the order of letters. Taking PQ as the reference axis, express each of the forces in vector form. Hence find the

(i) magnitude and direction of the resultant of forces

(ii) distance from P, where the line of action of the resultant cuts PQ.

Solution



PQ = $\begin{pmatrix}9\\0\end{pmatrix}$;

QR = $\begin{pmatrix}5cos60^0\\5sin60^0\end{pmatrix} = \begin{pmatrix}\frac{5}{2}\\\frac{5\sqrt{3}}{2}\end{pmatrix}$

RS= $\begin{pmatrix}-7cos60^0\\7sin60^0\end{pmatrix} = \begin{pmatrix}\frac{7}{2}\\\frac{7\sqrt{3}}{2}\end{pmatrix}$

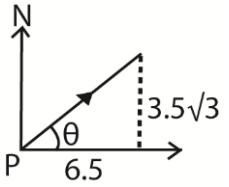
ST = $\begin{pmatrix}-3\\0\end{pmatrix}$

TU = $\begin{pmatrix}5cos60^0\\5sin60^0\end{pmatrix} = \begin{pmatrix}\frac{-1}{2}\\\frac{-\sqrt{3}}{2}\end{pmatrix}$

UP= $\begin{pmatrix}4cos60^0\\-4sin60^0\end{pmatrix} = \begin{pmatrix}2\\-2\sqrt{3}\end{pmatrix}$

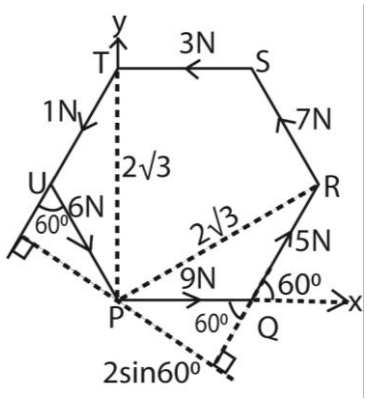
$$\mathrm{R} = \begin{pmatrix}9\\0\end{pmatrix} + \begin{pmatrix}\frac{5}{2}\\\frac{5\sqrt{3}}{2}\end{pmatrix} + \begin{pmatrix}\frac{7}{2}\\\frac{7\sqrt{3}}{2}\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix} + \begin{pmatrix}\frac{-1}{2}\\\frac{-\sqrt{3}}{2}\end{pmatrix} + \begin{pmatrix}2\\-2\sqrt{3}\end{pmatrix} = \begin{pmatrix}6.5\\3.5\sqrt{3}\end{pmatrix}$$

$|R| = \sqrt{(6.5)^2 + \left(3.5\sqrt{3}\right)^2}$ = 8.9N



$\theta = \tan^{-1}\left(\frac{3.5\sqrt{3}}{6.5}\right) = 43^0$

(ii) Equation of the line of the resultant is given by: G - xY + yX



P: G = $5\sin60^0 + 7 \times 2\sqrt{3} + 3 \times 2\sqrt{3} + 1 \times 2\sin60^0$

$= 5\sqrt{3} + 14\sqrt{3} + 6\sqrt{3} + \sqrt{3} = 26\sqrt{3}$Nm

By substitution

$26\sqrt{3} - 3.5\sqrt{3}x + 6.5y = 0$

The line cuts PQ when y = 0

⇨ $26\sqrt{3} - 3.5\sqrt{3}x = 0$; x = 7.43m

∴The lines cuts PQ at a distance 7.43m from P

Revision exercise

1. ABCD is a square of side 5m. Forces of 4N, 6N, 8N and 10N act along BD, DC, CA and CB respectively. When a force P acts along AD and force Q acts along AB, the whole system is equivalent a couple. Find the magnitude of P and Q and the moment of the couple. [12.8N, 2.49N, 65,9Nm]
2. ABCD is a square of side am. Forces of 1N, 4N, 3N and 6N act along AB, CB, DC and AD respectively. Calculate the magnitude and direction of a single force that could replace this system of forces and find where its line of action cuts AB. [4.47N, 26.6^0 to AB, 3.5a from A]
3. ABCD is a rectangle with AB = 3m and angle CAB = 30^0. Forces of 10N, 20N and 20N act along AC, AD and DB respectively. Calculate the magnitude and direction of a single force that could replace this system of forces and where its line of action cuts AB. [30N, 30^0 to AB, 2m from A]
4. ABCD is a rectangle with AB = 5m and BC = 3m. Forces of magnitude 2N, 4N, 3N and 1N act along the lines AB, BC, DC and DA respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
   (i) magnitude and direction of the resultant force [6N]
   (ii) equation of the line of action of the resultant
   (iii) distance from A where the line of action of the resultant force cuts AB. [1.57m]
5. ABCD is a rectangle with AB = 4m and BC = 3m. Forces of magnitude 3N, 5N, 6N, 4N and 7N act along the lines AB, BC, CD, DA and AC respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
   (i) magnitude and direction of the resultant force. [5.81N]
   (ii) distance from A where the line of action of the resultant force cuts AB[7.31m]
6. ABCD is a square of side 2m. Forces of magnitude 10N, 9N, 8N, 7N and 5N act along the lines AB, BC, CD, DA and AC respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
   (i) magnitude and direction of the resultant force [20.305N]

(ii) sum of moments about A [34Nm]

(iii) distance from A where the line of action of the resultant force cuts AB. [1.74m]

7. ABCDEF is a regular hexagon of side 4m. Forces of magnitude 6N, 4N, 7N, 8N, 4N and 2N act along the lines AB, BC, CD, DE, EF and FA respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
   (i) magnitude and direction of the resultant force [7.55N]
   (ii) distance from A where the line of action of the resultant force cuts AB. [21.71m]
8. ABCDEF is a regular hexagon of side 34m. Forces of magnitude 5N, 6N, 2N, 3N, 6N and 1N act along the lines AB, BC, CD, DE, EF and FA respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
   (i) magnitude and direction of the resultant force [4.583N]
   (ii) distance from A where the line of action of the resultant force cuts AB. [13.2m]
9. ABCDEF is a regular hexagon of side 2m. Forces of magnitude 2N, 3N, 4N and 5N act along the lines AC, AE, AF and ED respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
   (i) magnitude and direction of the resultant force [8.84N, 57.6^0 to AB]
   (ii) distance from A where the line of action of the resultant force cuts AB. [2.32m from A]
10. ABCDEF is a regular hexagon of side 2m. Forces of magnitude 3N, 4N, 2N, 1N, 2N and 6N act along the lines AB, BC, DC, ED, EF and AF respectively, in each case the direction of force being given by the order of the letters. Given AB is horizontal, find the
    (i) magnitude and direction of the resultant force [6N]
    (ii) Equation of the line of action of the resultant force [y = $\sqrt{3}x$]
11. ABCDE is a regular pentagon of side 2m. Forces of 5N act along AB, BC and AD. find the
    (i) magnitude and direction of the resultant force [12.5N,at 49.6^0 to AB]
    (ii) distance from A where the line of action of the resultant force cuts AB. [1m from A]

Thank You

Dr. Bbosa Science

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Linear interpolation and extrapolation

Linear interpolation

Deals with computations of values that lie within a given range

Example 1

The table below shows values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Find values of (i) f(1.88) (ii) x corresponding to f(x) = 0.4

Solution

1.8	1.88	2.0
0.532	y	0.484

$\frac{y-0.532}{1.88-1.8} = \frac{0.884-0.532}{2.0-1.8}$

y = 0.513

(ii)

2.2	x_0	2.4
0.436	0.4	0.384

$\frac{x_0-2.2}{0.4-0.436} = \frac{2.4-2.2}{0.384-0.436}$

x_0 =2.34

Example 2

Given the table below

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Using linear interpolation find

(i) f(x) when x = 10.15 (ii) f^{-1} (2.02)

Solution

10	10.15	11
2.42	y	2.18

$\frac{y-2.42}{10.15-10} = \frac{2.18-2.42}{11-10}$

y =2.384

11	x_0	12
2.18	2.02	1.92

$\frac{x_0-11}{2.02-2.18} = \frac{12-11}{1.92-2.18}$

x_0 = 11.62

Example 3

Given table below

x^0	40.0^0	40.4^0	40.8^0	50.4^0
sin x^0	0.6428	0.6481	0.6534	0.7705

Find (i) sin 40.5^0 (ii) $\sin^{-1}$ 0.6445

Solution

40.4^0	40.5^0	40.8^0
0.6481	y	0.6534

$$\frac{y-0.6481}{40.5-40.4} = \frac{0.6534-0.6481}{40.8-40.4}$$

y = 0.6494

40.0^0	x_0	40.4^0
0.6428	0.6445	0.6481

$$\frac{x_0-40.0}{0.6445-0.6428} = \frac{40.4-40.0}{0.6481-0.6428}$$

x_0 = 40.13

Linear extrapolation

This deals with computation of values that lie outside given values

Example 4

Given the table below

x	2.2	2.6	3.1
x^3	10.648	17.576	29.791

Find 3.4^3

2.6	3.1	3.4
17.576	29.791	y

$$\frac{y-29.791}{3.4-3.1} = \frac{29.791-17.576}{3.1-2.6}$$

y = 37.12

Example 5

The table below is an extract from table of sec x

x = 60^0	0’	12’	24’	36’	48’
$sec\ x$	2.0000	2.0122	2.0245	2.0371	2.0498

Use linear interpolation to determine

(i) sec $60^0$15’

(ii) angle whose secant is 2.0436 [$60^0$42’]

Solution

(i)

12’	15’	24’
2.0122	y	2.0245

$$\frac{y-2.0122}{15-12} = \frac{2.0245-2.0122}{24-12}$$

y = 2.03065

(ii)

36’	x	48’
2.0371	2.0436	2.0498

$$\frac{x-36}{2.0436-2.0371} = \frac{48-36}{2.0498-2.0371}$$

x = 42’ hence angle = $60^0$42’

Example 6

The table below shows the values of a function f(x)

x	1.8	2.0	2.2	2.4
f(x)	0.532	0.484	0.436	0.384

Use linear interpolation to find the value of

(i) F(2.08)

1.8	2.08	2.0
0.532	f(x)	0.484

$$\frac{f(x)-0.436}{2.08-2.0} = \frac{0.436-0.484}{2.2-2.0}$$

$$\frac{f(x)-0.436}{0.08} = \frac{-0048}{0.2}$$

f(x) = 0.4648 or 0.465 (3D)

(ii) x corresponding to f(x) = 0.5 (05marks)

1.8	x	2.0
0.532	0.5	0.484

$$\frac{0.5-0.532}{x-1.8} = \frac{0.484-0.532}{2.0-18}$$

$$\frac{-0.032}{x-1.8} = \frac{-0.048}{0.2}$$

$x = 1.9333$ or 1.9 (1D)

Example 7

Given the table below,

x	0	10	20	30
y	6.6	2.9	-0.1	-2.9

Use linear interpolation to find

(a) y when x = 16

Extract

x	10	16	20
y	2.9	y_0	-0.1

$$\frac{y_0-2.9}{16-10} = \frac{-0.1-2.9}{20-10}$$

$$\frac{y_0-2.9}{6} - \frac{-3.0}{10}$$

$y_0 = 1.1$

hence when x = 16, y =1.1

(b) x when y = -1

Extract

x	20	x_0	30
y	-0.1	-1	-2.9

$$\frac{x_0-20}{-1-(-0.1)} = \frac{30-20}{-2.9-(-0.1)}$$

$x_0 = 23.2$

Hence when y = -1; x = 23.2

Example 8

The table below shows the values of a function f(x) for given values of x.

x	9	10	11	12
f(x)	2.66	2.42	2.18	1.92

Use linear interpolation or extrapolation to find

(a) f(10.4)

Extract

10	10.4	11
2.42	f(x)	2.1

Using gradient approach

$\frac{2.18-f(x)}{11-10.4} = \frac{2.18-2.42}{11-10}$

$\frac{2.18-f(x)}{0.6} = \frac{-0.24}{1}$

f(x) = 2.18 + 0.24 x 0.6 = 2.324

(b) the value of x, corresponding to f(x) = 1.46 (05marks)

Extract

x	12	11
1.46	1.92	2.18

Using gradient approach

$\frac{2.18-1.46}{11-x} = \frac{2.18-1.92}{11-12}$

$\frac{0.72}{11-x} = \frac{0.26}{-1}$

X = 11 - $\frac{-1\ x\ 0.72}{0.26}$ = 13.769

Revision exercise

1. Table below is an extract from the table of cos x

x	0^0	10^0	20^0	30^0	40^0	50^0
Cos x	0.1736	0.1708	0.1679	0.1650	0.1622	0.1593

Use linear interpolation to determine: (i) cos 80^0 36′ [0.1633] cos $^{-1}$(0.1685) [$80^0$18′]

2. The table below shows variation of temperature with time in a certain experiment.

Time (s)	0	120	240	360	480	600
Temperature (^{0}C)	100	80	76	65	50	48

Use linear interpolation to determine

(i) value of ^{0}C corresponding to 400s [62^{0}C]

(ii) time at which the temperature is 77^{0}C [192s]

3. The table below shows the value of a function In(x) for given values of x

x	1.4	1.5	1.6	1.7
In(x)	0.3365	0.4055	0.4700	0.5306

Using linear interpolation or extrapolation, find

(i) In(1.66) [0.5064] (ii) find value of x corresponding to In(x) = 0.400 [1.492]

4. The table below shows variation of temperature with time in certain experiment.

Time (s)	0	10	15	20	30
Temperature (^{0}C)	80	70.2	65.8	61.9	54.2

Use linear interpolation to determine

(i) value of θ^0 corresponding to T= 18s [63.5^{0}C]

(ii) Time T at which the temperature θ^0 = 60^{0}C [22.5s]

5. Given the table below

x	-1.0	-0.5	-1.4
y	-1.0	-2.2	-3.7

Using linear interpolation or linear extrapolation to find

(i) y when x = 0.5 [-4.6] (ii) x when y = -4.5 [0.458]

6. In an examination, scaling is done such that candidate A who originally scored 35% gets 50% and candidate B with 40% gets 65%, determine the original mark for candidate C whose new mark is 80% [45%]
7. The table below is an extract of $\log_{10} x$

x	80.00	80.20	80.50	80.80
$\log_{10} x$	1.9031	1.9042	1.9058	1.9074

Using linear interpolation find

(i) $\log_{10} 80.759$[1.9072]

(ii) the number whose logarithm is 1.90388 [80.14]

8. The table below shows the values of a function f(x) for given values of x

x	2	3	4	5
$f(x)$	3.88	5.11	8.14	11.94

Use linear interpolation to determine

(i) f(2.15) [4.06]

(ii) the value of x corresponding to f(10.6) [4.68]

9. The table below shows distance in km a truck moves with a given amount of fuel in litres (l)

Distance (km)	20	28	33	42
$Fuel\ (l)$	10	13	21	24

Use linear interpolation or extrapolation to find

(i) How far the truck can move on 27.5l of fuel [52.5km]

(ii) the amount of fuel required to cover29.8km [15.88l]

10. The table below shows the values of a continuous f(t) with respect to t

t	0	0.3	0.6	1.2	1.6
$f(t)$	2.72	3.00	3.32	4.06	4.95

Use linear interpolation or extrapolation, find

(i) f(t) when t = 0.9 [3.69]

(ii) the value of t corresponding to f(t) = 4.48 [1.48]

11. The table below shows the delivery charges by courier company

Mass (g)	200	400	600
$charges\ (shs.)$	700	1200	300

Use linear interpolation or extrapolation, find

(i) the delivery charge of a parcel weighing 352g [1080]

(ii) mass of a parcel whose delivery charge is shs. 3,300 [633.33kg]

12. The table below shows the cost y shillings for hiring a motor cycle for a distance x kilometres.

Distance x (km)	10	20	30	40
Cost (shs.)	2800	3600	4400	5200

Use linear interpolation or extrapolation, find

(i) the cost of hiring the motor cycle for distance of 45km [shs. 5600]

(ii) distance travelled if he paid shs. 4000 [25km]

13. The table below shows the values of a function f(x) for given values of x

x	0.4	0.6	0.8
f(x)	-0.9613	-0.5108	-0.2231

Use linear interpolation to determine f^{-1}(-0.4308} correct to 2 decimal places [0.66]

14. The table below shows how T caries with S

T	-2.9	-0.1	2.9	3.1
S	30	20	12	9

Use linear interpolation/extrapolation to estimate values of

(a) T when S = 26 [-1.78] (ii) S when T = 3.4 [4.5]

15. The table below shows the commuter bus fare from stages A to B, C, D and E

Stage	A	B	C	D	E
Distance (km)	0	12	16	19	23
Fare (shs)	0	1300	1700	2200	2500

(a) Jane boarded from A and stopped at a place 2km after E. How much did she pay? (03marks) [shs. 2650]

(b) Okello paid shs 2000. How far from A did the bus leave him? (02marks) [17.km]

16. The table below shows the value of x and corresponding values of a function f(x)

The table below shows how T caries with S

x	0.3	0.6	0.9	1.2
f(x)	3.00	3.22	3.69	4.06

Use linear interpolation/extrapolation to estimate values of

(i) f(x) when x = 0.4 [3.0733] (ii) x when f(x) = 3.82 [1.0054]

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Linear momentum

This is the product of mass of a body and its velocity.
Momentum = mass x velocity

Collisions

Case 1: bodies separate after collision

Consider two bodies A and B with body A having a mass of M_A, initial velocity U_A, and body B having a mass f M_B, initial velocity U_B, after collision body A has a final velocity V_A and body B has a final velocity V_B.

$M_AU_A + M_BU_B = M_AV_A + M_BV_B$

Loss in k.e = $\left(\frac{1}{2}M_AU_A{}^2 + \frac{1}{2}M_BU_B{}^2\right) - \left(\frac{1}{2}M_AV_A{}^2 + \frac{1}{2}M_BV_B{}^2\right)$

%loss in k.e = $\frac{\left(\frac{1}{2}M_AU_A{}^2+\frac{1}{2}M_BU_B{}^2\right)-\left(\frac{1}{2}M_AV_A{}^2+\frac{1}{2}M_BV_B{}^2\right)}{\left(\frac{1}{2}M_AU_A{}^2+\frac{1}{2}M_BU_B{}^2\right)}x100\%$

Case2: bodies stick together and move with a common velocity after collision

Consider two bodies A and B with body A having a mass of M_A, initial velocity U_A, and body B having a mass f M_B, initial velocity U_B, after collision body A and body B stick together and move with common velocity V.

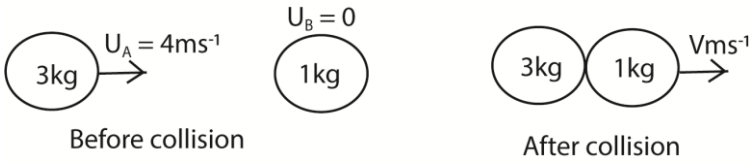
$M_AU_A + M_BU_B = (M_A + M_B)V$

Loss in k.e = $\left(\frac{1}{2}M_AU_A{}^2 + \frac{1}{2}M_BU_B{}^2\right) - \left(\frac{1}{2}(M_A + M_B)V^2\right)$

%loss in k.e = $\frac{\left(\frac{1}{2}M_AU_A{}^2+\frac{1}{2}M_BU_B{}^2\right)-\left(\frac{1}{2}(M_A+M_B)V^2\right)}{\left(\frac{1}{2}M_AU_A{}^2+\frac{1}{2}M_BU_B{}^2\right)}x100\%$

Example 1

A trolley of mass 3kg travelling at a velocity of 4ms^{-1} collide with another trolley of mass 1kg which is at rest. At what velocity do the two bodies move together after collision?



$M_AU_A + M_BU_B = (M_A + M_B)V$

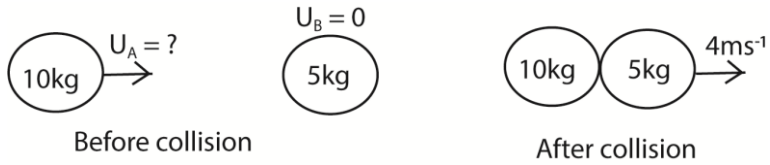
(3 x 4) + (1 x 0) = (3 + 1)V

12 = 4V

V = 3ms^{-1}

Example 2

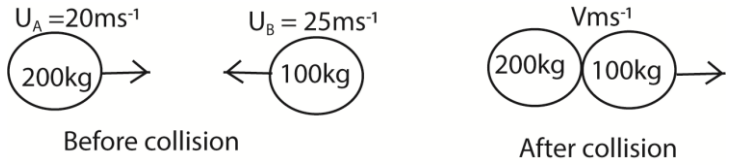
An object of mass 10kg collides with a stationary object of mass 5kg. If the objects stick together and move forward with a velocity of 4ms^{-1}. What was original velocity of the moving objects?



$M_AU_A + M_BU_B = (M_A + M_B)V$
$(10 \times U) + (5 \times 0) = (10 + 5) \times 4$
$U = 60ms^{-1}$

Example 3

Two bodies of masses 200kg and 100kg travel towards each other with velocities of $20ms^{-1}$ and $25ms^{-1}$ respectively and join to form one body on collision. Find the common velocity.



$M_AU_A + M_BU_B = (M_A + M_B)V$
$(200 \times 20) + (100 \times -25) = (200 + 100) \times v$
$V = 5ms^{-1}$

Example 4

A particle of mass 2kg moving with a speed $10ms^{-1}$ collides with a stationary particle of mass 7kg. Immediately after impact, the particle moves with the same speed but in opposite directions. Find the loss in kinetic energy

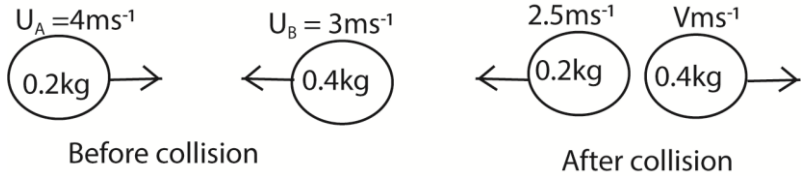
$M_AU_A + M_BU_B = M_AV_A + M_BV_B$
$2 \times 10 + 7 \times 0 = 2 \times -v + 7 \times v$
$v = 4ms^{-1}$
k.e before = $\frac{1}{2}\ x\ 2\ x\ 10^2 + \frac{1}{2}\ x\ 7\ x\ 0^2 = 100J$

k.e after = $\frac{1}{2}\ x\ 2\ x\ 4^2 + \frac{1}{2}\ x\ 7\ x\ 4^2$ = 72J

lossin k.e = 100 – 72 = 28J

Example 5

Two particles are moving towards each other along a straight line. The first particle has mass of 0.2kg and moving with velocity $4ms^{-1}$ and then the second has a mass of 0.4kg moving with a velocity of $3ms^{-1}$. On collision, the first particle reverses its direction and moves with a velocity of $2.5ms^{-1}$. Find the percentage loss in kinetic energy.



$M_AU_A + M_BU_B = M_AV_A + M_BV_B$
$0.2 \times 4 + 0.4 \times -3 = 0.2 \times -2.5 + 0.4V$
$V = 0.25ms^{-1}$
k.e before = $\frac{1}{2}\ x\ 0.2\ x\ 4^2 + \frac{1}{2}\ x\ 0.4\ x\ 3^2$
= 3.4J

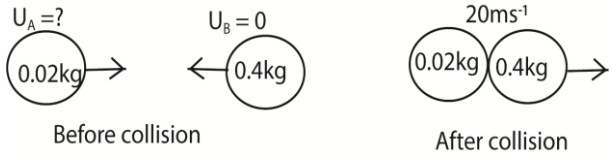
k.e after = $\frac{1}{2}\ x\ 0.2\ x\ -2.5^2 + \frac{1}{2}\ x\ 0.4\ x\ 0.25^2$
= 0.6375J

% loss in k.e = $\frac{(3.4-0.6375)}{3.4}\ x\ 100\%$ = 81.25%

Example 6

A bullet of mass 20g is fired into a block of wood of mass 400g lying on a smooth horizontal surface. If the bullet and the wood move together with the speed of $20ms^{-1}$. Calculate

(a) the speed with which the bullet hits the wood



$M_AU_A + M_BU_B = (M_A + M_B)V$
0.02 x UA + 0.4 x 0 = 0.42 x 20
UA = 420ms^{-1}

(b) The kinetic energy loss
k.e energy before = $\frac{1}{2}\ x\ 0.02\ x\ 420^2 = 1764J$
k.e after = $\frac{1}{2}\ x\ 0.42\ x\ 20^2 = 84J$
Loss in k.e = 1764 - 84 = 1680J

Example 7

Two bodies A and B of mass 4kg and 3kg moving with velocities (2i + 3j)ms^{-1} and (5i - 6j)ms^{-1} respectively collide. After collision A moves with a velocity (5i)ms^{-1}, Find the

(i) velocity of B after collision
$M_AU_A + M_BU_B = M_AV_A + M_BV_B$
$4(2i + 3j) + 3(5i – 6j) = 4\ (5i) + 3V_B$
$V_B = (i - 2j)\ ms^{-1}$

(ii) loss in kinetic energy
k.e before = $\frac{1}{2}\ x\ 4(2^2 + \ 3^2) + \frac{1}{2}\ x\ 3(5^2 + \ (-6)^2)$ = 117.5J
k.e after = $\frac{1}{2}\ x\ 4(5^2) + \frac{1}{2}\ x\ 3(1^2 + \ (-2)^2)$= 57.5J
Loss in k.e = 117.5 – 57.5 = 60J

Example 8

Two bodies A and B of mass 7.5kg and 5.0kg moving with velocities of (–i – 2j)ms^{-1} and (9i + 8j)ms^{-1} respectively. After collision the bodies stick together and move with a common velocity, find the

(i) common velocity
$M_AU_A + M_BU_B = (M_A + M_B)V$
$7.5(–i – 2j) + 5.0(9i + 8j) = 12.5V$
$V = (3i + 2j)ms^{-1}$

(ii) percentage loss in kinetic energy
k.e before = $\frac{1}{2}\ x\ 7.5((-1)^2 + \ (-2)^2) + \frac{1}{2}\ x\ 5(9^2 + \ 8^2)$ = 381.25J
k.e after = $\frac{1}{2}\ x\ 12.5(3^2 + \ 2^2)$= 81.25J
Loss in k.e = 381.25 – 81.25 = 300J
%loss in k.e = $\frac{300}{381.25}\ x\ 100\% = 78.69\%$

Example 9

Two bodies A and B of masses 3kg and 2kg respectively are 7m a part on a smooth horizontal surface. A moving directly towards B with a speed of 2ms^{-1} and acceleration of 0.3ms^{-2}. B is moving in the in the same direction as A with a speed of 5ms^{-1} and retardation of 0.2ms^{-2}. If the bodies collide and coalesce, calculate

(i) Time taken before collision occurs
let the distance travelled by B before collision = x
$s = ut + ½\ at^2$

For A: 7 + x = 2t + $\frac{1}{2}$ x $0.3t^2$ …… (i)

For B: x = 5t – $\frac{1}{2}$ x 0.2t^2 ……(ii)

Subtract (ii) from (i)

7 = -3t + $0.25t^2$

$t^2 - 12t - 28 = 0$

$t = 14s$

(ii) Common velocity immediately after collision

Initial velocity of A before collision V_A = 2 + 0.3 x 14 = 6.2ms^{-1}

Initial velocity of B before collision VB = 5 – 0.2 x 14 = 2.2ms^{-1}

Let the common velocity be V

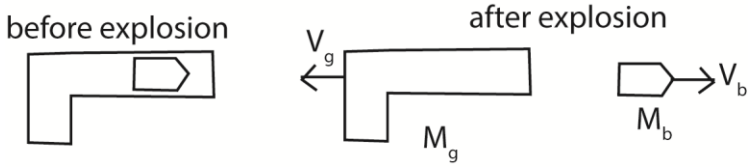
$M_AU_A + M_BU_B = (M_A + M_B)V$

3 x 6.2 + 2 x 2.2 = 5V

V = 4.6ms^{-1}

Recoil velocity of a gun and muzzle velocity of a bullet

When a bullet of mass Mb is fired with a muzzle vel0city of Vb, the gun of mass Mg jerks backward with a recoil velocity of Vg.



M_g x 0 + M_b x 0 = M_g x -V_g + M_bV_b

M_g x V_g = M_bV_b

Example 10

A bullet of mass 60g is fired from a gun of mass 3kg. The bullet leaves the gun with velocity of 400ms^{-1}. Find the initial speed of recoil of the gun and gain in kinetic energy of the system.

0.06 x 400 = 3 x V

V = 8ms^{-1}

Gain in k.e = k.e after = $\frac{1}{2}$ x 0.06 x 400^2 + $\frac{1}{2}$ x 3 x 8^2 =4896J

Example 11

A gun of mass 3000kg fires horizontally a shell at initial velocity of 300ms^{-1}. If the recoil of the gun is brought to rest by a constant opposing force of 9000N in 2 seconds, find the

(a)(i) Initial velocity of the recoil gun

F = ma	v = u + at
-9000 = 3000a	0 = u – 3 x 2
a = -3ms^{-2}	u = 6ms^{-1}

(ii) Gain in kinetic energy of the shell just after firing

$M_gV_g = M_bV_b$

$3000 \times 6 = M_b \times 300$

$M_b = 60kg$

(b)(i) displacement of the gun

$$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 6^2}{2\,x - 3} = 6m$$

(ii) work done against the opposing force

$W = Fs = 9000 \times 6 = 54{,}000J$

Revision exercise

1. A bullet of mass 50g travelling horizontally at 80ms-1 hits a block of wood of mass 10kg resting on a smooth horizontal plane. If the bullet emerges with a speed of $50ms^{-1}$, find the speed with which the block moves. [$0.15ms^{-1}$]
2. A bullet of mass 0.1kg travelling at 420ms-1 hits a block of wood of mass 2kg resting on a smooth horizontal plane. If the bullet becomes embedded on the block, find the speed with which the block moves after impact. [20ms-1]
3. A 2kg object moving with a velocity of $8ms^{-1}$ collides with a 3kg object moving with a velocity $6ms^{-1}$ along the same direction. If the collision is completely inelastic, calculate the decrease in kinetic energy.[2.4J]
4. A particle A of mass 150g lies at rest in a smooth horizontal surface. A second particle B of mass 100g is projected along the surface with the speed ums^{-1} and collides directly with A. On collision the masses coalesce and move with speed $4ms^{-1}$. Find the value of u and loss in the kinetic energy of the system during impact. [$10ms^{-1}$, 3J]
5. Two bodies A and B of masses 2kg and 4kg moving with velocities of $8ms^{-1}$ and $5ms^{-1}$ respectively collide and move in the same direction. Object A's new velocity is $6ms^{-1}$.
   (i) find the velocity of B after collision [$6ms^{-1}$]
   (ii) calculate the percentage loss in kinetic energy [5.26%]
6. Two bodies A and B of mass 2kg and 3kg moving with velocities of $4ms^{-1}$ and $3ms^{-1}$ respectively in the same direction collide and coalesce, find he
   (i) common speed after collision [$3.4ms^{-1}$]
   (ii) loss in kinetic energy [0.6J]
   (iii) percentage loss in kinetic energy [2.03%]
7. A bullet of mass 30g is fired horizontally at $200ms^{-1}$ and hits a block of wood of mass 2kg resting on smooth horizontal plane. If the bullet becomes embedded on the block, find the
   (i) common velocity of bullet and wood. [$2.96ms^{-1}$]
   (ii) percentage loss in kinetic energy. [98.52%]
8. Two smooth sphere A and B of equal radii and mass 3kg and 1.5kg respectively are travelling along the same horizontal line in opposite direction. The speeds of A and B are $6ms^{-1}$ and $2ms^{-1}$ respectively. The sphere A collides and after collision B reverses its direction and moves with speed of $4ms^{-1}$. Find the velocity of A after collision. [$3ms^{-1}$]
9. Two smooth spheres A and B of equal radii and masses 180g and 100g respectively travelling along the same horizontal line. The initial speeds of A and B are $2ms^{-1}$ and $6ms^{-1}$ respectively. The spheres collide and after collision both spheres reverse their directions and B moves with speed of $3ms^{-1}$. Find the speed of A after collision and loss in kinetic energy of the system. [$3ms^{-1}$, 0.9J]

10. A particle of mass 2kg moving with speed $10ms^{-1}$ collides with stationary particles of mass 7kg. Immediately after impact the particles move with the same speeds but in opposite direction. Find the loss in kinetic energy during collision. [28J].
11. Two identical rail way truck are travelling in the same direction along the same straight piece of track with constant speed of $6ms^{-1}$ and $2ms^{-1}$. The faster truck catches up with the other one on collision, the two trucks couple together. Find the common speed of the trucks after collision. [$4ms^{-1}$]
12. A 2kg object moving with a velocity of $6ms^{-1}$ collides with a stationary object of mass 1kg. If the collision is perfectly elastic, calculate the velocity of each object after collision. [$2ms^{-1}$, $8ms^{-1}$]
13. A van of mass of mass 1200kg and a lorry of mass 3200kg collide. Just before the crash they are moving directly towards each other and each has a speed of $12ms^{-1}$. Immediately after the crash they move with the same velocity. Find the loss in kinetic energy. [251kJ]
14. A body of mass 6kg moving with velocity $(8i - 4j)ms^{-1}$ collides with a body of mass 2kg which is at rest. On collision the two bodies coalesce, find the
    (i) common velocity after collision [$(6i - 3j)ms^{-1}$]
    (ii) loss in kinetic energy [960J]
15. A body of mass 2kg moving with velocity $(-2i + 4j)ms^{-1}$ collides with a body of mass 3kg moving with velocity $(3i + 4j)ms^{-1}$. On collision the two bodies coalesce, find the
    (i) common velocity after collision [$(i + 4j)\ ms^{-1}$]
    (ii) loss in kinetic energy [15J]
16. A body of mass 500g moving with velocity $(2i - 4j)ms^{-1}$ collides with a body of mass 1500g moving with velocity $(6i + 8j)ms^{-1}$. On collision the two bodies coalesce, find the
    (i) common velocity after collision [$(5i + 5j)ms^{-1}$]
    (ii) loss in kinetic energy [30J]
17. Two bodies A and B of mass 2kg and 5kg moving with velocities of $(-2i + 3j)ms^{-1}$ and $(6i - 10j)ms^{-1}$ respectively collide. After collision A moves with velocity $(3i - 2j)ms^{-1}$, find the
    (i) velocity of B after collision [$(4i - 8j)ms^{-1}$]
    (ii) loss in kinetic energy [140J]
18. Two bodies A and B of mass 5kg and 2kg moving with velocities of $(-4i + 3j)ms^{-1}$ and $(3i - 10j)ms^{-1}$ respectively collide. After collision A moves with a velocity $(-2i + j)ms^{-1}$, find the
    (i) velocity of B after collision [$(-2i + 4j)ms^{-1}$]
    (ii) loss in kinetic energy [40J]
19. A body X, of mass 250g moving with velocity $(-2i + 3j)ms^{-1}$ collide with a body Y of mass 750g moving with velocity $(5i + 8j)ms^{-1}$. After collision X moves with a velocity (-2i + j)ms-1, find the
    (i) velocity of Y after collision [$(4i + 6j)ms^{-1}$]
    (ii) loss in kinetic energy [5.25J]
20. A shell of mass 5kg is fired from a gun of mass 2000kg. The shell leaves the gun with a velocity of $400ms^{-1}$. find the initial speed of recoil of the rifle and gain in kinetic energy of the system [$1ms^{-1}$, 2520J)
21. A bullet of mass 20g is fired from a rifle of mass 2.5kg. The bullet leaves the gun with a velocity of $500ms^{-1}$. Find the initial of the recoil of rifle and gain in kinetic energy of the system. [$4ms^{-1}$, 2520J]
22. A bullet of mass 5kg is fired from a rifle of mass 2000g. The bullet leaves the gun with a velocity of $400ms^{-1}$. Find the recoil velocity of the rifle. [$1ms^{-1}$]

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Location of real roots

The range where the root of an equation lie can be located using the following methods

(i) sign change
(ii) Graphical method

(a) Sign change method

Example 1

Show that equation $x^3 + 6x^2 + 9x + 2 = 0$ has a root between -1 and 0

Solution

$f(x) = x^3 + 6x^2 + 9x + 2$

$f(-1) = (1)^3 + 6(1)^2 + 9(-1) + 2 = -14$

$f(0) = (0)^3 + 6(0)^2 + 9(0) + 2 = 2$

Since there is a sign change the root lies between 0 and -1.

Example 2

Show that the equation $e^{2x} \sin x - 1 = 0$ has a root between 0 and 1

Solution

Note that in trigonometric function the calculator must be in radian mode

$f(x) = e^{2x} \sin x - 1$

$f(0) = e^{2(0)} \sin 0 - 1 = -1$

$f(1) = e^2 \sin 1 - 1 = 5.2177$

Since there is a sign change the root lies between 0 and 1.

(b) Using graphical method

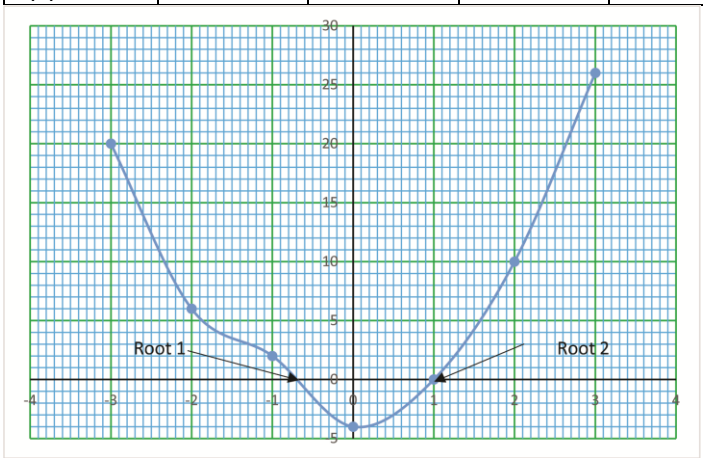
One or two graph(s) can be drawn to locate the root.

(i) Single graph method
When one graph is drawn then the root lies between the two points where the curve crosses the xaxis.

Example 3

Using a suitable graph locate the interval over which the root of the equation $3x^2 + x - 4 = 0$ lie.

x	-3	-2	-1	0	1	2	3
f(x)	20	6	-2	-4	0	10	26

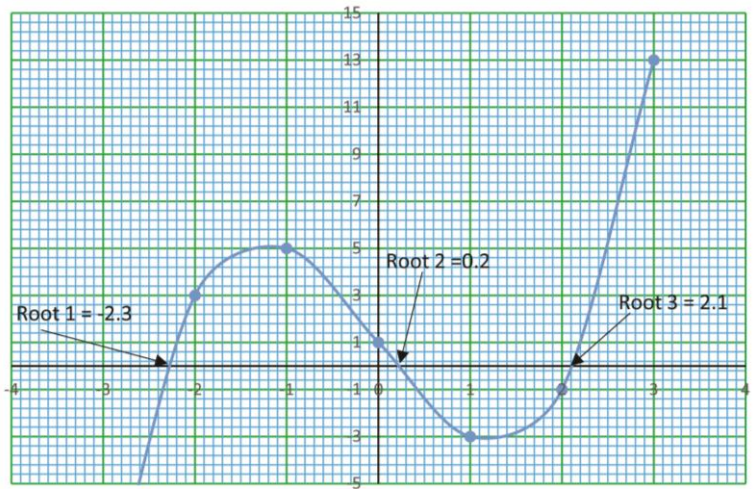


The root lies between -1 and 1

Example 4

Show graphically that there is positive real root of equation $x^3 - 5x + 1 = 0$

x	-3	-2	-1	0	1	2	3
f(x)	-11	3	5	1	-3	-1	13



(ii) Double graph method

When two graphs are drawn, the root lies between the points where the two curves meet.

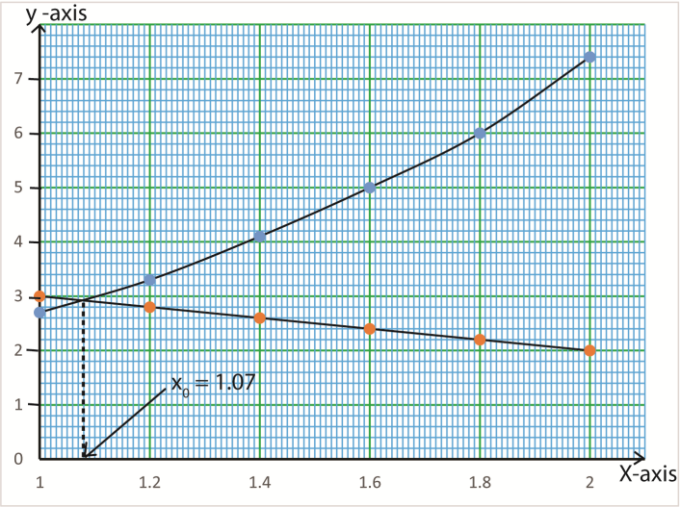
Note

(i) Both curves must have a consistent scale and should be labelled.
(ii) A line must be drawn using a ruler while a curve must be drawn using a freehand
(iii) Both graphs must be labelled
(iv) The initial approximation of the root must be located and indicated in the graph

Example 5

By plotting graph of y = e^x and y = 4 - x on the same axes, show the root of the equation e^x +x - 4 = 0 lie between 1 and 2

x	1	1.2	14	1.6	1.8	2.0
y = e^x	2.7	3.3	4.1	5.0	6.0	7.4
y = 4 - x	3.0	2.8	2.6	2.4	2.2	2

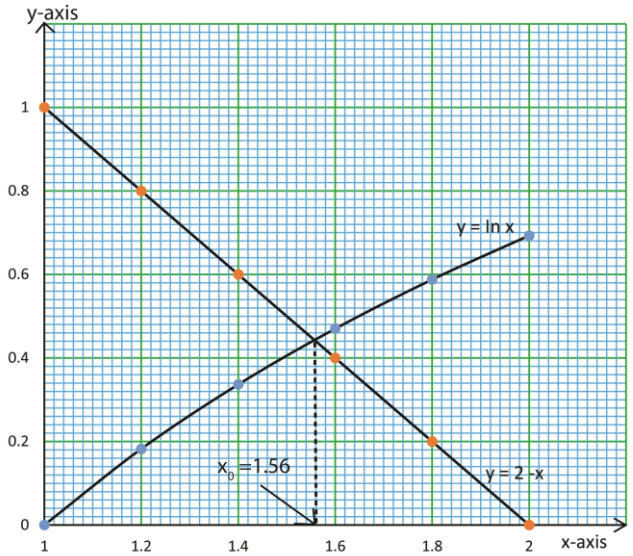


Therefore the root(1.07) lies between 1 and 2.

Example 6

Show that the equation Inx + x – 2 = 0 has a real root between x = 1 and x = 2

x	1	1.2	1.4	1.6	1.8	2.0
y = Inx	0	0.1823	0.3365	0.4700	0.5878	0.6731
y = 2-x	1	0.8	0.6	0.4	0.2	0

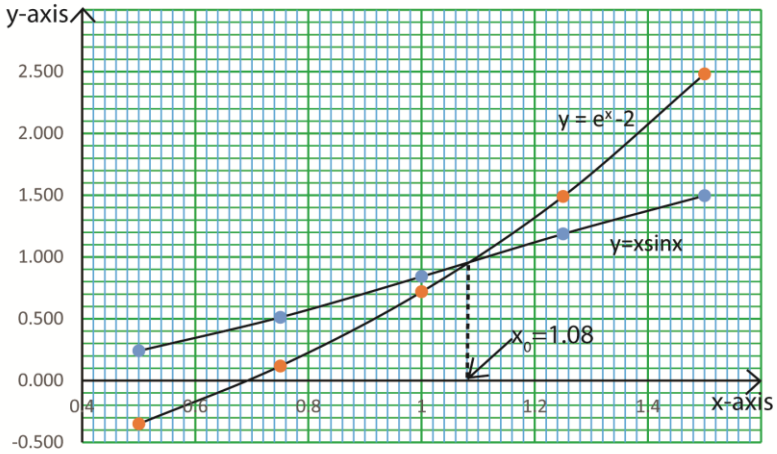


Therefore the root lies between x =1 and x =2

Example 7

By plotting graphs $y = e^x - 2$ an $y = x \sin x$ on the same axis show that the root of the equation $e^x - 2 - x\sin x = 0$ lies between x = 0.5 and x = 1.5

x	0.5	0.75	1.00	1.25	1.5
y = xsinx	0.240	0.511	0.841	1.186	1.496
$y = e^x - 2$	-0.351	0.117	0.718	1.490	2.481



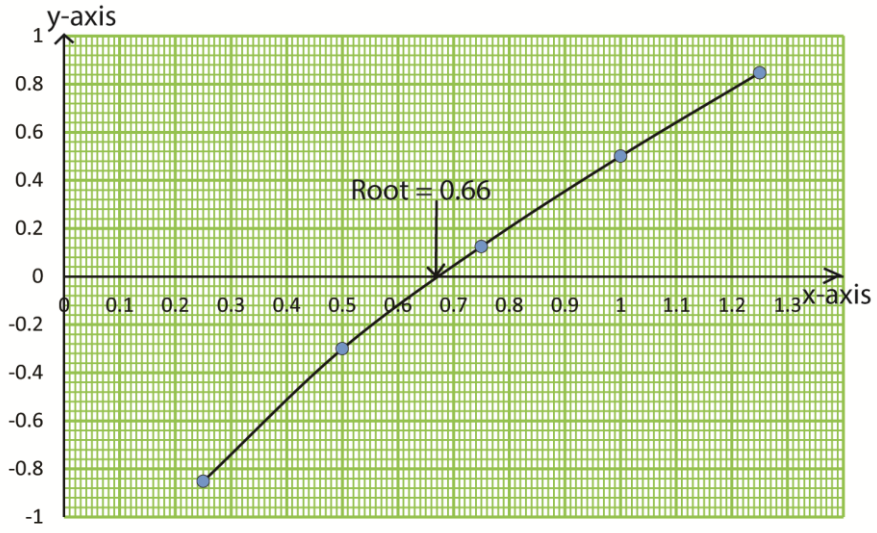
Example 8

Show graphically the equation x + logx = 0.5 has only one real root that lie between 0.5 and 1.

Solution

let y = x + logx - 0.5

x	0.25	0.5	0.75	1.00	1.25
y	-.852	-0.301	0.125	0.5	0.847



Therefore the root (0.66) lies between 0.5 and 1

Revision exercise 1

1. By sketching graphs of y = 2x and y = tanx show that the equation 2x = tanx has only one root between x = 1.1 and 1.2. Use linear interpolation to find the value of the root correct to 2dp.
2. Given the equation y = sinx - $\frac{x}{3}$, show by plotting two suitable graphs on the same axis that positive root lies between $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$.
3. Show graphically that the positive real root of the equation $2x^2 + 3x - 3 = 0$ lies between 0 and 1 [0.7]
4. Use a graphical method to show that the equation $e^x - x - 2 = 0$ has only one real root between 2 and -1 by drawing two graphs $y = e^x$ and y = x + 2 [-1.8]
5. On the same axes, draw graphs of y = 3 – 3x and $y = 2x^2$ to show that the root of the equation $2x^2 + 3x\ \ 3 = 0$ lies between -3 and -2 [-2.2]
6. Show graphically that the positive real root of the equation $x^3 - 3x - 1 = 0$, lies between 1 and 2 [1.6]
7. on the same axes, draw graph y = 3x -1 and $y = x^3$ to show that the root of the equation $X^3 - 3x - 1 = 0$lies between 0 and 1.[0.35]
8. Using suitable graphs and plotting them on the same axes. Find the root of the equation e^{2x}sinx -1 = 0, in the interval x = 0.1 and x = 0.8. [0.44]
9. Show graphically that equation $e^{-x} = x$ has only one real root between 0.5 and 1. [0.56]
10. Show graphically that equation $e^x = -2x + 2$ has only one real root between 0 and 1.0.
11. on the same axes, draw graphs of y = 9x -4 and $y = x^3$ show that the root of equation $x^3 - 9x + 4 = 0$ lie between 2.5 and 3
12. Show that the positive real root of equation $4 + 5x^2 - x^3 = 0$ lies between 5 and 6.
13. On the same axes, draw graphs of y = x +1 and y = tanx to show that the equation tanx - x -1 = 0 lie between 1 and 1.5.
14. Using suitable graphs and plotting them on the same axes, find the roots of the equation $5e^x = 4x + 6$ in the interval x = 2 and x = -1.
15. On the same axes, draw graphs of y = 2x + 1 and y = $\log_e(x + 2)$ to show that the root of equation $\log_e(x + 2) - 2x - 1 = 0$ lies between 1 and 0.
16. Using suitable graphs and plotting them on the same axes, find the real root of the equation $9\log_{10} x$ = 2(X -1) in the interval x = 3 and x =4.

Method of solving for roots

The following methods can be used

(a) Interpolation

Example 9

Show that the equation $x^4 - 12x^2 + 12 = 0$ has root between 1 and 2. Hence use linear interpolation to get the first approximation of the root.

Solution

$f(x) = x^4 - 12x^2 + 12$

$f(1) = 1^4 - 12(1)^2 + 12 = 1$

$f(2) = 2^4 - 12(2)^2 + 12 = -20$

Since there is a sign change,

then the root lies between

1 and 2.

x	1	x_0	2
f(x)	1	0	-20

$\frac{x_0-1}{0-1} = \frac{2-1}{-20-1}$

$x_0 = 1.05$

Example 10

Show that the equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between 1 and 2.Hence use linear interpolation twice to get the approximation of the root.

solution

Note: for trigonometric functions the

calculator must be strictly in radian mode

$f(x) = 2x - 3\cos\left(\frac{x}{2}\right)$

$f(1) = 2 \times 1 - 3\cos\left(\frac{1}{2}\right) = -0.633$

$f(2) = 2 \times 1 - 3\cos\left(\frac{2}{2}\right) = 2.379$

Since there is a sign change,

then the root lies between

1 and 2

x	1	x_0	2
f(x)	-0.633	0	2.379

$\frac{x_0-1}{0--0.633} = \frac{2-1}{2.379--0.633}$

$x_0 = 1.2102$

x	1.2102	x_0	2
f(x)	-0.047	0	2.379

$\frac{x_0-1}{0--0.047} = \frac{2-1}{2.379--0.047}$

$x_0 = 1.226$

Example 11

Show that the equation $3x^2 + x - 5 = 0$ has a real root between x = 1 and x = 2. Hence use linear interpolation twice to calculate the root to 2 dp.

Solution

$f(x) = 3x^2 + x - 5$

$f(1) = 3(1)^2 + 1 - 5 = 1$

$f(2) = 3(2)^2 + 2 - 5 = 9$

Since there is a sign change,

then the root lies between

x	1	x_0	2
f(x)	-1	0	9

$\frac{x_0-1}{0--1} = \frac{2-1}{9--1}$

$x_0 = 1.1$

x	1.1	x_0	2
f(x)	-0.27	0	9

$\frac{x_0-1.1}{0--0.27} = \frac{2-1.1}{9--0.27}$

$x_0 = 1.13$

(b) General iterative method

This involves generating equation by splitting the original equation into several equations by making x the subject.

Example 12

Given $x^2 + 4x – 2 = 0$. Find the possible equations for estimating the roots

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{2}{x_n} - 4$ | $x_{n+1} = \sqrt{(2 - 4x_n)}$ | $x_{n+1} = \frac{2-x^2}{4}$

Example 13

Given $f(x) = x^3 – 3x – 12 = 0$. Generate equations in form of $x_{n+1} = g(x_n)$ that can be used to solve the equation f(x) =0

Solution

Let x_{n+1} be a better approximation

x_n be the next approximation

$x_{n+1} = \frac{x_n^3-12}{3}$ | $x_{n+1} = \sqrt[2]{(3x_n + 12)} = \frac{12}{x_n^2-3}$ | $x_{n+1} = \sqrt{\left(3 + \frac{12}{x_n}\right)} = \frac{3x_n+12}{x_n^2}$

Testing for convergence

From the several iterative equations obtained, the equation whose $|f^1(x_n| < 1$ is the one which converges the correct root.

Example 14

Given the two iterative formulas

(i) $x_{n+1} = \frac{x_n^3-1}{5}$ (ii) $x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

Using $x_0 = 2$ deduce a more suitable formula for solving the equation. Hence find the root correct to 2dp

$x_{n+1} = \frac{x_n^3-1}{5}$

$f(x_n) = x_{n+1} = \frac{x_n^3-1}{5}$; $f^1(x_n) = \frac{3x_n^2}{5}$

$f^1(2) = \frac{3(2)^2}{5} = 2.4$

since $|f^1(2)|>1$ it will not converge

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$

f(x_n) = $\sqrt{\left(5 + \frac{1}{x_n}\right)}$; $f^1(x_n) = -\frac{1}{2}x_n{}^{-2}\left(5 + \frac{1}{x_n}\right)$

$f^1(2) = -\frac{1}{2}(2)^{-2}\left(5 + \frac{1}{2}\right)$ = -0.0533

since $|f^1(2)|$<1 it will converge so this equation gives the root

$x_{n+1} = \sqrt{\left(5 + \frac{1}{x_n}\right)}$, $|e|$= 0.005, x_0 = 2

$x_1 = \sqrt{\left(5 + \frac{1}{2}\right)}$ = 2.3452

$|x_1 - x_0|$ = 2.3452 – 2 = 0.3452 > 0.005

$x_2 = \sqrt{\left(5 + \frac{1}{2.3452}\right)}$ = 2.3295

$|x_2 - x_1|$ = 2.3452 – 2.3295 = 0.0157 > 0.005

$x_3 = \sqrt{\left(5 + \frac{1}{2.3295}\right)}$ = 2.3301

$|e|$=$|2.3301 - 2.3295|$= 0.0006 < 0.005

Hence root is 2.33

Example 15

Show that the iterative formula for solving the equation $x^3 = x + 1$ is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ starting with x_0 = 1 find the solution of the equations to 3sf.

Solution

$x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$ $|e|$= 0.005, x_0 = 1

$x_1 = \sqrt{\left(1 + \frac{1}{1}\right)}$ =1.41421

$|x_1 - x_0|$ = $|1.41421 - 1|$ = 0.41421 > 0.005

$x_2 = \sqrt{\left(1 + \frac{1}{1.41421}\right)}$ =1.30656;

$|x_2 - x_1|$ = $|1.30656 - 1.41421|$ = 0.10765 > 0.005

$x_3 = \sqrt{\left(1 + \frac{1}{1.30656}\right)}$ = 1.32869

$|x_3 - x_2|$ = $|1.32869 - 1.30656|$ =0.03691 > 0.005

$x_4 = \sqrt{\left(1 + \frac{1}{1.32869}\right)}$ =1.32389

$|e|=|1.32389 - 1.32869|$= 0.0048 < 0.005

Hence the root is 1.32

Example 16

Given two iterative formulae I and II (shown below) for calculating the positive root of the quadratic equation f(x) = 0

$x_{n+1} = \frac{1}{2}(x_n^2 - 1)$ and $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$ for n = 1, 2, 3 ……

Taking x_0 = 2.5, use each formula thrice to two decimal places to decide which is the more suitable formula. Give a reason for your answer.

Solution

Iterative formula $x_{n+1} = \frac{1}{2}(x_n^2 - 1)$

x_0 = 2.5

$x_1 = \frac{1}{2}(2.5^2 - 1)$ = 2.625

$|x_1 - x_0|$ = 0.125

$x_2 = \frac{1}{2}(2.625^2 - 1)$ 2.99453125

$|x_2 - x_1|$ = 0.3200125

$x_3 = \frac{1}{2}(2.99453125^2 - 1)$ = 3.837432861

$|x_3 - x_2|$ = 0.89212036

Iterative formula $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$

x_0 = 2.5

$x_1 = \frac{1}{2}\left(\frac{2.5^2+1}{2.5-1}\right)$ = 2.416666667

$|x_1 - x_0|$ = 0.083333

$x_2 = \frac{1}{2}\left(\frac{2.416666667^2+1}{2.416666667-1}\right)$ = 2.414215686

$|x_2 - x_1|$ = 0.002450781

$x_2 = \frac{1}{2}\left(\frac{2.414215686^2+1}{2.414215686-1}\right)$ = 2.414215686

$|x_3 - x_2|$ = 0.000002124

The more suitable formula is $x_{n+1} = \frac{1}{2}\left(\frac{x_n^2+1}{x_n-1}\right)$.

Because the absolute difference between $x_3 - x_2$ is less than absolute error, where as in the first formula the absolute difference between$x_3 - x_2$ is greater than absolute error. In all the 2nd formula converge whereas the first formula diverges.

Example 17

(a) (i) Show that the equation $e^x - 2x - 1 = 0$ has a root between x = 1 and x = 1.5.
(ii) Use linear interpolation to obtain an approximation for the root

(b) (i) Solve the equation in (a)(i), using each formula below twice
Take the approximation in (a)(i) as the initial value
Formula I: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$.
Formula II: $x_{n+1} = \frac{e^{x_n}(x_n-1) + 1}{e^{x_n}-2}$
(ii) Deduce with a reason which of the two formulae is appropriate for solving the given equation in (a)(i). Hence write down a better approximate root, correct to two decimal places

Solution

(a) (i) using sign change method
let f(x) = e^x – 2x – 1
f(1) = e^1 – 2(1) – 1 = -2.817
f(1.5) = $e^{1.5}$ – 2(1.5) – 1 = 0.4817

Since f(1).f(1.5)< 0, the root lies between x = 1 and x = 1.5

(a)(ii) Extract

1	x_0	1.5
-0.2817	0	0.4817

$\frac{x_0-1}{0--0.2817} = \frac{1.5-1}{0.4817--0.2817}$; x_0 = 1.1845

Hence the approximation to the root is 1.18 (2 dp)

(b)(i)

Solution

formula 1: $x_{n+1} = \frac{1}{2}(e^{x_n} + 1)$

x_0 = 1.18

$x_1 = \frac{1}{2}(e^{1.18} + 1)$= 2.1272

$|x_1 - x_0|$ = 0.9472

$x_2 = \frac{1}{2}(e^{2.127187} + 1)$= 4.6956

$|x_2 - x_1|$ = 2.5684

formula 2: $x_{n+1} = \frac{e^{x_n}(x_n-1)+1}{e^{x_n}-2}$

x_0 = 1.18

$x_1 = \frac{e^{1.18}(1.18-1)+1}{e^{1.18}-2}$= 1.2642

$|x_1 - x_0|$ = 0.0842

$x_2 = \frac{e^{1.2642}(1.2642-1)+1}{e^{1.2642}-2}$= = 1.2565

$|x_2 - x_1|$ = 0.0077

Formula 1, the sequence 1.18, 2.1272, 4.6956 diverge, hence the formula is not suitable

Formula 2, the sequence 1.18, 1.2642, 1.2565 converge, hence the formula is suitable solving the equation

A better approximation = 1.26 (2 dp)

Revision exercise 2

1. Given the following iterative formula
   (i) $x_{n+1} = 5 - \frac{3}{x_n}$ (ii) $x_{n+1} = \frac{1}{5}(x_n^2 + 3)$

   Taking x_0 = 5 deduce a more suitable iterative formula for solving the equation

2. Show that the iterative formula for solving the equation x^2 -5x + 2 = 0 can be written in two ways as $x_{n+1} = 5 - \frac{2}{x_n}$ or $x_{n+1} = \frac{x_n^2+2}{5}$.
   Starting with x_0 = 4, deduce the more suitable formula for the equation and hence find the root correct to 2 dp [4.56]

3. Show that the iterative formula for solving the equation $x^3 - x - 1 = 0$ is $x_{n+1} = \sqrt{\left(1 + \frac{1}{x_n}\right)}$. Starting with xo = 1 find the root of the equation correct to 3 s.f. [1.33]
4. (a) Show that the iterative formula for solving the equation $2x^2 - 6x - 3 = 0$ is $x_{n+1} = \frac{2x_n^2+3}{4x_n+6}$
   (b) Show that the positive root for $2x^2 - 6x - 3 = 0$ lies between 3 and 4. find the root correct to 2 decimal places [3.44]
5. (a) If b is the first approximation to the root of equation x2 = a, show that the second approximation to the root is given by $\frac{b+\frac{a}{b}}{2}$. Hence taking b = 4, estimate $\sqrt{17}$ correct to 3 dp [4.123]
   (b) Show that the positive real root of the equation $x^2 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 dp

(c) Newton Raphson's Method

It is given by $x_{n+1} = x_n - \left[\frac{f(x_n)}{f^1(x_n)}\right]$ n = 1, 2, 3 …

Example 18

Use Newton Raphson's method to find the root of equation $x^3 + x - 1 = 0$ using x0 = 0.5 as the initial approximation, correct your answer to 2 decimal places

Solution

$f(x) = x^3 + x - 1$, $f^1(x) = 2x^2 + 1$

$x_{n+1} = x_n - \left[\frac{(x^3+x-1)}{3x^2+1}\right]$

$x_1 = 0.5 - \left[\frac{((0.5)^3+0.5-1)}{3(0.5)^2+1}\right]$ =0.7142

$|x_1 - x_0| = 0.7142 - 0.5 = 0.2142 > 0.005$

$x_2 = 0.7142 - \left[\frac{((0.7142)^3+0.7142-1)}{3(0.7142)^2+1}\right]$ =0.6831

$|x_2 - x_1| = |0.6831 - 0.7142|$

$= 0.0311 > 0.005$

$x_2 = 0.6831 - \left[\frac{((0.6831)^3+0.6831-1)}{3(0.6831)^2+1}\right]$ =0.6824

$|3 - x_2| = |0.6824 - 0.6831|$

$= 0.0007 < 0.005$

∴ Root = 0.68

Example 19

Show that the equation 5x - 3cos2x = 0 has a root between 0 and 1. Hence use Newton Raphson's method to find the root of equation correct to 2 decimal places using $x_0 = 0.5$.

Solution

Using sign change method to locate The roots. Note for trigonometric functions the calculator is used in radians mode

f(x) =5x - 3cos2x

f(0) = 5(0) – 3cos2(0) = -3

f(1) = 5(1) – 3cos2(1) = 2.455

Since there is change sign the root lies between x = 0 and x = 1

f(x) = 5x - 3cos2x, $f^1(x) = 5 + 6\sin 2x$

$x_{n+1} = x_n - \left[\frac{(5x_n - 3\cos 2x_n)}{5 + 6\sin 2x_n}\right]$

$x_0 = 0.5, |e| = 0.005$

$x_1 = 0.5 - \left[\frac{(5(0.5) - 3\cos 2(0,5))}{5 + 6\sin 2(0.5)}\right] = 0.4125$

$|x_1 - x_0| = |0.4125 - 0.5| = 0.0875 > 0.005$

$x_1 = 0.4125 - \left[\frac{(5(0.4125) - 3\cos 2(0.4125))}{5 + 6\sin 2(0.4125)}\right] = 0.4096$

$|x_2 - x_1| = |0.4096 - 0.4125| = 0.0029 < 0.005$

∴ Root = 0.41

Example 20

Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$. Hence taking $x_0 = 2.5$ determine $\sqrt[3]{10}$ correct to 3 dp.

Solution

$x = N^{\frac{1}{3}}$

$x^3 - N = 0$

$f(x) = x^3 - N; f^1(x) = 3x^2$

$x_{n+1} = x_n - \left[\frac{(x_n{}^3 - N)}{3x_n{}^2}\right] = \frac{x_n(3x_n{}^2) - (x_n{}^3 - N)}{3x_n{}^2}$

$= \frac{2x_n{}^3 + N}{3x_n{}^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right).$

$x_0 = 2.5, N = 10, |e| = 0.005$

$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$

$x_1 = \frac{1}{3}\left(2(2.5) + \frac{N}{2.5^2}\right) = 2.2$

$|x_1 - x_0| = |2.2 - 2.5| = 0.3 > 0.005$

$x_2 = \frac{1}{3}\left(2(2.2) + \frac{N}{2.2^2}\right) = 2.1554$

$|x_2 - x_1| = |2.1554 - 2.2| = 0.0446 > 0.005$

$x_3 = \frac{1}{3}\left(2(2.1554) + \frac{N}{2.1554^2}\right) = 2.1544$

$|x_3 - x_2| = |2.1544 - 2.1554| = 0.001 < 0.005$

∴ Root = 2.154

Example 21

(a) Show that the equation x -3sinx = 0 has a root between 2 and 3. (03marks)

f(x) = x – 3sinx
f(2) = 2 – 3sin 2 = -0.7279
f(3) = 3 – 3sin3 = 2.5766
since f(2).f(3) = -1.8755<0
there exist a root of x-3sinx =0 between 2 and 3

(b) Show that Newton- Raphson iterative formula for estimating the root of the equation in (a) is given by

$$X_{n+1} = \frac{3(\sin x_n - x_n \cos x_n)}{1 - 3\cos x_n}, n = 0, 1, 2 \ldots.$$

Hence find the root of the equation corrected to 2 decimal places (09 marks)

f'(x) = 1 – 3cos x

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$= x_n - \frac{x_n - 3\sin x_n}{1 - 3\cos x_n}$

$= \frac{x_n - 3x_n\cos x_n - x_n + 3\sin x_n}{1 - 3\cos x_n}$

$x_{n+1} = \frac{3(\sin x_n - x_n\cos x_n}{1 - 3\cos x_n}$

Taking $x_0 = \frac{2+3}{2} = 2.5$

$x_1 = \frac{3(\sin 2.5 - 2.5\cos 2.5}{1 - 3\cos 2.5} = 2.293$

Error = $|2.293 - 2.5| = 0.207 > 0.005$

$x_2 = \frac{3(\sin 2.293 - 2.5\cos 2.293}{1 - 3\cos 2.293} = 2.279$

Error = $|2.279 - 2.293| = 0.014 > 0.005$

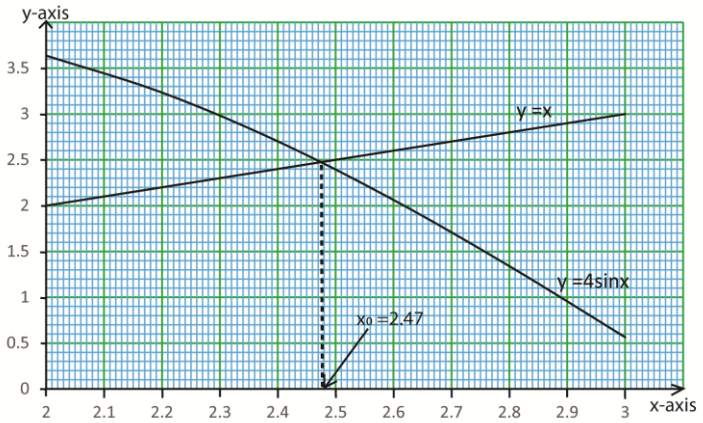
$x_3 = \frac{3(\sin 2.279 - 2.5\cos 2.279}{1 - 3\cos 2.279} = 2.279$

Error = $|2.279 - 2.279| = 0.000 < 0.005$

∴ root = 2.279 = 2.28(2D)

Example 22

(a) On the same axis, draw graphs of y = x and y = 4sinx to show that the root of the equation x-4sinx = 0 lies between x= 2 and x = 3



Therefore the root (2.47) lies between x = 2 and x = 3

(b) Use Newton Raphson's method to calculate the root of the equation x – 4sinx = 0, taking approximate root in (a) as the initial approximation to the root. correct your answer to 3 decimal places.

f(x) = x – 4sinx

f'(x) = 1 – 4cos x

$x_{n+1} = x_n - \frac{x_n - 4\sin x_n}{1 - 4\cos x_n}$

$Taking\ x_0 = 2.47$

$x_1 = 2.47 - \frac{2.47 - 4\sin 2.45}{1 - 4\cos 2.47} = 2.4746$

$Error = |2.4746 - 2.47| = 0.0046 > 0.0005$

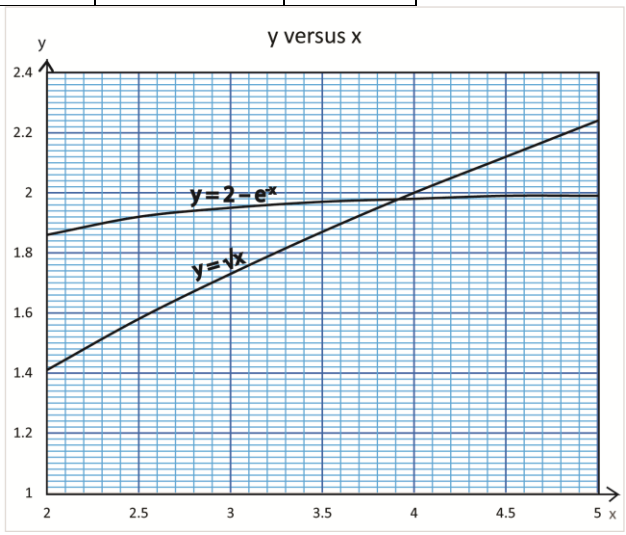
$x_2 = 2.4746 - \frac{2.4746 - 4\sin 2.4546}{1 - 4\cos 2.4746} = 2.4746$

$Error = |2.4746 - 2.4746| = 0.000 < 0.0005$ ∴ the root = 2.475 (3D)

Example 23

(a) Draw on the same axes the graphs of the curves y = 2 – e^{-x} and y = $\sqrt{x}$values 2≤ x ≤ 5. (04marks)

x	y = 2 – e^{-x}	y = $\sqrt{x}$
2.0	1.86	1.41
2.5	1.92	1.58
3.0	1.95	1.73
3.5	1.97	1.87
4.0	1.98	2.00
4.5	1.99	2.12
5.0	1.99	2.24



(b) Determine from your graph the interval within which the roots of the equation

$e^{-x} + \sqrt{x-2} = 0$ lies

Hence, use Newton-Raphson's method to find the root of the equation correct to 3 decimal places (07marks)

Root lies between 3.9 and 4

f(x) = 2 – e^{-x} - $\sqrt{x}$

f'(x) = $e^{-x} - \frac{1}{2\sqrt{x}}$

f(x_n) = $e^{-x_n} - \frac{1}{2\sqrt{x_n}}$

$x_{n+1} = x_n - \frac{2 - e^{x_n} - \sqrt{x_n}}{2e^{-x_n}\sqrt{x_n} - 1}$

$x_0 = \frac{3.9+4}{2}$ = 3.95

$x_1 = 3.95 - \frac{2\sqrt{3.95}(2 - e^{-3.95} - \sqrt{3.95})}{2e^{-3.95}\sqrt{3.95} - 1}$ = 3.9211

$Error = |3.9211 - 3.95| = 0.0289$

$x_2 = 3.9211 - \frac{2\sqrt{3.9211}(2 - e^{-3.9211} - \sqrt{3.9211})}{2e^{-3.9211}\sqrt{3.9211} - 1}$ = 3.9211

∴ Root = 3.921 (3dp)

Example 24

Given the equation $X^3 - 6x^2 + 9x + 2 = 0$

(a) Show that the equation has a root between -1 and 0.

Let $f(x) = X^3 - 6x^2 + 9x + 2$

$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2$

$= -1 - 6 - 9 + 2 = -14$

$f(0) = 0 + 0 + 0 + 2$

$= 2$

$f(-1).f(0) = -14 \times 2 = -28$

since $f(-1).f(0) < 0$; the root exist between -1 and 0.

(b) (i) Show that the Newton Raphson formula approximating the root of the equation is given by $X_{n+1} = \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$

$f(x) = X^3 - 6x^2 + 9x + 2$

$f(x_n) = x_n^3 - 6x_n^2 + 9x_n + 2$

$f'(x_n) = 3x_n^2 - 12x_n + 9$

$$x_{n+1} = x_n - \left(\frac{x_n^3 - 6x_n^2 + 9x_n + 2}{3x_n^2 - 12x_n + 9}\right)$$

$$= \frac{x_n(3x_n^2 - 12x_n + 9) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$$

$$= \frac{(3x_n^3 - 12x_n^2 + 9x_n) - (x_n^3 - 6x_n^2 + 9x_n + 2)}{3x_n^2 - 12x_n + 9}$$

$$= \frac{2x_n^3 - 6x_n^2 - 2}{3x_n^2 - 12x_n + 9}$$

$$= \frac{2}{3}\left[\frac{x_n^3 - 3x_n^2 - 1}{x_n^2 - 4x_n + 3}\right]$$

(ii) Use the formula in (b)(i) above, with initial approximation of $x_0 = -0.5$, to find the root of the given equation correct to two decimal places

Taking x = -0.5

$x_1 = \frac{2}{3}\left[\frac{(-0.5)^3 - 3(-0.5)^2 - 1}{(-0.5)^2 - 4(-0.5) + 3}\right]$ = -0.2381

$|e| = |0.2381 - (-0.5)|$ = 0.2619

$x_2 = \frac{2}{3}\left[\frac{(-0.2381)^3 - 3(-0.2381)^2 - 1}{(-0.2381)^2 - 4(-0.2381) + 3}\right]$ = -0.1968

$|e| = |0.1968 - (-0.2381)|$ = 0.0.0413

$x_3 = \frac{2}{3}\left[\frac{-0.1968^3 - 3(-0.1968)^2 - 1}{(-0.1968)^2 - 4(-0.1968) + 3}\right]$ = -0.1958

$|e| = |-0.1958 - (-0.1968)|$ = 0.001< 0.005

Hence the root = -0.20 (2D)

Revision Exercise 3

1. Using the Newton Raphson's formula, show that the reciprocal of a number N is $x_n(2 - Nx_n)$
2. Use Newton Raphson's iterative formula to show that the cube root of a number N is given by $\frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right)$. Hence use the iterative formula to find $\sqrt[3]{96}$ correct to3 decimal places. use $x_0 = 5$. [4.579]
3. (a) Show that the equation 3x3 + x -5 = 0 has real root between x = 1 and x = 2.
   (b) Using linear interpolation, find the first approximation for this root to 2dp. [1.04]

(c) Using Newton Raphson's method twice find the value of this root correct to 2 dp. [1.09]

4. (a) Show graphically that there is a positive real root of equation $xe^{-x} - 2x + 5 = 0$ between x = 2 and x = 3
   (b) Using Newton Raphson's method, find this root correct to 1 dp. [2.6]
5. Using the iterative formula for NRM, show that the fourth root of a number N is $\frac{3}{4}\left(x_n + \frac{N}{3{x_n}^3}\right)$. Starting with $x_0 = 2.5$ show that $(45.7)^{\frac{1}{4}} = 2.600$ (3dp)
6. On the same axes, draw graphs of $y = x^3$ and $y = 2x + 5$. Using NRM twice find the positive root of the equation $x^3 - 2x - 5 = 0$ correct to 2 decimal places. [2.09]
7. (a) Show that the Newton Raphson's formula for finding the smallest positive root of the equation $3\tan x + x = 0$ is $\frac{6x_n - 3\sin 2x_n}{6 + 2\cos 2x_n}$
   (b) By sketching the graphs of $y = \tan s$, $y = \frac{-x}{3}$ 0r otherwise, find the first approximation to the required root and use it to find the actual root correct to 3 dp . [2.456]
8. (a) Show that the root of the equation $f(x) = e^x + x^3 - 4x = 0$ has a root between x =1 and x = 2
   (b) Use the Newton Raphson's method to find the root of equation in (a) correct to 2 decimal places. [$x_0 = 1$, root =1.12]
9. (a) Show that the iterative formula for approximation of the root of f(x) = 0 by NRM process for the equation $xe^x + 5x - 10 = 0$ is $x_{n+1} = \frac{{x_n}^2 e^{x_n} + 10}{x_n e^{x_n} + e^{x_n} + 5}$.
   (b) Show that the root of the equation in (i) above lies between x = 1 and x = 2. Hence find the root of the equation correct to 2 dp. [1.20]
10. (a) Use a graphical method to find a first approximation to the real root of $x^3 + 2x - 2 = 0$.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 dp. [0.77]
11. (a) Show that equation $x = \ln(8 - x)$ has a root between x= 1 and x = 2.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 decimal places [1.82]
12. (a) Use graphical method to find the first approximation to the root of $x^3 - 3x + 4 = 0$. [-2]
    (b) Use NRM to find the root of the equation in (a) correct to 2 d.p. [-2.20]
13. Show graphically that equation $e^x + x - 4 = 0$ has only one root between x = 1 and x = 2. Use NRM to find the approximation of the equation correct to 3dp. [1.07]
14. Show that the NRM for approximating the K^{th} root of a number N is given by $\frac{1}{K}\left((K-1)x_n + \frac{N}{{x_n}^{K-1}}\right)$. Hence use your formula to find the positive square root of 67 correct to 4 s.f. [8.185].
15. (a) Show that equation $x^3 + 3x - 9 = 0$ has a root between x= 1 and x = 2.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to 2 0ne places [1.6]
16. (a) Show graphically that there is a positive real root of equation $xe^{-x} - 2x - 1 = 0$ between x = 1 and x = 2
    (b) Using Newton Raphson's method, find this root for the equation in (a) correct to 2 dp. [1.26]
17. (a) Show that equation $2x - 3\cos\left(\frac{x}{2}\right) = 0$ has a root between x= 1 and x = 2.
    (b) Use the Newton Raphson's method to find the root of the equation in (a) correct to one places [1.23]

18. (a) If a is the first approximation to the root of the equation $x^5 - b = 0$, show that the second approximation is given by $\frac{4a+\frac{b}{a^4}}{5}$.
    (b) Show that the positive real root of the equation $x^5 - 17 = 0$ lies between 1.5 and 1.8. Hence use the formula in (a) above to determine the root to 3 decimal places. [1.762]
19. (a)(i) On the same axes, draw graphs of $y = x^2$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$ at intervals of $\frac{\pi}{8}$.
    (ii) Use your graphs, to find to 1 decimal place an approximate root of the equation $x^2 - \cos x = 0$ [0.8]
    (b) Use the NRM to calculate the root of the equation $x^2 - \cos x = 0$ taking the approximate root in (a) as the initial approximation. Correct your answer to 3 dp. [0.824]
20. (a) (i) Draw on same axes the graphs of equation $y = x\sin x$ and $y = e^x - 2$ for $0 \leq x \leq 1.5$.
    (ii) Use your graphs to find an approximate root of the equation $2 - e^x + x\sin x =$ [1.1]
(c) Use the Newton Raphson's method to find the root of the equation in (a)(ii) correct to three decimal places [1.085]
21. Show graphically that equation $e^x + x - 8 = 0$ has only one real root between x = 1 and x = 2. Use NRM to find approximation of x = In(x 8) correct to 3 dp [1.821]
22. Draw using the same axes, graphs of $y = x^2$ and $y = \sin 2x$ for $0 \leq x \leq \frac{\pi}{2}$. From the graphs obtain to one decimal place an approximation of the non-zero root of the equation $x^2 - \sin 2x = 0$. Using NRM, calculate to 2 dp a more suitable approximation. [0.97]
23. Given the equation $\ln(1 + 2x) - x = 0$.
    (i) show the root of the equation above lies between x = 1 and x = 1.5
    (ii) Use NRM twice to estimate the root of the equation, correct to 2 dp. [1.26]

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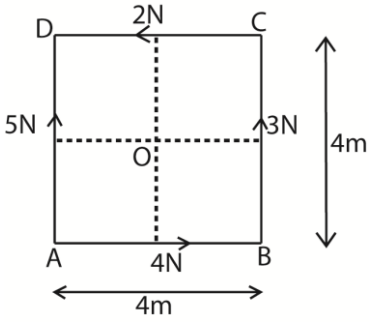
Moment of a force acting on a polygon

Example 1

ABCD is a square of side 4m. Forces of magnitude 4N, 3N, 2N and 5N act along AB, BC, CD and AD respectively in each case the direction of force being given by the order of the letters. Given that AB is horizontal, find the moment of force about

(i) center of the square
(ii) point A

Solution



$\overset{O}{\curvearrowleft}$ G = 4 x 2 + 3 x 2+ 2 x 2 – (5 x 2)

=8Nm anticlockwise

$\overset{A}{\curvearrowleft}$ G = (4 x 0) + (3 x 4) + (2 x 2) – (5 x 0)

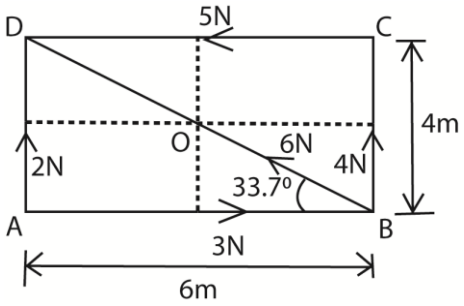
= 20Nm anticlockwise

Example 2

ABCD is a rectangle where AB = 6m and BC = 4m. Forces of magnitude 3N, 4N, 5N, 2N and 6N act along line AB, BC, CD, AD, and BD respectively. In each case the direction of force being given by the order of the letters. Find the moment of forces about

(i) center of the rectangle (ii) point A

Solution



$\overset{O}{\curvearrowleft}$ G = 3x3 + 4 x 3+ 5 x 2 – (3 x 2) + (6sin33.7^0 x 3) -

(6sin33.7^0 x 3) = 22Nm anticlockwise

$\overset{A}{\curvearrowleft}$ G = (3 x 0) + (4 x 6) + (5 x 4) – (2 x 0) +

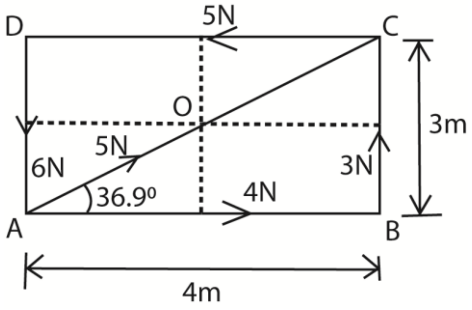
(6sin33.7^0 x 0) - (6cos33.7^0 x 0)

= 63.97Nm anticlockwise

Example 3

ABCD is a rectangle where AB = 4m and BC = 3m. Forces of magnitude 4N, 3N, 5N, 6N and 5N act along line AB, BC, CD, DA, and Ac respectively. In each case the direction of force being given by the order of the letters. Find the moment of forces about

(ii) center of the rectangle



(ii) point A

$\circlearrowleft_O$ G = 4 x 1.5 + 3 x 2 + 5 x 1.5 + (6 x 2) -

(5sin36.9^0 x 2) + (5sin36.9^0 x 1.5)

= 31.49Nm anticlockwise

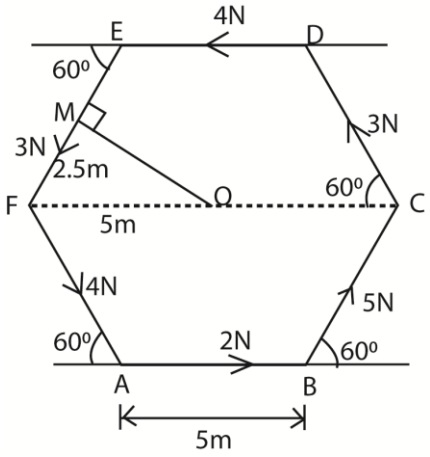
$\circlearrowleft_A$ G = (3 x 4) + (5 x 3)

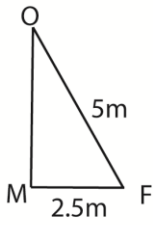
=27Nm anticlockwise

Example 4

ABCDEF is a regular hexagon of side 5m. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 4N act along the lines AB, BC, CD, DE, EF and FA respectively. In each case the direction of force being given by the order of the letters. Given that AB is horizontal, find the sum of moments of forces about

(i) center O of the hexagon

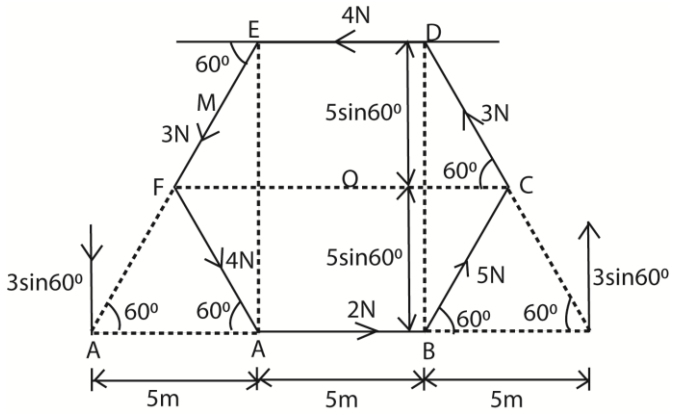




OM = $\sqrt{5^2 - (2.5)^2}$ = 4.33m

$\circlearrowleft_O$ G = (2 + 5 + 3 + 4 + 3 + 4) x OM = 21 x 4.33 = 90.93Nm anticlockwise

(ii) Point A



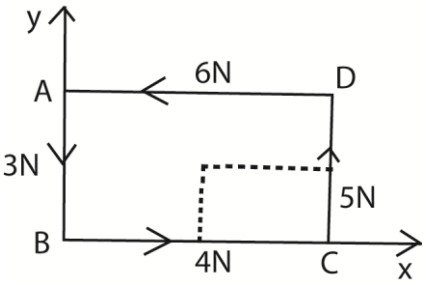
$\overset{\curvearrowleft}{A}$ G = (5sin600 x 5) + (3sin600 x 10) + (4 x 10sin600) + (3sin600 x 5)

= 95.26 Nm anticlockwise

Example 5

Forces of magnitude 3N, 4N, 5N and 6N act on a rectangle along lines AB, BC, CD and DA of a rectangle. Their direction is the order of the letters. BC is horizontal. Find the resultant force and the couple at the centre of a rectangle of sides 2m and 4m.

Solution



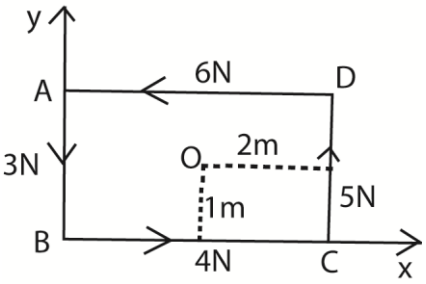
→, X = 4 - 6 = -2

↑, Y = 5 – 3 = 2

The resultant force, R = $\sqrt{(-2)^2 + 2^2}$

=2.83N

For a couple



$\overset{\curvearrowleft}{O}$ G = 4 x 1+ 5 x 2 +6 x 1 + 3 x 2

= 26Nm anticlockwise

For the system of forces to reduce to couple, an additional force equal in magnitude to the resultant but acting in opposite direction must be added. Hence the moment of the couple about the center is 26Nm in the sense BADC or clockwise.

Revision exercise

1. Forces of 2N, 3N, 4N and 5N act along the sides of a square ABCD of side 4m in direction AB, BC, CD and AD respectively. Find the sum of moments of the forces about
   (i) the center of square [8Nm]
   (ii) Point A [28Nm]
2. Forces of 5N, 6N, 4N, 7N, 6N and 8N act in direction AB, BC, CD, DA, AC, and DB respectively of a square ABCD of side 6m. Find the sum of moments of forces about
   (i) centre of the square [66Nm]
   (ii) point A [26Nm]
3. ABCD is a rectangle with AB = 8cm and BC = 6 cm. Forces of 4N, 5N, 3N, 6N and 8N act in the direction AB, BC, CD, AD and BD respectively of the rectangle ABCD, find the sum of moments of the forces about
   (i) the center of the rectangle [17Nm]
   (ii) point A [96.4Nm]
4. ABCDEF is a regular hexagon 2m. Forces of magnitude 5N, 2N, 6N, 4N, 8N and 3N acting along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal. Find the sum of moments of the force about point A. [50.23Nm]
5. ABCDEF is a regular hexagon with side 3m. Forces of magnitude 4N, 5N, 1N, 3N, 7N and 2N act along line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Find the sum of moments of the forces about point A [$30\sqrt{3}$Nm]
6. ABDCEF is a regular hexagon of 4m. Forces of magnitude 8N, 4N, 7N, 4N, 6N and 5N act along AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, find the sum of moments of the forces about point A [$64\sqrt{3}$Nm]
7. ABDCEF is a regular hexagon of 4m. Forces of magnitude 5N, 6N, 7N, 4N, 5N and 8N act along AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, find the sum of moments of the forces about
   (i) centre of hexagon [$70\sqrt{3}$Nm]
   (ii) point A [$66\sqrt{3}$Nm]
8. ABDCEF is a regular hexagon of 3m. Forces of magnitude 3N, 1N, 2N, 5N, 6N and 4N act along AB, BC, CD, ED, EF and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, find the sum of moments of the forces about
   (iii) centre of hexagon [$4.5\sqrt{3}$Nm]
   (iv) point A

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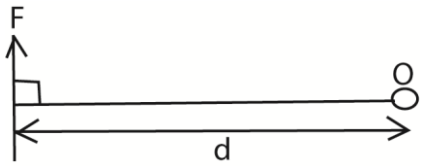
Moment of a force

This is the product of a force and perpendicular distance from the pivot to the line of action of the force. The unit of moments is Nm.



The moment of force about point O is F x d

Matrix approach of finding sum of moments about the origin

If forces $(a_1\text{i} + b_1\text{j})$N, $(a_2\text{i} + b_2\text{j})$N, ……$(a_n\text{i} + b_n\text{j})$ act on the body at point $(x_1 + y_1)$, $(x_2 + y_2)$, ….. $((x_n + y_n)$. The sum of the moments about the origin is

$$G = \begin{vmatrix} x_1 & a_1 \\ y_1 & b_1 \end{vmatrix} + \begin{vmatrix} x_2 & a_2 \\ y_2 & b_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & a_n \\ y_n & b_n \end{vmatrix}$$

$$G = (b_1x_1 - a_1y_1) + (b_2x_2 - a_2y_2) + \cdots + (b_nx_n - a_ny_n)$$

Note

If G is positive, the sum of moments will be anticlockwise and if G is negative the sum of moments will be clockwise.

Example 1

Find the moment about the origin of a force of 4jN acting at a point which has position vector -5iN

Solution

$G = \begin{vmatrix} -5 & 0 \\ 0 & 4 \end{vmatrix}$= -5x4 – 0x0 = -20Nm clockwise

Example 2

Find the moment about the origin of a force of 4jN acting at a point which has position vector 5iN

$G = \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix}$= 5x4 – 0x0 = 20Nm anticlockwise

Example 3

Forces of (2i-3j)N, (4i + j)N and (5i -3j)N act on a body at points with Cartesian co-ordinates (1,1), (2, 4), and (-1, 3) respectively. Find the sum of moments of the forces about the origin.

Solution

$G = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 5 \\ 3 & -3 \end{vmatrix}$ = (1 x -3 – 2 x 1)+(2 x 1 – 4x 4)+(-1 x -3 – 3 x 5) = -31Nm

= 31Nm clockwise

Example 4

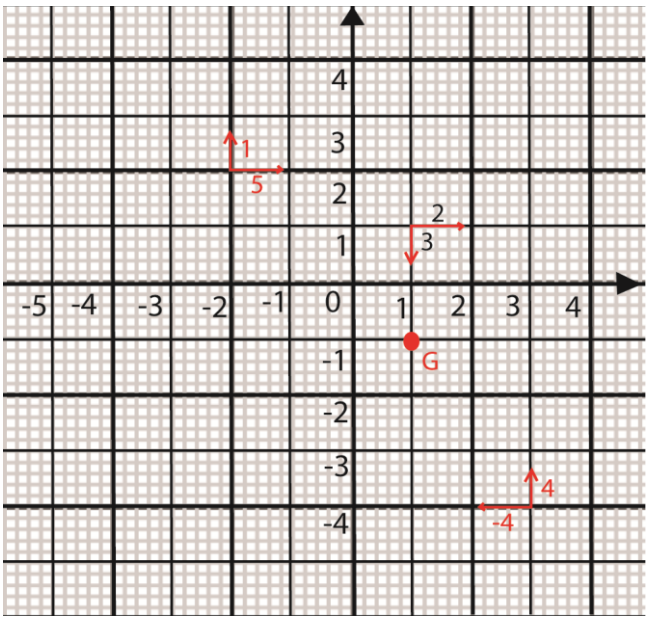
Forces (2i -3j)N, (5i + j)N and (-4i+4j) act on a body at points with position vector (i + j), (-2i +2j) and (3i -4j) respectively. Find the sum of moments of forces about the

(i) origin

$G = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} -2 & 5 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ -4 & 4 \end{vmatrix}$ = (1 x -3 -2 x 1) + (-2 x 1 – 2 x 5) + (3 x 4 - –4x -4)

= -21Nm = 21Nm clockwise

(ii) point with position vector (i –j)



G = (5 x 3) + (1 x 3) +(2 x 0) + (2 x 2) + (4 x 3) – (4 x 2) = 26Nm clockwise

Revision exercise

1. Find the moment about the origin of a force of 3i acting at a point which has position vector (2i + 3j)m. [9Nm clockwise]
2. Find the moment about the origin of force (4i + 2j)N acting at a point which has position vector (3i + 2j)m. [2Nm clockwise]
3. A force of (3i -2j)N act at a point which has position vector (5i + j)m. Find the moment about the point which has a position vector (i +2j)m. [5Nm clockwise]
4. A force of (2i + j)N act at a point which has position vector (2i + 2j)m and a force of 5iN at a point which has position vector (-2i + j)m. Find the sum of moments of these forces about the origin. [7Nm clockwise]
5. A force of (3i + 2j)N act at a point which has position vector (5i + j)m and a force of (I + j)N act at a point which has position vector (2i + j)m. Find the sum of moments of these forces about the point which has position vector (i + 3j)m. [17Nm anticlockwise]

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Parallel forces in equilibrium

Conditions for a body to be in equilibrium

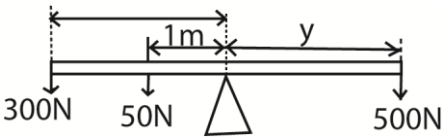
When a system of parallel forces act on a body then it will be in equilibrium when;

(i) the sum of forces acting in one direction are equal to the sum of forces acting in opposite direction.

(ii) sum of clockwise moments about a point are equal to the sum of anticlockwise moment about the same point

Example 1

Given the diagram below. Find the value of y

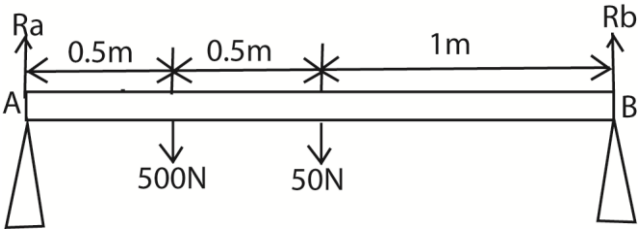


500 x y = (300 x 3) + (50 x 1)

y = $\frac{950}{500} = 1.9m$

Example 2

A uniform beam of weight 50N and length 2m rests horizontally on two supports pivoted at each end. A load of weight 500N is placed 0.5m from one end. Find the reaction on each support.



B: Ra x 2 =50 x 2 + 500 x 1.5

2Ra = 50 + 750

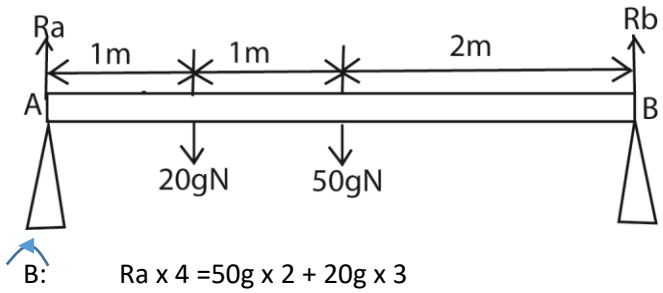
Ra = 400N

Also Ra + Rb = 500N + 50 N

Rb = 550N – 400N = 150N

Example 3

A uniform beam of mass 50kg and length 4m rests horizontally on two supports pivoted at each end. A load of 20kg is placed 1m from one end. Find the reaction on each support



B: Ra x 4 =50g x 2 + 20g x 3

4Ra = 100g + 60g = 160 x 9.8

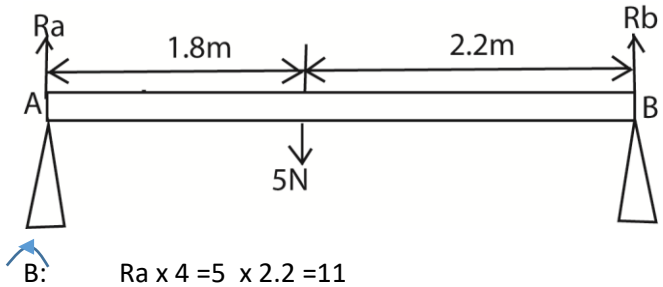
Ra = 392N

Also Ra + Rb = 500N + 50 N

Rb = 20gN + 50gN – 392N = 294N

Example 4

A non- uniform beam AB of length 4m has its weight 5N acting at a point 1.8m from end A. The beam rests horizontally on two supports pivoted at each end. Find the reaction on each support.



B: Ra x 4 =5 x 2.2 =11

Ra = 2.75N

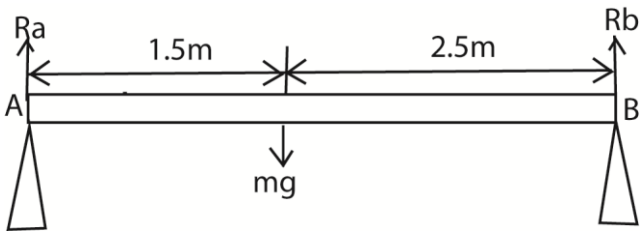
Also Ra + Rb = 5N

Rb = 5N – 2.75N = 2.25N

Example 5

A non- uniform beam AB of length 4m rests in horizontal position on vertical support at A and B. The centre of gravity is at 1.5m from end A. The reaction at B is 37.5N fine the

(a) mass of the beam (b) reaction at B



A: 37.5 x 4 = mg x 2.5

m = 10.2kg

Also Ra + 37.5 = 10.2 x 9.8

Ra = 62.5N

Revision exercise

1. A uniform beam AB of length 10m rests horizontally on two supports A and B. If the beam has a mass of 20g, find the reaction on each support. [98N, 9N]
2. A uniform beam of length 14m and mass 20kg rests horizontally on two supports, one at A and another at C which is 4m from B. find the reactions at each support [58.8N, 137.2N]
3. A uniform beam AB of length 10m and mass 20kg rests horizontally on two supports, one at A and another at C which is 2m from B. If a weight of mass 20kg is attached to the beam at a point 6m from A. Find the reaction on the supports. [392N, 196.2N]
4. A uniform beam AB of length 4m and mass 10kg rests horizontally on two supports at A and the other at C which is 1m from B. Where must a body of mass 50kg stand on the beam so that the reaction on each support is equal? [1.4m]
5. A uniform beam AB of length 12m and mass 12kg rests on two supports A and B. At what distance must a particle of mass 4g be tied so that the reaction of each support is equal. [9m from A]
6. A playground sea saw consists of a uniform beam of length 4m supported at its mid-point. If a girl of mass 25kg sits at one end of the sea saw, find where her brother of mass 40kg must sit if the sea saw is to balance horizontally. [75cm from other end]
7. A broom consists of a uniform broom stick of length 120cm and mass 4kg and a broom head of mass 6kg attached at the other end. Find where a support should be placed so that the broom balances horizontally. [24cm from the head]
8. A non-uniform beam AB of length 4m rests horizontally on two supports, one at A and the other at B. The reaction at the supports are 5gN and 3gN respectively. If instead the rod the rod were to rest horizontally on one support, find how far from A this support would have to be placed. [1.5m from A]
9. A uniform beam AB of mass 80g and of length 100cm is pivoted at 30cm from A, a force of 10N is placed on the beam at the 80cm from end A and a string is tied at the 40cm from end B so that the beam rests horizontally. Find the tension in the string. [17.2N]
10. A uniform beam AB of length 100cm is pivoted at 60cm from end B. The beam rests horizontally when a mass at A is 35g. Calculate the mass 9m) of the beam. [0.14kg]
11. A uniform meter rule pivoted at 10cm mark balances when a mass of 400N is suspended at the 0cm mark. If the system is in equilibrium. Find the mass of the ruler [10kg]
12. Two boys are carrying a uniform ladder of weight 800N, if the boys hold the ladder at 2m and 3m respectively from the centre of gravity, calculate the weight that each boy support. [480N, 320N]

Thank You

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Probability theory

Probability is the measure of chance of an event occurring or not occurring

Sample space

A sample space(s) is the set of all possible outcomes of an experiment

Example 1

When a dice is rolled once s = (1, 2, 3, 4, 5, 6)

When a coin is tossed once s = (H, T)

Events

An event (E) is a subset of a sample space.

Example2

When a coin is tossed twice, s = (HH, TT, HT, TH)

When interested in getting one head, E = (TH, HT)

Probability of an event

Given an event, E over a sample space, S

$P(E) = \frac{n(E)}{n(S)}$

Example 3

Find the probability of getting two heed when an ordinary coin is tossed thrice

S = (HHH, HHT, HTH, HTT, THH, THT, TTH, TTT)

E= (HTH, HHT, THH)

$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$

Example 4

Find the probability of getting a number greater than 3 when ordinary die is tossed once

S = (1, 2, 3, 4, 5, 6)

E = (4, 5, 6)

P(E) = $\frac{1}{3}$

Intersection of events

For any two events A and B, the probability that A and B occur together is (A∩B)

Union of event

For any two events A and B, the probability that event A or B or both occur is P(A∪ B)

P(A∪ B) = $P(A) + P(B) - P(A \cap B)$

P(A or B) = P(A) + P(B) – P(A and B)

Example 5

The probability that a student passes economics is $\frac{2}{3}$, the probability that he passes mathematics is $\frac{4}{9}$. If the probability that he passes at least one of them is $\frac{4}{5}$. Find the probability that he passes both subjects.

P(E∪ M) = P(E) + P(M) – P(E∩ M)

P(E∩ M) = $\frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$

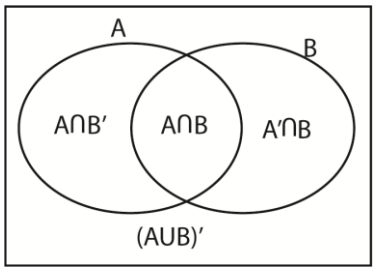
Complement of events

A' denotes event A does not occur

For events A and B

(i) P(A) + P(A') = 1
(ii) P(B) + P(B') = 1
(iii) P(A∪ B) + $P(A \cup B)' = 1$
(iv) P(A∩ B) + $P(A \cap B)' = 1$

Venn diagram



(i) P(A) = P(A∩ B') + $P(A \cap B)$
(ii) P(B) = P(B∩ A') + $P(A \cap B)$

Contingency table

	B	B'	
A	P(A∩ B)	P(A∩ B')	P(A)
A'	P(A' ∩ B)	P(A'∩ B')	P(A')
	P(B)	P(B')	1

(i) P(A) = P(A∩ B) + P(A∩ B')
(ii) $P(A') = P(A' \cap B) + P(A' \cap B')$
(iii) P(B) = P(A∩ B) + P(A' ∩ B)
(iv) P(B') = P(A∩ B') + P(A' ∩ B')

Demorgan's rule

(i) P(A′ ∩ B') = P(A∪ $B)' = 1 -$ P(A ∪ B)
P(neither A nor B = 1 – P(A or B)

(ii) P(A′∪ B') $= P(A \cap B)' = 1 - P(A \cap B)$

Types of events

- Undefined events
- Mutual events
- Independent events
- Exhaustive events

Unidentified events

For unidentified events, there is no restriction on $P(A \cap B)$

Example 6

Event A and B are such that P(A) = $\frac{19}{30}$, P(B) = $\frac{2}{5}$ and P(A∪ $B) = \frac{4}{5}$ find

(i) P(A∩ B)
Using, P(A∪ $B) = P(A) + P(B) - P(A \cap B)$
P(A∩ $B) = \frac{19}{30} + \frac{2}{5} - \frac{4}{5} = \frac{19+12-24}{30} = \frac{7}{30}$

(ii) P(A′ ∩ B')
Using, P(A′ ∩ B') = P(A∪ $B)' = 1 -$ P(A ∪ B)
P(A′ ∩ B')= 1 - $\frac{4}{5} = \frac{1}{5}$

(iii) P(B∩ A')
Using, P(B) = P(B∩ A') $+ P(A \cap B)$
P(B∩ $A') = \frac{2}{5} - \frac{7}{30} = \frac{5}{30} = \frac{1}{6}$

(iv) P(A′∪ B)
P(A′∪ $B) = P(A')$ + P(B) – P(A′∩ B)
= $\left(1 - \frac{19}{30}\right) + \frac{2}{5} - \frac{1}{6} = \frac{3}{5}$

(v) P(A∩ B')
P(A∩ B') = P(A) – P(A∩ B)
= $\frac{19}{30} - \frac{7}{30} = \frac{12}{30} = \frac{2}{5}$

Example 7

Event X and Y are such that P(C) = 0.3 P(Y) = 0.4 and P(X∩ $Y) = 0.1$, find

(i) P(Y′)
P(Y′) = 1- P(Y) = 1 – 0.4 = 0.6

(ii) P(X∩ Y')
P(X∩ Y') = P(X) – P(X∩ Y) = 0.3 – 0.1 = 0.2

(iii) P(X′∩ Y')
P(X′∩ Y') = P(X′) – P(X′∩ Y) = (1-0.3)- 0.3 = 0.4

(iv) P(X′∪ Y')
P(X′∪ Y')= P(X∩ $Y)' = 1 - 0.1 = 0.9$

Example 8

Events A and B are such that P(A) = 0.4, P(A∩ B) $= 0.1$ and $P(A \cup B) = 0.9$, find

(i) P(B)

P(A∪ B) = P(A) + P(B) – P(A∩ B)

0.9 = 0.4 + (P(B) – 0.1

P(B) = 0.6

(ii) P(A'∩ B)

P(A'∩ B) $= P(B) - P(A \cap B)$

= 0.6 – 0.1

= 0.5

(iii) P(A∩ B')

P(A∩ B') $= P(A) - P(A \cap B)$

= 0.4 – 0.1

= 0.3

(iv) P(A∪ B')

P(A∪ B') = P(A) + P(B') – P(A∩ B')

= 0.4 +(1-0.6) – 0.3

= 0.5

Example 9

Events A and B are such that P(A) = 0.7, P(A∩ B) = 0.45 and P(A'∩ B') $= 0.18$, find

(i) P(B')

P(A') =P(A'∩ B) $+ P(A' \cap B')$

1 - 0.7 = P(A'∩ B) + 0.18

P(A'∩ B) = 0.3 – 0.18 = 0.12

P(B) = P(A'∩ B) + P(A∩ B)

1-P(B') =0.12 + 0.45

P(B') = 0.43

Alternatively

P(A'∩B') = P(AUB)' =1- P(AUB)

1 – P(AUB) = 0.18

P(AUB)= 0.82

P(AUB) = P(A) + P(B) – P(A∩B)

0.82 = 0.7 + P(B) – 0.43

P(B) = 0.57

P(B') = 1- 0.57 = 0.43

(ii) P(A or B, but not both A and B)

= P(A'∩B) + P(A∩B') = 0.12 + P(A) – P(A∩B) = 0.12 + 0.7 – 0.45 = 0.37

Example 10

The probability that Anne reads the NewVision is 0.75 and the probability that she reads NewVision and bot Daily-monitor is 0.65. The probability that she reads neither of the papers is 0.15. Find the probability that she reads Daily-monitor.

P(N) = 0.75, P(N∩D') = 0.65, P(N'∩D') = 0.15

P(D') = P(N∩D') + P(N'∩D')

1-P(D') =0.65 + 0.15

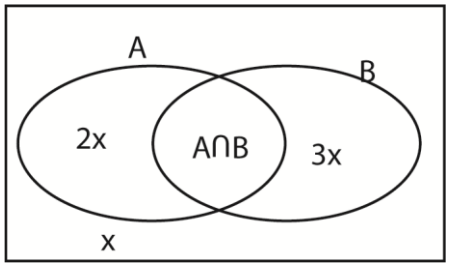
P(D') = 0.2

Example 11

Event A and B are such that P(A'∩B) = 3x, P(A∩B') = x and P(B) = $\frac{4}{7}$. Use a Venn diagram to find the value of

(i) x
(ii) P(A∩B)

Solution



(i) P(AUB) = P(A∩B') + P(B)

1 – P(AUB)' = 2x + $\frac{4}{7}$

1 – x =2x + $\frac{4}{7}$

x = $\frac{1}{7}$

(ii) P(A∩B) = P(B) – P(A'∩B)

= $\frac{4}{7} - 3\left(\frac{1}{7}\right) = \frac{1}{7}$

Exercise A

1. Event C and D are such that P(C) = 0.5, P(D) = 0.7 and PCUD) = 0.8, find
   (i) P(C∩D)
   (ii) (ii) P(C∩D')
2. Events A and B are such that P(A) = 0.36, P(B) = 0.25 and P(A'∩B) =0.24. Find
   (i) P(A') (ii) P(A∩B) (iii) P(AUB) (iv) P(A∩B') (v) P(A'UB')
3. Events C and D are such that P(C) = 0.7, P(C∩D)= 0.3 and P(CUD) = 0.9. Find
   (i) P(D), (ii) P(C∩D') (iii) P(C'∩D) (iv) P(C'∩D')
4. Events A and B are such that P(A') = $\frac{2}{3}$, P(B) = $\frac{1}{2}$ and P(A∩B) = $\frac{1}{12}$. Find
   (i) P(AUB) (ii) P(A∩B')
5. Events A and B are such that P(A) = $\frac{2}{3}$, P(A∩B) = $\frac{5}{12}$ and P(AUB) = $\frac{3}{4}$. Find
   (i) P(B) (ii) P(A'∩B)
6. Events A and B are such that P(A) =P(B), P(A∩B) = 0.1 and P(AUB) = 0.7. Find

(i) P(A) (ii) P(A'UB)

7. Event A and B are such that P(A) = $\frac{7}{12}$, P(B) =$\frac{3}{4}$ and PA∩B) =$\frac{2}{5}$. Find
   (i) P(AUB) (ii) P(A∩B')

Solutions to exercise A

1. Event C and D are such that P(C) = 0.5, P(D) = 0.7 and PCUD) = 0.8, find
   (i) P(C∩D)
   P(C∪ D) $= P(C) + P(D) - P(C \cap D)$
   0.8 = 0.5 + 0.7 $-P(C \cap D)$
   $P(C \cap D) = 0.4$
   (ii) (ii) P(C∩D')
   P(C) = P(C∩ D) + P(C∩ D')
   P(C∩ D') = 0.5 – 0.4 = 0.1

2. Events A and B are such that P(A) = 0.36, P(B) = 0.25 and P(A'∩B) =0.24. Find
   (i) P(A') = 1 – P(A) = 1 – 0.36 = 0.64
   (ii) P(A∩B)
   P(B) = P(A∩ B) + P(A′ ∩ B)
   P(A∩ B) = 0.25 -0.24 – 0.01
   (iii) P(AUB)
   P(AUB) = P(A)+ P(B) – P(A∩B)
   = 0.36 + 0.25 – 0.01 = 0.6
   (iv) P(A∩B')
   P(A) = P(A∩B) + P(A∩B')
   P(A∩B') = 0.36 -0.01 = 0.35
   (v) P(A'UB')
   P(A'∪ B') $= P(A \cap B)' = 1 - P(A \cap B)$
   = 1 – 0.01 = 0.99
3. Events C and D are such that P(C) = 0.7, P(C∩D)= 0.3 and P(CUD) = 0.9. Find
   (i) P(D)
   P(CUD)= P(C) + P(D)- P(C∩D)

   0.9 = 0.7 + P(D) – 0.3

   P(D) = 0.5

   (ii) P(C∩D')
   P(C) = P(C∩D) + P(C∩D')
   P(C∩D')= 0.7 – 0.3 = 0.4
   (iii) P(C'∩D)
   P(D) = P(C∩D) + P(C′ ∩D)
   P(C′ ∩D) = 0.5 – 0.3 = 0.2
   (iv) P(C'∩D')
   P(C'∩D')= 1 - P(CUD) = 1 – 0.9 = 0.1

4. Events A and B are such that P(A') = $\frac{2}{3}$, P(B) = $\frac{1}{2}$ and P(A∩B) = $\frac{1}{12}$. Find
   (i) P(AUB)

P(A) = 1- P(A') = 1 - $\frac{2}{3}$ = $\frac{1}{3}$

P(AUB) = P(A) + P(B) - P(A∩B) = $\frac{1}{3} + \frac{1}{2} - \frac{1}{12} = \frac{9}{12} = \frac{3}{4} = 0.75$

(ii) P(A∩B')

P(A) = P(A∩B) + P(A∩B')

P(A∩B') = $\frac{1}{3} - \frac{1}{12} = \frac{3}{12}$ =$\frac{1}{4} = 0.25$

5. Events A and B are such that P(A) = $\frac{2}{3}$, P(A∩B) = $\frac{5}{12}$ and P(AUB) = $\frac{3}{4}$. Find

(i) P(B)

P(AUB) = P(A) + P(B) - P(A∩B)

P(B) = $\frac{3}{4} + \frac{5}{12} - \frac{2}{3} = \frac{6}{12} = \frac{1}{2} = 0.5$

(ii) (ii) P(A'∩B)

P(B) = P(A∩B) + P(A′ ∩B)

P(A′ ∩B) = $\frac{1}{2} - \frac{5}{12} = \frac{1}{12}$

6. Events A and B are such that P(A) =P(B), P(A∩B) = 0.1 and P(AUB) = 0.7. Find

(i) P(A)

P(AUB) = P(A) + P(B) - P(A∩B)

2P(A) = 0.7 + 0.1 =0.8

P(A) = 0.4

(ii) P(A'UB)

P(A) = P(A∩B) + P(A∩B')

P(A∩B') = 0.4 -0.1 = 0.3

P(A∪ B') = P(B) + P(A') – P(A∩ B')

= 0.4 +(1-0.4) – 0.3

= 0.57

7. Event A and B are such that P(A) = $\frac{7}{12}$, P(B) =$\frac{3}{4}$ and PA∩B) =$\frac{2}{5}$. Find

(i) P(AUB)

P(AUB) = P(A) + P(B) - PA∩B)

= $\frac{7}{12} + \frac{3}{4} - \frac{2}{5} = \frac{56}{60} = \frac{14}{15}$

(ii) (ii)P(A∩B')

P(A) = PA∩B)+ P(A∩B')

P(A∩B') = $\frac{7}{12} - \frac{2}{5}$ =$\frac{11}{60}$]

Mutually exclusive events

Two events are mutually exclusive if they do not occur together i.e. P(A∩B)=0

P(AUB) = P(A) + P(B)

Example12

Events A and B are mutually exclusive such that P(B) =$\frac{1}{2}$, P(A) = $\frac{2}{5}$, find

(i) P(AUB)

P(AUB) = P(A) + P(B) = $\frac{1}{2} + \frac{2}{5} = \frac{9}{10} = 0.9$

(ii) P(A'∩B)

P(A'∩B) = P(B) - P(A∩B) = 0.5 – 0 = 0.5

(iii) P(A′∩B′)

P(A′∩B′) = P(AUB)′ = 1 – P(AUB) = 1- 0.9 = 0.1

Example 13

Event A and B are mutually exclusive such that P(AUB) = $\frac{7}{10}$, P(A) = $\frac{3}{5}$; find

(i) P(B)

P(B) = P(AUB) + P(A∩B) – P(A)= 0.7 + 0 – 0.6 = 0.1

(ii) P(A′∩B)

P(B) = P(A∩B) + P(A′ ∩B)

P(A′ ∩B) = 0.1 – 0 = 0.1

(iii) P(A∩B′)

P(a) = P(A∩B) + P(A∩B′)

P(A∩B′) = 0.6 – 0 = 0.6

(iv) P(A′∩B′)

P(A′∩B′) = P(AUB)′ = 1-P(AUB) = 1- 0.7 = 0.3

(v) P(A′UB′)

P(A′UB′)= P(A∩B)′ = 1 - P(A∩B) = 1- 0 = 1

(vi) P(AUB′)

P(AUB′) = P(A) + P(B′) –P(A∩B′)

= 0.6 + 1- 0.1 -0.6 = 0.9

(vii) P(A′UB)

P(A′UB) = P(B) + P(A′) –P(A′ ∩B)

= 0.1 + 1- 0.6 -0.1 = 0.4

Example 14

In an athletics competition in which there no dead heats, the probability that Kiplimo wins is 0.5, the probability that Bekele wins is 0.2, the probability that Cheptegei wins is 0.1.

Find the probability that

(i) Bakele or Kiplmo wins

P(BUK) = P(B) + P(K) = 0.2 + 0.5 = 0.7

(ii) Neither Kiplimo nor Cheptegei wins

P(K′∩B′) = P(KUB)′ = 1-P(KUB) = 1- (0.5 + 0.1) = 0.4

Revision exercise B

1. Events A and B are mutually exclusive such that P(A) = $\frac{1}{2}$ and P(B) = $\frac{2}{5}$; find

(i) P(AUB) (ii) P(A∩B′) (ii) P(A′∩B′)

2. Events A and B are mutually exclusive such that P(B) = $\frac{3}{10}$, P(A) = $\frac{3}{5}$, find

(i) P(AUB) (ii) P(A′) (iii) P(A′∩B)

3. Events A and B are mutually exclusive such that P(B) = 0.4, P(A) = 0.5, find

(i) P(A′UB) (ii) P(B′) (ii) P(A′∩B′)

4. Events A and B are mutually exclusive such that P(AUB) = $\frac{8}{10}$, P(A) = $\frac{2}{5}$, find

(i) P(B) (ii) P(A'∩B) (iii) P(A∩B') (iv) P(A'∩B') (v) P(AUB') (vi) P(A'UB)

5. Events A and B are mutually exclusive such that P(A'∩B) = 0.3, P(A'UB) = 0.45, find

(i) P(B) (ii) P(A) (iii) P(A'∩B) (iv) P(A'∩B') (v) P(AUB') (vi) P(A'UB)

6. Event A and B are mutually exclusive such P(AUB) = 0.9, P(AUB') = 0.6, find

(i) P(B) (ii) P(A) (iii) P(A'UB) (iv) P(A'∩B') (v) P(A'UB')

Solutions to exercise B

1. Events A and B are mutually exclusive such that P(A) = $\frac{1}{2}$ and P(B) = $\frac{2}{5}$; find

(i) P(AUB) = P(A) + P(B) = $\frac{1}{2} + \frac{2}{5} = \frac{9}{10} = 0.9$

(ii) P(A∩B') = P(A) – P(A∩B) = 0.4 – 0 = 0.4

(iii) P(A'∩B') = P(AUB)' = 1-P(AUB) = 1- 0.9 = 0.1

2. Events A and B are mutually exclusive such that P(B) = $\frac{3}{10}$, P(A) = $\frac{3}{5}$, find

(i) P(AUB) = P(A) + P(B) = $\frac{3}{10} + \frac{3}{5} = \frac{9}{10} = 0.9$

(ii) P(A') = 1- P(A) = 1 – 0.6 = 0.4

(iii) P(A'∩B) = P(B) – P(A∩B) = 0.3- 0 = 0.3

3. Events A and B are mutually exclusive such that P(B) = 0.4, P(A) = 0.5, find

(i) P(A'UB)

P(A'UB) = P(B) + P(A') –P(A' ∩B)
= 0.4 + 1- 0.5 -0.4 = 0.5

(ii) P(B') = 1- P(B) = 1 – 0.4 =0.6

(iii) P(A'∩B') = P(AUB)' = 1 – P(AUB) = 1- (P(A) + P(B) = 1-(0.4 + 0.5) =0.1

4. Events A and B are mutually exclusive such that P(AUB) = $\frac{8}{10}$, P(A) = $\frac{2}{5}$, find

(i) P(B) =P(AUB) – P(A) = 0.8- 0.4 = 0.4

(ii) P(A'∩B) = P(B) – P(A∩B) =0.4 – 0 =0.4

(iii) P(A∩B') = P(A) - P(A∩B) = 0.4 – 0 = 0.4

(iv) P(A'∩B') = P(AUB)' = 1- P(AUB) = 1- 0.8 = 0.2

(v) P(AUB') =P(A) + P(B') – P(A∩B') = 0.6 + (1-0.4) – 0.4 = 0.6

(vi) P(A'UB) = P(A') + P(B) – P(A∩B') = (1-0.4) + 0.4 – 0.4 = 0.6

5. Events A and B are mutually exclusive such that P(A'∩B) = 0.3, P(A'UB) = 0.45, find

(i) P(B) = P(A∩B) + P(A'∩B) = 0+ 0.3 = 0.3

(ii) P(A)

$P(A'UB) = P(A') + P(B) – P(A'\cap B)$

$0.45 = P(A') + 0.3 – 0.3$

$P(A') = 0.45$

$P(A) = 1- P(A') = 1- 0.45 = 0.55$

(iii) $P(A'\cap B) = 0.55$

$P(B)= P(A\cap B) + P(A'\cap B)$

$P(A'\cap B) = P(B) - P(A\cap B) = 0.3 – 0 = 0.3$

(iv) $P(A'\cap B') = P(AUB)' = 1 – P(AUB) = 1 – (P(A) + P(B) = 1- (0.55 + 0.3) = 0.15$

(v) $P(A'UB') = P(A\cap B)' = 1- P(A\cap B) = 1-0 = 1$

(vi) $P(A'UB) = P(A') + P(B) – P(A'\cap B)$

$=(1 – 0.55) + 0.3 - 0.3 =0.45$

6. Event A and B are mutually exclusive such P(AUB) = 0.9, P(AUB') = 0.6, find

(i) P(B) = 0.4

$P(A) + P(B') – P(A\cap B') =0.6$

$P(A) + [1 – P(B)] – P(A) = 0.6$

$1-P(B) = 0.6$

$P(B) = 0.4$

(ii) P(A) = 0.5

$P(A) + P(B) = 0.9$

$P(A) = 0.9 – 0.4 = 0.9$

(iii) $P(A'UB) = P(A') + P(B)– P(A'\cap B)$

$= (1 - 0.5) + 0.4 – 0.4 = 0.5$

(iv) $P(A'\cap B') = P(AUB)' = 1- P(AUB) = 1 – 09 = 0.1$

(v) $P(A'UB') = P(A\cap B)' = 1 - P(A\cap B) = 1 - 0 = 1$

Independent event

Two events A and B are independent if the occurrence of one does not affect the other

(i) $P(A\cap B) = P(A) \times P(B)$
(ii) $P(A' \cap B) = P(A') \times P(B)$
(iii) $P(A\cap B') = P(A) \times P(B')$
(iv) $P(A' \cap B') = P(A') \times P(B')$

Example 15

Event A and B are independent such that P(AUB) = $\frac{8}{10}$, P(A) = $\frac{1}{2}$, $find$

(i) P(B)

P(AUB)= P(A) + P(B) – P(A) x P(B)

0.8 = 0.5 + x – 0.5x

0.5x = 0.3; x = 0.6.

P(B) = 0.6

(ii) P(A'∩B)

P(A'∩B) = P(A') x P(B) = (1-P(A)) x P(B) = (1-0.5)(0.6) = 0.3

(iii) P(A∩B')

P(A∩B') = P(A) x P(B') = 0.5(1-0.6) = 0.2

(iv) P(A'∩B')

P(A'∩B') = P(A') x P(B') = (1-0.5)(1-0.6) = 0.2

(v) P(A'UB)

P(A'UB) = P(A') + P(B) – P(A∩B) = 0.5 + 0.6 – (0.5 x 0.6) = 0.8

Example 16

Events A and B are independent such that P(A∩B) = $\frac{1}{12}$, P(A) = $\frac{1}{3}$, find

(i) P(B)

P(A∩B) = P(A) x P(B)

$\frac{1}{12} = \frac{1}{3}\ x\ P(B)$; P(B) = $\frac{1}{4}$

(ii) P(AUB)

P(AUB) = P(A) + P(B) – P(A∩B) = $\frac{1}{3} + \frac{1}{4} - \frac{1}{12} = 0.5$

(iii) Show that a' and B are independent

P(A'∩B) = P(B) – P(A∩B)

P(B) – P(A) x P(B)

= P(B)(1-P(A)

= P(A') x P(B)

Example 17

Events A and B are independent

(i) Show that the events A and B' are also independent.

P(A∩B') =P(A) – P(A∩B)

= P(A) – P(A) x P(B)

= P(A)(1-P(B)

= P(A) x P(B')

(ii) Find P(B) given that given that P(A) = 0.4 and P(AUB) = 0.8

P(AUB) = P(A) + P(B) – P(A∩B)

0.8 = 0.4 + y – 0.4y

0.6y = 0.4

y = $\frac{0.4}{0.6} = \frac{2}{3}$

Example 17

Events A and B are independent such that P(AUB') = $\frac{9}{10}$, P(A) = $\frac{2}{5}$, find

(i) P(B)

P(AUB') = P(A) + P(B') – P(A∩B')

0.9 = 0.4 + (1-y) – 0.4(1- y) = 0.4 + 1 –y – 0.4 + 0.4y = 1 – 0.6y

0.6y = 0.1

y = $\frac{0.1}{0.6} = \frac{1}{6}$

(ii) P(AUB)

P(AUB) = P(A) + P(B)- P(A∩B)= $\frac{2}{5} + \frac{1}{6} - \frac{2}{5} x \frac{1}{6} = \frac{12+5-2}{30} = \frac{15}{30} = 0.5$

(iii) Show that A' and B' are independent

P(A' ∩B') = 1 – P(AUB)

= 1- [P(A) + P(B)- P(A)x P(B)]

= 1- [P(A) + P(B) (1-P(A)]

= 1 – P(A) - P(B)x P(A')

= P(A') - P(B)x P(A')

= P(A')[1-P(B)]

= P(A') x P(B')

Example 18

The probability of two independent events A and B occurring together is $\frac{1}{8}$. The probability that either or both events occur is $\frac{5}{8}$. Find

(i) P(A)

(ii) P(B)

Solution

Given P(AUB) = $\frac{5}{8}$ and P(A∩B) = $\frac{1}{8}$

Let P(A) = x and P(B) = y

x + y – xy = $\frac{5}{8}$ ..... (i)

xy = $\frac{1}{8}$

x = $\frac{1}{8y}$ ..... (ii)

(i) and (ii)

$\frac{1}{8y} + y - \frac{1}{8} = \frac{5}{8}$

$\frac{1}{8y} + y = \frac{6}{8}$

Multiplying through by 8y

$1 + 8y^2 = 6y$

$8y^2 - 6y + 1 = 0$

Using quadratic equation

$y = \frac{6 \pm \sqrt{(-6)^2 - 4(8)(1)}}{2(8)} = \frac{6 \pm 2}{16}$

Either y = $\frac{8}{16} = 0.5$ Or $y = \frac{4}{16} = \frac{1}{4}$ = 0.25

When y = 0.5; x = $\frac{1}{8\ x\ 0.5} = 0.25$

When y = 0.25; x = $\frac{1}{8\ x\ 0.25} = 0.5$

Hence P(A) = 0.25 and P(B) = 0.5 or P(A) = 0.5 and P(B) = 0.25

Example 19

Abel, Bob and Charles applied for the same job in a certain company. The probability that Abel will take the job is $\frac{3}{4}$, the probability that Bob takes it is $\frac{1}{2}$ while the probability that Charles will take the job is $\frac{2}{3}$; what is the probability that

(i) None of them will take the job

P(none takes it) = P(A'∩B'∩C') = P(A') x P(B') x P(C') = $\frac{1}{4}\ x\ \frac{1}{2} x\ \frac{1}{3} = \frac{1}{24}$

(ii) (one takes)

P(one takes it) = P(A∩B'∩C') + P(A'∩B∩C') + P(A'∩B'∩C)

$\left(\frac{3}{4}\ x\ \frac{1}{2} x\ \frac{1}{3}\right) + \left(\frac{1}{4}\ x\ \frac{1}{2} x\ \frac{1}{3}\right) + \left(\frac{1}{4}\ x\ \frac{1}{2} x\ \frac{2}{3}\right) = \frac{1}{4}$

Revision exercise C

1. Events A and B are independent such that P(A) = 0.4, P(B) = 0.25. Find
   (i) P(A∩B) (i) P(A∩B') (iii) P(A'∩B')
2. Events A and B are independent such that P(A) = 0.3, P(B) = 0.5. Find
   (i) P(A∩B) (ii) P(AUB) (iii) P(A'∩B')
3. Events A and B are independent such that P(A) = 0.4, PAUB) = 0.7 Find
   (i) P(B) (ii) P(A∩B) (iii) P(A'∩B)
4. Events A and B are independent such that P(A) = $\frac{1}{3}$, P(B) = $\frac{3}{4}$. Find the probability that
   (i) Both occur
   (ii) Only one occurs
5. The probability that two independent events A and B occur together = $\frac{2}{15}$ and the probability that either A or B or both occur is $\frac{3}{5}$. Find
   (i) P(A) (ii) P(B)
6. A mother and her daughter both enter a competition. The probability that a mother wins a prize = $\frac{1}{6}$ and the probability that her daughter wins the prize is $\frac{2}{7}$. Assuming that the two events are independent, find the probability that
   (i) Either the mother of the daughter but not both wins the prize
   (ii) At least one of them wins the prize

7. Two athletes, Kiprotich and Chebet attempt to qualify for Olympics games. The probabilityof Kiprotich qualifying is 0.8 and the probability that both Kiprotich and Chebet qualifying is 0.6. given that the probability of the athletes qualifying are independent events, find the probability that only one qualifies
8. The probability of two independent events A and B occurring together = $\frac{1}{10}$. The probability that either or both events occur is $\frac{8}{10}$. Find
   (i) P(A)
   (ii) P(B)

9. The probability that a certain types of computer will break down on the first month of use = 0.1. If the school has two such computer bought at the same time, find the probability that at the end of the first month just one has broken. Assume that the performance of the two computers are independent
10. Three athletes enter a marathon race. The respective probabilities of them completing the race are 0.9, 0.7 and 0.6. assuming that their performance are independent, find the probability that
    (i) they all complete the race
    (ii) at least two complete the race
11. The probability that two twins pass an interview are $\frac{1}{3}\ and\ \frac{2}{5}$. Assuming that their performance are independent, find the probability
    (i) They all pass the interview
    (ii) Only one passes the interview
12. The probability that Angela can solve a certain number is 0.4 and the probability that Jane can solve the same number is 0.5, find the probability the number will be solved if both students try the number independently
13. Three target men take part in a shooting competition, their chances of hitting the target are $\frac{1}{4}, \frac{1}{3}\ and\ \frac{1}{2}$. Assuming that their performance are independent, find the probability that
    (i) Only one will hit the target
    (ii) Target will be hit
14. Three football teams Noa, Kitende and Budo are playing in Nationals. The probability that Noa, Kitende and Budo qualify for the finals is $\frac{2}{3}, \frac{3}{5}\ and\ \frac{1}{4}$. Find the probability that only two teams will qualify for the finals
15. Three Athletes Kiprop, Chebet and Aloysius are competing for a place in Olympics games. The probability that Kiprop, Chebet and Aloysius will qualify for the Olympics games is $\frac{2}{3}, \frac{2}{5}\ and\ \frac{5}{6}$. Find the probability that only one athlete will qualify for the Olympics games .
16. The probability that three girls Faith, Jane and Angela will pass exams is $\frac{2}{3}, \frac{2}{5}\ and\ \frac{3}{4}$ respectively. Find the probability that
    (i) All the three will fail
    (ii) All the three will pass
    (iii) Only two will pass
17. The interview involves written, oral and practical tests. The probability than an interviewee passes written = 0.8, oral is 0.6 and practical is 0.7. what is the probability that the interviewee will pass
    (i) The entire interview
    (ii) Exactly two of the interview test

18. The probabilities that the players A, B and C score in a netball game are $\frac{1}{5}, \frac{1}{4}\ and\ \frac{1}{3}\ respectively$. If the player play together in a game, find the probability that
    (i) Only one score
    (ii) At least one player scores
    (iii) Two and only two player score
19. Events a and B are such that P(A) = $\frac{1}{5}$, P(B) = $\frac{1}{2}$. Find P(AUB) when A and B are
    (i) Independent events
    (ii) Mutually exclusive events

Solutions to revision exercise C

1. Events A and B are independent such that P(A) = 0.4, P(B) = 0.25. Find
    (i) P(A∩B) P(A) x P(B) = 0.4 x 0.25 = 0.1
    (ii) P(A∩B') =P(A) x P(B') = 0.4 x 0.75 = 0.3
    (iii) P(A'∩B') = P(A') x P(B') = 0.6 x 0.75 =0.45
2. Events A and B are independent such that P(A) = 0.3, P(B) = 0.5. Find
    (i) P(A∩B) = P(A) x P(B) = 0.3 x 0.5 = 0.15
    (ii) P(AUB) =P(A0 + P(B) - P(A∩B) = 0.3 + 0.5 – 0.16 = 0.65
    (iii) P(A'∩B') = 1 – P(AUB) = 1 – 0.65 = 0.35
3. Events A and B are independent such that P(A) = 0.4, PAUB) = 0.7 Find
    (i) P(B)
    P(AUB) = P(A) + P(B) – P(A) x P(B)
    0.7 = 0.4 + x – 0.4x
    0.6x = 0.3
    x = 0.5
    hence P(B) = 0.5
    (ii) P(A∩B) = P(A) x P(B) = 0.4 x 0.5 =0.2
    (iii) P(A'∩B) = 1 – P(AUB) = 1 – 0.7 = 0.3
4. Events A and B are independent such that P(A) = $\frac{1}{3}$, P(B) = $\frac{3}{4}$. Find the probability that
    (i) Both occur
    P(A∩B) = P(A) x P(B)= $\frac{1}{3}\ x\ \frac{3}{4} = \frac{1}{4}$
    (ii) Only one occurs
    P(A∩B') + P(A'∩B) = P(A) x P(B') + P(A') x P(B) = $\frac{2}{3}\ x\ \frac{3}{4} + \frac{1}{3}\ x\ \frac{1}{4} = \frac{7}{12}$
5. The probability that two independent events A and B occur together = $\frac{2}{15}$ and the probability that either A or B or both occur is $\frac{3}{5}$. Find
    Let the P(A) = x and P(B) = y
    xy = $\frac{2}{15}$ ..... (i)
    x + y - xy= $\frac{3}{5}$ ..... (ii)
    Eqn (i) and eqn. (ii)
    $\frac{2}{15y} + y - \frac{2}{15} = \frac{3}{5}$
    $2 + 15y^2 – 2y= 9y$
    $15y^2 – 11y + 2 = 0$
    Using quadratic equation

y = $\frac{11\pm\sqrt{(-11)^2-4(15)(2)}}{2(15)} = \frac{11\pm1}{30}$

y = $\frac{10}{30} = \frac{1}{3}\ Or\ y = \frac{12}{30} = 0.4$

Either P(A) = 0.4 and P(B) = $\frac{1}{3}$ or P(A) = $\frac{1}{3}$ and P(B) =0.4

6. A mother and her daughter both enter a competition. The probability that a mother wins a prize = $\frac{1}{6}$ and the probability that her daughter wins the prize is $\frac{2}{7}$. Assuming that the two events are independent, find the probability that

   (i) Either the mother of the daughter but not both wins the prize =$\frac{5}{14}$

   P(M∩D') + P(M'∩D) = P(M)x P(D') + P(M') x P(D) = $\frac{1}{6}\ x\ \frac{5}{7} + \frac{5}{6}\ x\ \frac{2}{7} = \frac{15}{42} = \frac{5}{14}$

   (ii) At least one of them wins the prize

   P(AUB) = P(A) + P(B) – P(A∩B) = $\frac{1}{6} + \frac{2}{7} - \frac{1}{6}\ \text{x}\ \frac{2}{7} = \frac{17}{42}$

7. Two athletes, Kiprotich and Chebet attempt to qualify for Olympics games. The probabilityof Kiprotich qualifying is 0.8 and the probability that both Kiprotich and Chebet qualifying is 0.6. given that the probability of the athletes qualifying are independent events, find the probability that only one qualifies = 0.35

   P(K∩B) = P(K) x P(C) = 0.8 x P(C) = 0.6

   P(C) = $\frac{0.6}{0.8} = \frac{3}{4}$

   P(K∩C') + P(K'∩C) = 0.8 x 0.25 + 0.2 x 0.75 = 0.35

8. The probability of two independent events A and B occurring together = $\frac{1}{10}$. The probability that either or both events occur is $\frac{8}{10}$. Find

   (iii) P(A)

   (iv) P(B)

   Solution

   Let the P(A) = x and P(B) = y

   xy = $\frac{1}{10}$ ..... (i)

   x + y - xy= $\frac{8}{10}$ ..... (ii)

   Eqn (i) and eqn. (ii)

   $\frac{1}{10y} + y - \frac{1}{10} = \frac{8}{10}$

   $1 + 10y^2 – y= 8y$

   $10y^2 – 9y + 1 = 0$

   Using quadratic equation

   y = $\frac{9\pm\sqrt{(-9)^2-4(10)(1)}}{2(10)} = \frac{9\pm6.4}{20}$

   y = $\frac{2.6}{20} = 0.13\ Or\ y = \frac{12}{30} = 0.77$

   Either P(A) = 0.13 and P(B) = 0.77 or P(A) = 0.77 and P(B) = 0.13

9. The probability that a certain types of computer will break down on the first month of use = 0.1. If the school has two such computer bought at the same time, find the probability that at the

end of the first month just one has broken. Assume that the performance of the two computers are independent= 0.18

Let the computers be A and B

P(one has broken down) = P(A∩B') + P(A'∩B)

=0.1 x 0.9 + 0.9 x 0.1

= 0.18

10. Three athletes enter a marathon race. The respective probabilities of them completing the race are 0.9, 0.7 and 0.6. Assuming that their performance are independent, find the probability that

(i) they all complete the race

Let the athletes be A, B, C

P(all complete the race) = P(A∩B∩C) = 0.9 x 0.7 x 0.6 = 0.378

(ii) at least two complete the race

P(At least two complete the race) = P(A'∩B∩C) + P(A∩B'∩C) + P(A∩B∩C') + P(A∩B∩C)

= 0.1 x 0.7 x 0.6 + 0.9 x 0.3 x 0.6 + 0.9 x 0.7 x 0.4 + 0.378

= 0.834

11. The probability that two twins pass an interview are $\frac{1}{3}$ and $\frac{2}{5}$. Assuming that their performance are independent, find the probability

(i) They all pass the interview

Let the twins be A and B

P(all pass) =$P(A \cap B)$ = $\frac{1}{3}$ x $\frac{2}{5}$ = $\frac{2}{15}$

(ii) Only one passes the interview

P(A∩B') + P(A'∩B) = $\frac{2}{3}$ x $\frac{2}{5}$ + $\frac{1}{3}$ x $\frac{3}{5}$ = $\frac{7}{15}$

12. The probability that Angela can solve a certain number is 0.4 and the probability that Jane can solve the same number is 0.5, find the probability the number will be solved if both students try the number independently = 0.7

P(the number solved) = P(A ∩J') + P(A'∩J) + P(A∩J)

= 0.4 x 0.5 + 0.6 x 0.5 + 0.4 x 0.5

= 0.7

13. Three target men take part in a shooting competition, their chances of hitting the target are $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$. Assuming that their performance are independent, find the probability that

Let the men be A, B and V

(i) Only one will hit the target =

P(one hit the target) = P(A'∩B'∩C) + P(A'∩B∩C') + P(A∩B'∩C')

= $\frac{3}{4}x$ $\frac{2}{3}$ x $\frac{1}{2}$ + $\frac{3}{4}x$ $\frac{1}{3}$ x $\frac{1}{2}$ + $\frac{1}{4}x$ $\frac{2}{3}$ x $\frac{1}{2}$

=$\frac{11}{24}$

(ii) Target will be hit

P(target hit)= P(A'∩B'∩C) + P(A'∩B∩C') + P(A∩B'∩C') + P(A'∩B∩C) + P(A∩B∩C') + P(A∩B'∩C) + P(A∩B∩C)

= $\frac{11}{24}$ + $\frac{3}{4}x$ $\frac{1}{3}$ x $\frac{1}{2}$ + $\frac{1}{4}x$ $\frac{1}{3}$ x $\frac{1}{2}$ + $\frac{1}{4}x$ $\frac{2}{3}$ x $\frac{1}{2}$ + $\frac{1}{4}x$ $\frac{1}{3}$ x $\frac{1}{2}$

= $\frac{18}{24}$ = 0.75

14. Three football teams Noa, Kitende and Budo are playing in Nationals. The probability that Noa, Kitende and Budo qualify for the finals is $\frac{2}{3}, \frac{3}{5}\ and\ \frac{1}{4}$. Find the probability that only two teams will qualify for the finals

    P(only two teams qualify) = P(N′∩K∩B) + P(N∩K′∩B) + P(N∩K∩B′)

    $= \frac{1}{3}\ x\ \frac{3}{5}\ x\ \frac{1}{4} + \frac{2}{3}\ x\ \frac{2}{5}\ x\ \frac{1}{4} + \frac{2}{3}\ x\ \frac{3}{5}\ x\ \frac{3}{4}$

    $= \frac{25}{60} = \frac{5}{12}$

15. Three Athletes Kiprop, Chebet and Aloysius are competing for a place in Olympics games. The probability that Kiprop, Chebet and Aloysius will qualify for the Olympics games is $\frac{2}{3}, \frac{2}{5}\ and\ \frac{5}{6}$. Find the probability that only one athlete will qualify for the Olympics games .

    P(only one qualifies) = P(K∩C′∩A′)+ P(K′∩C∩A′) + P(K′∩C′∩A)

    $= \frac{2}{3}\ x\ \frac{3}{5}\ x \frac{1}{6} + \frac{1}{3}\ x\ \frac{2}{5}\ x \frac{1}{6} + \frac{1}{3}\ x\ \frac{3}{5}\ x \frac{5}{6}$

    $= \frac{23}{90}$

16. The probability that three girls Faith, Jane and Angela will pass exams is $\frac{2}{3}, \frac{2}{5}\ and\ \frac{3}{4}$ respectively. Find the probability that

    (i) All the three will fail

    P(all fail) = P(F′∩J′∩A′) = $\frac{1}{3} x\ \frac{3}{5}\ x\ \frac{1}{4} = \frac{1}{20}$

    (ii) All the three will pass

    P(all pass) = P(F∩J∩A) = $\frac{2}{3} x\ \frac{2}{5}\ x\ \frac{3}{4} = \frac{1}{5}$

    (iii) Only two will pass

    P(only two pass) = P(F′∩J∩A) + P(F∩J∩A′) + P(F∩J′∩A)

    $= \frac{1}{3} x\ \frac{2}{5}\ x\ \frac{3}{4} + \frac{2}{3} x\ \frac{2}{5}\ x\ \frac{1}{4} + \frac{2}{3} x\ \frac{3}{5}\ x\ \frac{3}{4} = \frac{28}{60} = \frac{7}{15}$

17. The interview involves written, oral and practical tests. The probability than an interviewee passes written = 0.8, oral is 0.6 and practical is 0.7. what is the probability that the interviewee will pass

    (i) The entire interview

    P(pass entire interview) = P(W∩O∩P) = 0.8 x 0.6 x 0.7 =0.336

    (ii) Exactly two of the interview test

    P(pass exactly two) = P(W′∩O∩P) + P(W∩O′∩P) + P(W∩O∩P′)

    = 0.2 x 0.6 x 0.7 + 0.8 x 0.4 x 0.7 + 0.8 x 0.6 x 0.3

    = 0.452

18. The probabilities that the players A, B and C score in a netball game are $\frac{1}{5}, \frac{1}{4}\ and\ \frac{1}{3}\ respectively$. If the player play together in a game, find the probability that

    (i) Only one score

    P(only one scores) = P(A∩B′∩C′) + P(A′∩B∩C′) + P(A′∩B′∩C)

    $= \frac{1}{5} x\ \frac{3}{4}\ x\ \frac{2}{3} + \frac{4}{5} x\ \frac{1}{4}\ x\ \frac{2}{3} + \frac{4}{5} x\ \frac{3}{4}\ x\ \frac{1}{3} = \frac{26}{60} = \frac{13}{30}$

    (ii) At least one player scores

    P(at least one scores) = P(A∩B′∩C′) + P(A′∩B∩C′) + P(A′∩B′∩C) + P(A∩B∩C′) + P(A′∩B∩C) + P(A∩B′∩C) + P(A∩B∩C)

    $= \frac{13}{30} + \frac{1}{5} x\ \frac{1}{4}\ x\ \frac{2}{3} + \frac{4}{5} x\ \frac{1}{4}\ x\ \frac{1}{3} + \frac{1}{5} x\ \frac{3}{4}\ x\ \frac{1}{3} + \frac{1}{5} x\ \frac{1}{4}\ x\ \frac{1}{3}$

    $= \frac{36}{60} = \frac{3}{5}$

    (iii) Two and only two player score

    = P(A∩B∩C′) + P(A′∩B∩C) + P(A∩B′∩C)

$=\frac{1}{5}x\ \frac{1}{4}\ x\ \frac{2}{3}+\frac{4}{5}x\ \frac{1}{4}\ x\ \frac{1}{3}+\frac{1}{5}x\ \frac{3}{4}\ x\ \frac{1}{3}$

$=\frac{9}{60}=\frac{3}{20}$

19. Events a and B are such that P(A) = $\frac{1}{5}$, P(B) = $\frac{1}{2}$. Find P(AUB) when A and B are

(i) Independent events

P(AUB) = P(A) + P(B) – P(A∩B) = $\frac{1}{5}+\frac{1}{2}-\ \frac{1}{5}x\frac{1}{2}=\ \frac{6}{10}=\ \frac{3}{5}$

(ii) Mutually exclusive events = $\frac{7}{10}$

P(AUB) = P(A) + P(B) – P(A∩B) = $\frac{1}{5}+\frac{1}{2}-\ 0=\ \frac{7}{10}$

Conditional probability

If A and B are two events, then the conditional probability that A occurs given that B has already occurred is P(A/B)

P(A/B) = $\frac{P(A\cap B)}{P(B)}$

Example 20

Events A and B are such that P(A) = $\frac{1}{5}$, P(B) = $\frac{1}{4}$ and P(A/B) = $\frac{2}{5}$. Find

(i) P(A∩B)

P(A/B) = $\frac{P(A\cap B)}{P(B)}$

P(A∩B) = $\frac{2}{5}\ x\ \frac{1}{4}=\ \frac{2}{20}=0.1$

(ii) P(AUB)

P(AUB) = P(A) + P(B) - P(A∩B) = $\frac{1}{5}+\ \frac{1}{4}-0.1=0.35$

(iii) P(B/A)

P(B/A) = $\frac{P(A\cap B)}{P(A)}=\ \frac{0.1}{0.2}=0.5$

Example 21

Events A and B are such that P(A) = $\frac{4}{7}$, P(A∩B') = $\frac{1}{3}$ and P(A/B)= $\frac{5}{14}$. Find

(i) P(A∩B)

P(A∩B) = P(A) - P(A∩B') = $\frac{4}{7}-\ \frac{1}{3}=\ \frac{12-7}{21}=\frac{5}{21}$

(ii) P(B)

P(A/B) = $\frac{P(A\cap B)}{P(B)}$

P(B) = $\frac{5}{21}\div\frac{5}{14}=\frac{5}{21}x\frac{14}{5}=\frac{2}{3}$

(iii) P(AUB)

P(AUB) = P(A) + P(B) - P(A∩B) = $\frac{4}{7}+\frac{2}{3}-\ \frac{5}{21}=1$

Example 22

Events A and B are independent. Given that P(A∩B') = $\frac{1}{4}$ and P(A'/B) = $\frac{1}{6}$. Find

(i) P(A)

$P(A'/B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(A') \, x \, P(B)}{P(B)}$

$P(A') = \frac{1}{6}$ and $P(A) = 1 - \frac{1}{6} = \frac{5}{6}$

(ii) P(B)

$P(A \cap B') = P(A) \times P(B') =$

$P(B') = P(A \cap B') \div P(A) = \frac{1}{4} \div \frac{5}{6} = \frac{1}{4} \, x \frac{6}{5} = \frac{3}{10}$

$P(B) = 1 - P(B') = 1 - 0.3 = 0.7$

(iii) $P(A \cap B) = P(A) \times P(B) = \frac{5}{6} \, x \, \frac{7}{10} = \frac{7}{12}$

(iv) P(AUB)'

$P(AUB) = P(A) + P(B) - P(A \cap B) = \frac{5}{6} + \frac{7}{10} - \frac{7}{12} = \frac{19}{20}$

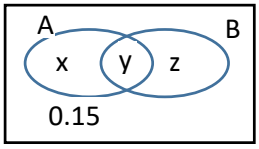
$P(AUB)' = 1 - P(AUB) = 1 - \frac{19}{20} = \frac{1}{20}$

Revision exercise D

1. Events X and Y are such that $P(X') = \frac{3}{5}$, $P(Y/X') = \frac{1}{3}$ and $P(Y'/X) = \frac{1}{4}$. Find
   (i) $P(Y) = \frac{1}{2}$ (ii) P(X'/Y)
2. Events A and B are such that $P(A) = \frac{2}{5}$, $P(A/B) = \frac{1}{2}$, $P(B/A) = \frac{2}{3}$. Find
   (i) $P(A \cap B) = \frac{4}{15}$ (ii) $P(B) = \frac{8}{15}$ (iii) $P(AUB) = \frac{2}{3}$
3. Events A an B are such that $P(A) = \frac{1}{3}$, $P(B/A) = \frac{1}{4}$ and $P(B'/A') = \frac{4}{5}$. Find
   (i) $P(B'/A) = \frac{3}{4}$ (ii) $P(A \cap B) = \frac{1}{12}$ (iii) $P(B) = \frac{13}{60}$ (iv) $P(AUB) = \frac{7}{15}$
4. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B') = \frac{1}{3}$. Find
   (i) $P(A'UB') = \frac{5}{6}$ (ii) $P(A'/B') = \frac{1}{2}$
5. Events A and B are such that $P(A \cap B) = \frac{1}{12}$, $P(B/A) \frac{1}{3}$ and $P(B) = \frac{1}{6}$. Find
   (i) $P(A) = \frac{1}{4}$ (ii) $P(A/B) = \frac{1}{2}$ (iii) $P(A/B') = \frac{1}{5}$
6. Events A and B are such that P(AUB) = 0.8, P(A/B) = 0.2 and $P(A' \cap B') = 0.4$. Find
   (i) $P(A \cap B) = 0.1$ (ii) $P(B) = 0.5$ (ii) $P(A) = 0.4$ (iv) $P(A/B') = 0.6$ (v) $P(A'/B') = 0.4$
7. Events A and B are independent. Given that P(A) = 0.2 and P(B) = 0.15. find
   (i) $P(A \cap B) = 0.03$ (ii) P(A/B) = 0.2 and P(AUB) = 0.32
8. Events A and B are such that P(A) = 0.2, P(A/B) = 0.4 and P(B) = 0.25. Find
   (i) $P(A \cap B) = 0.1$ (ii) $P(B/A) = 0.5$ (ii) $P(AUB) = 0.35$
9. Events A and B are such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A/B) = \frac{2}{5}$. Find
   (i) $P(A \cap B) = 0.1$ (ii) $P(B/A) = 0.3$
10. Events A and B are such that $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{4}$ and $P(A/B) = \frac{2}{3}$. Find
    (i) $P(A \cap B) = \frac{1}{6}$ (ii) $P(B/A) = \frac{1}{4}$
11. Events A and B are such that $P(B) = \frac{1}{3}$, $P(A) = \frac{1}{2}$ and $P(A \cap B') = \frac{1}{3}$. Find
    (i) $P(A'UB') = \frac{5}{6}$ (ii)$P(B'/A') = \frac{2}{3}$
12. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{8}$ and $P(A/B) = \frac{7}{12}$. Find
    (i) $P(A \cap B) = \frac{7}{32}$

(ii) P(B/A') = $\frac{3}{8}$

13. A and B are intersecting sets as shown in the Venn diagram below



Given that P(A) = 0.6, P(A'/B) = $\frac{5}{7}$ and PAUB) = 0.85. Find

(i) the values x, y and z (x = 0.5, y = 0.1 and z = 0.25)

(ii) P(A/B) = $\frac{2}{7}$ (iii) P(A/B') = $\frac{10}{13}$

14. Events A and B are independent with A twice likely to occur as B. If P(A) = 0.5, find

(i) P(AUB) = $\frac{5}{8}$ (ii) P$\left(\frac{A\cap B}{A}\right)$ = $\frac{1}{8}$

15. Two events A and B are such that P(A/B) = $\frac{2}{5}$, P(B) = $\frac{1}{4}$ and P(A) = $\frac{1}{5}$. Find

(i) P(A∩B) = 0.1 (ii) P(AUB) = 0.35

16. Two events a and B are such that P(B/A) = $\frac{1}{3}$, P(B) = $\frac{1}{8}$ and P(A∩B) = $\frac{1}{10}$. Find

(i) P(A) = 0.3 (ii) P(AUB) = 0.325 (iii) P(A/B) = 0.8

17. Two events A and B are such that P(A/B)= 0.1, P(A) = 0.7 and P(B) = 0.2. Find

(i) P(AUB) = 0.88 (ii) P(A∩B') = 0.68

Solutions to revision exercise D

1. Events X and Y are such that P(X') = $\frac{3}{5}$, P(Y/X') = $\frac{1}{3}$ and P(Y'/X) = $\frac{1}{4}$. Find

P(Y) = $\frac{1}{2}$

P(Y/X') = $\frac{P(Y\cap X')}{P(X')}$

P(Y∩X') = $\frac{1}{3}\ x\ \frac{3}{5} = \frac{1}{5}$

2. Events A and B are such that P(A)= $\frac{2}{5}$, P(A/B) = $\frac{1}{2}$, P(B/A) = $\frac{2}{3}$. Find

(i) P(A∩B)

P(B/A) = $\frac{P(A\cap B)}{P(A)}$

P(A∩B)) = $\frac{2}{3}\ x\frac{2}{5} = \frac{4}{15}$

(ii) P(B)

P(A/B) = $\frac{P(A\cap B)}{P(B)}$

P(B) = $\frac{4}{15} \div \frac{1}{2} = \frac{4}{15}\ x\ 2 = \frac{8}{15}$

(iii) P(AUB)

P(AUB) = P(A) + P(B) – P(A∩B) = $\frac{2}{5} + \frac{8}{15} - \frac{4}{15} = \frac{10}{15} = \frac{2}{3}$

3. Events A an B are such that P(A) = $\frac{1}{3}$, P(B/A) = $\frac{1}{4}$ and P(B'/A') = $\frac{4}{5}$. Find

(i) P(B'/A) = $\frac{3}{4}$

P(B/A) =

(ii) (ii) P(A∩B) = $\frac{1}{12}$ (iii) P(B) = $\frac{13}{60}$ (iv) P(AUB) = $\frac{7}{15}$

4. Events A and B are such that P(A) = $\frac{1}{2}$, P(B) =$\frac{1}{3}$ and P(A∩B') = $\frac{1}{3}$. Find

   (i) P(A'UB') = $\frac{5}{6}$ (ii) P(A'/B') = $\frac{1}{2}$

5. Events A and B are such that P(A∩B) = $\frac{1}{12}$, P(B/A) $\frac{1}{3}$ and P(B) = $\frac{1}{6}$. Find

   (i) P(A) = $\frac{1}{4}$ (ii) P(A/B) = $\frac{1}{2}$ (iii) P(A/B') = $\frac{1}{5}$

6. Events A and B are such that P(AUB) = 0.8, P(A/B) = 0.2 and P(A'∩B') = 0.4. Find

   (i) P(A∩B)

   (ii) P(B) = 0.5

   (iii) P(A) = 0.4

   (iv) P(A/B') = 0.6

   (v) P(A'/B') =0.4

7. Events A and B are independent. Given that P(A) = 0.2 and P(B) = 0.15. find

   (i) P(A∩B) = P(A) x P(B) = 0.2 x 0.15 = 0.03

   (ii) P(A/B) =$\frac{P(A\cap B)}{P(B)} = \frac{0.03}{0.15}$= 0.2

   (iii) P(AUB) = P(A) + P(B) – P(A∩B) = 0.2 + 0.15 – 0.03 = 0.32

8. Events A and B are such that P(A) = 0.2, P(A/B) = 0.4 and P(B) = 0.25. Find

   (i) P(A∩B)

   P(A/B) =$\frac{P(A\cap B)}{P(B)}$

   P(A∩B) = 0.4 x 0.25= 0.1

   (ii) P(B/A)

   P(B/A) =$\frac{P(A\cap B)}{P(A)} = \frac{0.1}{0.2} = 0.5$

   (iii) P(AUB)

   P(AUB) = P(A) + P(B) - P(A∩B) = 0.2 + 0.25 – 0.1= 0.35

9. Events A and B are such that P(A) = $\frac{1}{3}$, P(B) = $\frac{1}{4}$ and P(A/B) = $\frac{2}{5}$. Find

   (i) P(A∩B)

   P(A/B) =$\frac{P(A\cap B)}{P(B)}$

   P(A∩B) = $\frac{2}{5}\ x\frac{1}{4} = \frac{1}{10}$

   (ii) P(B/A) = 0.3

   P(B/A) =$\frac{P(A\cap B)}{P(A)} = \frac{1}{10} \div \frac{1}{3} = \frac{3}{10}$

10. Events A and B are such that P(A) = $\frac{2}{3}$, P(B) = $\frac{1}{4}$ and P(A/B) = $\frac{2}{3}$. Find

    (i) P(A∩B)

    P(A/B) =$\frac{P(A\cap B)}{P(B)}$

    P(A∩B) = $\frac{2}{3}\ x\frac{1}{4} = \frac{1}{6}$

    (ii) P(B/A)

    P(B/A) =$\frac{P(A\cap B)}{P(A)} = \frac{1}{6} \div \frac{2}{3} = \frac{1}{6}\ x\frac{3}{2} = \frac{1}{4}$

11. Events A and B are such that P(B) = $\frac{1}{3}$, P(A) =$\frac{1}{2}$ and P(A∩B') = $\frac{1}{3}$. Find

    (i) P(A'UB')

    P(A'UB') = 1 – P(A∩B) = 1 - $\frac{1}{2}\ x\ \frac{1}{3}$ = $\frac{5}{6}$

    (ii) P(B'/A')

P(B′/A′) = $\frac{P(A' \cap B')}{P(A')} = \frac{\frac{2}{3} x \frac{1}{2}}{\frac{1}{2}} = \frac{2}{3}$

12. Events A and B are such that P(A) =$\frac{1}{2}$, P(B) = $\frac{3}{8}$ and P(A/B) = $\frac{7}{12}$. Find

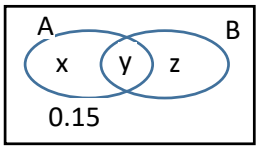
(i) P(A∩B)

P(A/B) = $\frac{P(A \cap B)}{P(B)}$

P(A∩B) = $\frac{7}{12}\ x\ \frac{3}{8} = \frac{7}{32}$

(ii) P(B/A′) =$\frac{P(B \cap A')}{P(A')} = \frac{\frac{3}{8} x \frac{1}{2}}{\frac{1}{2}} = \frac{3}{8}$

13. A and B are intersecting sets as shown in the Venn diagram below



Given that P(A) = 0.6, P(A′/B) =$\frac{5}{7}$ and PAUB) = 0.85. Find

(i) the values x, y and z

from the Venn diagram

x + y + z = 0.85 …….. (i)

x + y = 0.6 ……. (ii)

subtracting eqn. (ii) from eqn. (i)

z = 0.85 – 0.6 = 0.25

P(A′∩B) = $\frac{P(A' \cap B)}{P(B)} = \frac{5}{7}$

P(A′∩B) = P(B) only = 0.25

⇨ 0.25 = $\frac{5}{7} P(B)$

P(B) = $\frac{0.25\ x\ 7}{5} = 0.35$

But P(B) = y + z = y + 0.25 = 0.35

y = 0.1

Substituting y in eqn.(i)

x + 0.1 = 0.6; x = 0.5

(ii) P(A/B) = $\frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.35} = \frac{10}{35} = \frac{2}{7}$

(iii) P(A/B′) = $\frac{P(A \cap B')}{P(B')} = \frac{0.5}{0.65} = \frac{50}{65} = \frac{10}{13}$

14. Events A and B are independent with A twice likely to occur as B. If P(A) = 0.5, find

(i) P(AUB)

P(B) = 0.25

P(AUB) = P(A) + P(B) – P(A∩B) = 0.5 + 0.25 – 0.5 x 0.25 = 0.625

(ii) $P(A \cap B/A) = \frac{0.125\ x\ 0.5}{0.5}$=0125

15. Two events A and B are such that P(A/B) =$\frac{2}{5}$, P(B) = $\frac{1}{4}$ and P(A) = $\frac{1}{5}$. Find

(i) P(A∩B)

P(A/B) = $\frac{P(A \cap B)}{P(B)}$

P(B) = 0.4 x 0.25 = 0.1

(ii) P(AUB)

P(AUB) = P(A) + P(B) – P(A∩B) = 0.2 + 0.25 – 0.1 = 0.35

16. Two events a and B are such that P(B/A) = $\frac{1}{3}$, P(B) = $\frac{1}{8}$ and P(A∩B) = $\frac{1}{10}$. Find

(i) P(A) = 0.3

P(B/A) = $\frac{P(A \cap B)}{P(A)}$

P(A) = $\frac{1}{10} \div \frac{1}{3} = \frac{3}{10} = 0.3$

(ii) P(AUB)

P(AUB) = P(A) + P(B) – P(A∩B) = 0.3 + 0.125 – 0.1 = 0.325

(iii) P(A/B)

P(A/B) = $\frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.125} = 0.8$

17. Two events A and B are such that P(A/B)= 0.1, P(A) = 0.7 and P(B) = 0.2. Find

(i) P(AUB) =

P(A/B) = $\frac{P(A \cap B)}{P(B)}$

P(A∩B) = 0.1 x 0.2 = 0.02

P(AUB) = P(A) + P(B) - P(A∩B)

= 0.7 + 0.2 – 0.02 =0.88

(ii) P(A∩B')

P(A∩B') = P(A) – P(A∩B) = 0.7 – 0.02 = 0.68

Combinations

The number of combinations of r objects from n unlike objects is nc_r where

$$nc_r = \frac{n!}{(n-r)!r!}$$

Example 23

Find the value of $5c_2 = \frac{5!}{(5-3)!.2!} \frac{5\ x\ 4\ x\ 3\ x\ 2\ x\ 1}{(3\ x\ 2\ x\ 1)(2\ x\ 1)} = 10$

Example 24

Find the number of ways of selecting a football team from 15 players

$15c_{11} = 1365\ ways$

Example 25

A committee of 4 men and 3 women is to be formed from 10 men and 8 women. In how many ways can the committee be formed?

$10c_4\ x\ 8C_3 = 210\ x\ 56 = 11760\ ways$

Example 26

A group of 9 has to be selected from 8 girls it can consist of either 5 boys or 4 girls of 4 boys and 5 girls. Find how many different groups can be chosen

$10c_5\ x\ 8c_4 + 10c_4\ x\ 8c_5 = 252\ x\ 70 + 210\ x\ 56 = 29400\ ways$

Exercise E

1. A bag contains 5 Pepsi and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type.
2. In a group of 12 international referees, there are 3 from Africa, 4 from Asia and 5 from Europe. To officiate at a tournament 3 referees are chosen at random from the group, find the probability that:
   (i) A referee is chosen from each continent
   (ii) Exactly 2 referees are chosen from Asia
   (iii) 3 referees are chosen from the same continent
3. Box P contains 4 red and 3 green sweets and box Q contains 7 red and 4 green sweets. A box is selected and 2 sweets are randomly picked from it, one a time without replacement. If P is twice likely to be picked as Q, find the probability that both sweets are
   (i) same colour
   (ii) of different colour
   (iii) from P given that they are different colours.
4. A bag contains 20 good and 4 bad oranges. If 5 oranges are selected at random without replacement. Find the probability that 4 are good and the other is bad = 0.456

Solutions to exercise E

1. A bag contains 5 Pepsi and 4 Mirinda bottle tops. Three bottle tops are picked at random from the bag one after the other without replacement. Find the probability that the bottle tops picked are of the same type.
   Solution
   n(S) = 3 tops from the 9 = $9c_3 = 84\ ways$
   n(E) = 3 Pepsi from 5 + 3 Mirinda from 4 = $5c_3\ x\ 4c_0 + \ 5c_0\ x\ 4c_3 = 10 + 4 = 14\ ways$
   P(same type) = $\frac{14}{84} = \frac{1}{6}$
2. In a group of 12 international referees, there are 3 from Africa, 4 from Asia and 5 from Europe. To officiate at a tournament 3 referees are chosen at random from the group, find the probability that:
   (iv) A referee is chosen from each continent
   (v) Exactly 2 referees are chosen from Asia
   (vi) 3 referees are chosen from the same continent

   Solution

   (i) n(S) = 3 referees from 12 = $12c_3 = 220ways$

   n(E) =$3c_1x\ 4c_1\ x\ 5c_1 = 60\ ways$

   P(1 from each) = $\frac{60}{220} = \frac{3}{11}$

   (ii) $n(E) = 4c_2\ x\ 3c_1x\ 5c_0 + 4c_2\ x\ 3c_0x\ 5c_1 = 18 + 30 = 48\ ways$
   P(2 from Asia) =$\frac{48}{220} = \frac{12}{55}$

   (iii) $n(E) = 4c_3\ x\ 3c_0x\ 5c_0 + 4c_0\ x\ 3c_3x\ 5c_0 + \ 4c_0\ x\ 3c_0x\ 5c_3$= 4 + 1 + 10 = 15ways
   P(3 from the same) =$\frac{15}{220} = \frac{3}{44}$

3. Box P contains 4 red and 3 green sweets and box Q contains 7 red and 4 green sweets. A box is selected and 2 sweets are randomly picked from it, one a time without replacement. If P is twice likely to be picked as Q, find the probability that both sweets are
   (iv) same colour
   (v) of different colour
   (vi) from P given that they are different colours.

   Solution

   (i) P(both the same colour) = $\frac{2}{3}\left[\frac{4c_2 \; x \; 3c_0}{7c_2}\right] + \frac{2}{3}\left[\frac{4c_0 \; x \; 3c_2}{7c_2}\right] + \frac{1}{3}\left[\frac{7c_2 \; x \; 4c_0}{11c_2}\right] + \frac{1}{3}\left[\frac{7c_0 \; x \; 4c_2}{11c_2}\right]$
   = 0.4494

   (ii) P(both different colour) =$\frac{2}{3}\left[\frac{4c_1 \; x \; 3c_1}{7c_2}\right] + \frac{1}{3}\left[\frac{7c_1 \; x \; 4c_1}{11c_2}\right] = 0.5506$

   (iii) P(from P/different colour) = $\frac{\frac{2}{3}\left[\frac{4c_1 \; x \; 3c_1}{7c_2}\right]}{0.5506}$ = 0.6919

4. A bag contains 20 good and 4 bad oranges. If 5 oranges are selected at random without replacement. Find the probability that 4 are good and the other is bad

   P(X= 4) = $\frac{20c_4.4c_1}{24c_5}$ = 0.456

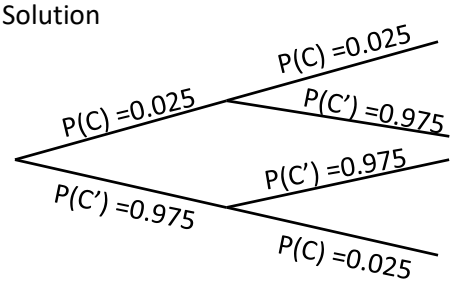
Probability tree diagram

Example 27

A factory makes cakes. When an inspector tests a random sample of cakes, the probability of any cake being contaminated is 0.025. Jane bought two of the cakes made from the factory. Find the probability

(i) both are contaminated
(ii) only one is contaminate

Solution



(i) P(both contaminated) = P(C∩ C) =0.025 x 0.025 = 0.000625

(ii) P(one contaminated) =P(C∩ C') + P(C′ ∩ C) =0.025 x 0.975 + 0.975 x 0.025 = 0.04875

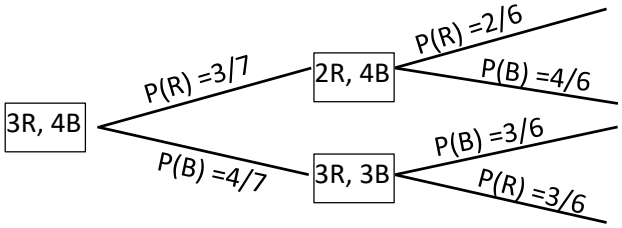
Example 28

A box contains 3 red balls and 4 blue balls. Two balls are randomly drawn one after the other without replacement. Find the probability that

(i) 1st ball is blue
(ii) 2nd ball is red
(iii) 2nd ball is red given that the 1st was blue

(iv) Both balls are of the same colour

(v) Different colour



(i) P(1st is blue) = $\frac{4}{7}x\ \frac{3}{6}+\ \frac{4}{7}x\ \frac{3}{6}$ = 0.5714

(ii) P(2nd is red) = $\frac{3}{7}x\ \frac{2}{6}+\ \frac{4}{7}x\ \frac{3}{6}$ = 0.4286

(iii) P(2nd is red given 1st is blue) = $\frac{3}{6}$ = 0.5

(iv) P(both same colour) = $\frac{3}{7}x\ \frac{2}{6}+\ \frac{4}{7}x\ \frac{3}{6}$ = 0.4286

(v) P(different colour) = $\frac{3}{7}x\ \frac{4}{6}+\ \frac{4}{7}x\ \frac{3}{6}$ = 0.5714

Or = 1- 0.4286 = 0.5714

Example 29

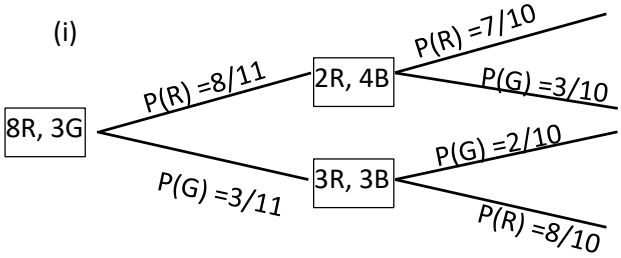
A bag contains 8red pens and 3 green pens. Two pens are randomly picked one after the other, find the probability of drawing two pens of different colours, if the

(i) first pen is not replaced

(ii) first pen is replaced

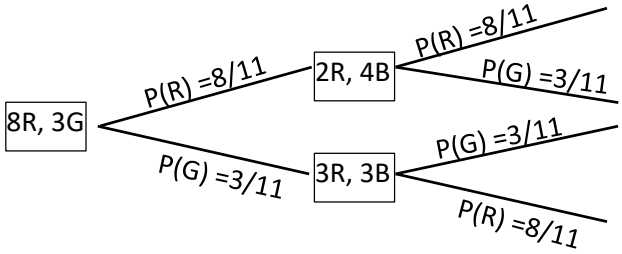
Solution

(i)



P(different colour) = $\frac{8}{11}\ x\ \frac{3}{10}+\ \frac{3}{11}\ x\ \frac{8}{10}=\frac{24}{55}$

(ii)

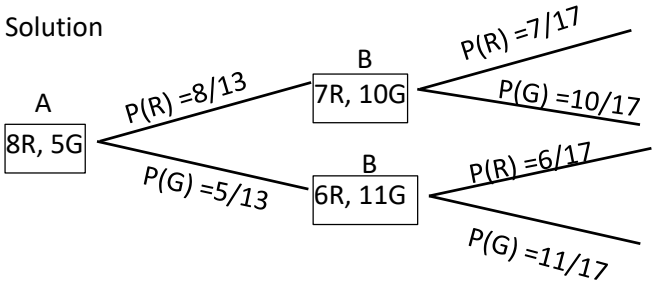


P(different colour) = $\frac{8}{11}\ x\ \frac{3}{11}+\ \frac{3}{11}\ x\ \frac{8}{11}=\frac{48}{121}$

Example 30

Bag A contains 8 red pens and 5 green pens. Bag B contains 6 red and 10 green pens. a pen is randomly picked from bag A and placed in bag B. A pen is then randomly picked from bag B, fins the probability that it will be red

Solution

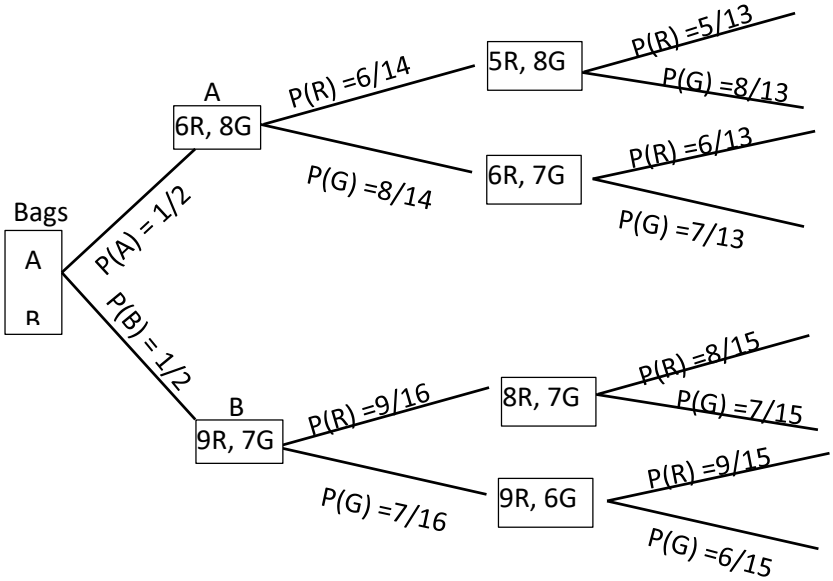


P(Red pen) = $\frac{8}{13}\ x\ \frac{7}{17} + \frac{5}{13}\ x\ \frac{6}{17}$ = $\frac{86}{221}$ =0.3891

Example 31

Box A contains 6 red and 8 green sweets and box B contains 9 red and 7 green sweets. A box is randomly picked and 2 sweets are randomly picked from it, one at a time without replacement. If A is likely to be picked as B, find the probability that both sweets are

(i) same colour
(ii) from A given that are of same colour
(iii) of different colours



(i) P(same colour) = $\frac{1}{2}\ x\ \frac{6}{14}\ x\ \frac{5}{13} + \frac{1}{2}\ x\ \frac{8}{14}\ x\ \frac{7}{13} + \frac{1}{2}\ x\ \frac{9}{16}\ x\ \frac{8}{15} + \frac{1}{2}\ x\ \frac{7}{16}\ x\ \frac{6}{15}$ = 0.4738

(ii) P(both different colour) = 1 - 0.4738 = 0.5262

(iv) P(from A given that are of same colour) = $\frac{\frac{1}{2}x\frac{6}{14}x\frac{5}{13}+\frac{1}{2}x\frac{8}{14}x\frac{7}{13}}{0.4738}$ = 0.4987

Exercise F

1. A box contains 15 red and 5 black balls. Two balls are picked at random one after the other without replacement. find the probability that:
   (i) both are red = $\frac{21}{38}$
   (ii) are different colour = $\frac{15}{38}$
   (iii) both are black, given that the second ball is black
2. A box contains 4 red balls and 6 black balls two balls are randomly drawn one after the other without replacement. Find the probability that
   (i) second ball is red given that the first ball is red = $\frac{1}{3}$
   (ii) both balls are red = $\frac{2}{15}$
   (iii) both balls of different colour = $\frac{8}{15}$
3. A box contains 3 black balls and 5 white balls. Two balls are randomly drawn one after the other without replacement. find the probability
   (i) second ball is white = $\frac{5}{8}$
   (ii) first ball is white given that the second ball is white = $\frac{4}{7}$
4. A box contains 3 red sweets, 8 blue sweets and 7 green sweets. Three sweets are randomly drawn one after the other without replacement. Find that
   (i) all sweets are blue = $\frac{7}{102}$
   (ii) all sweets are red = $\frac{1}{816}$
   (iii) one of each colour = $\frac{7}{34}$
5. A box contains 7 black sweets and 3 white sweets. Three sweets are randomly drawn one after the other with replacement. Find the probability that:
   (i) all three sweets are black = 0.34
   (ii) a white, black and a white sweet in that order are chosen = 0.063
   (iii) two white and one black sweets are drawn = 0.189
   (iv) at least one black sweet drawn = 0.97
6. A coin is tossed four times, find the probability of obtaining less than two heads. = $\frac{5}{16}$
7. The probability that I am late for work is 0.05. Find the probability that on two consecutive mornings:
   (i) I am late for work twice = 0.0025
   (ii) I am late for work once = 0.095
8. A box A contains 3 red balls and 4 brown balls while box B contains 3 red balls and 2 brown balls. A box is drawn at random and one ball is randomly drawn from it. Find the probability that;
   (i) the ball is red = $\frac{18}{35}$
   (ii) the ball came from box A given that it is red = $\frac{5}{12}$
9. a bag contains 10 white and 6 red balls two balls are randomly drawn one after the other without replacement. Find the probability that the second ball drawn is
   (i) red given that the first one was white = 0.4
   (ii) white $= 0.675$

10. Box P contains 2 red balls and 2 blue balls while Box Q contains 2 red balls and 3 blue balls. A box is drawn at random and two balls are randomly drawn from it, one after the other without replacement. Find the probability that the balls are of different colour. = $\frac{19}{30}$.
11. A box contains 4 white balls and 1 black ball. A second box contains 1 white and 4 black balls. A ball is drawn at random from the first bag and put into the second bag, then a ball is taken from the second bag and put into the first bag. Find the probability that a white ball will be picked when a ball is selected from the first bag = $\frac{7}{10}$
12. (a) a box contains 7 red balls and 6 blue balls . Three balls are selected at random without replacement. find the probability that:
    (i) they are the same colour = 0.1923
    (ii) at most two are blue = 0.9301

    (b) Two boxes P and Q contain white and brown cards. P contains 6 white and 4 brown. Q contains 2 white and 3 brown. A box is selected at random and a card is selected at random. Find the probability that

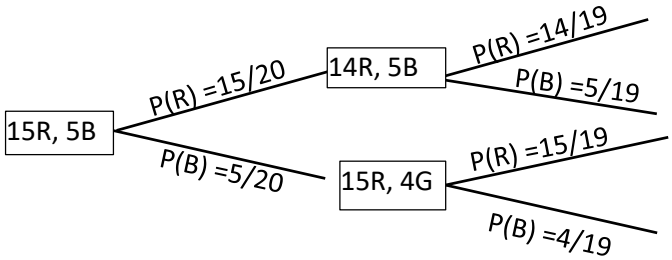
    (i) a brown card is selected = 0.5

    (ii) Box Q is selected given that the card is white = 0.4

13. A box contains two types of balls, red and black. When a ball is picked from the box, the probability that it is red is $\frac{7}{12}$. Two balls are picked at random from the box without replacement. find the probability that
    (i) The second ball is black = $\frac{5}{11}$
    (ii) The first ball is red, given that the second one is black = $\frac{7}{11}$
14. A bag contains 30 white, 20 blue and 20 red balls. Three balls are selected at random without replacement. Find the Probability that the first ball is white and the ball is also white = 0.18
15. A box A contains 4 white and 2 re ball. Box B contains 3 white and 2 red balls. A box is selected at random and two balls are picked one after the other without replacement.
    (i) Find the probability that the two balls picked are red. = 0.1333
    (ii) Given that two white balls are picked, what is the probability that they are from box B = 0.3333

Solutions to exercise F

1. A box contains 15 red and 5 black balls. Two balls are picked at random one after the other without replacement. find the probability that:



    (i) both are red

    P(both red) = P(R∩) = $\frac{15}{20}\ x\ \frac{14}{19} = \frac{21}{38}$

(ii) are different colour

P(different colour) = P(R∩B) + P(B∩R) = $\frac{15}{20}\ x\ \frac{5}{19} + \frac{5}{20}\ x\ \frac{15}{19} = \frac{15}{38}$

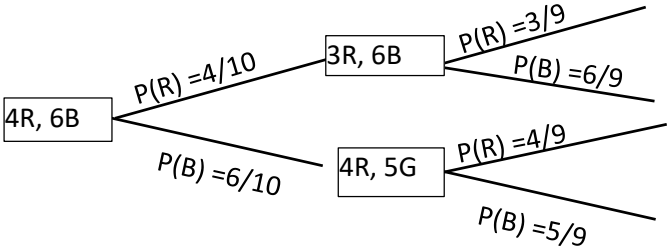
(iii) Both are black, given that the second ball is black = $\frac{20}{83}$

P(2nd black) = P(R∩B) + P(B∩B) = $\frac{15}{20}\ x \frac{5}{19} + \frac{5}{20}\ x \frac{4}{19} = \frac{95}{380}$

P(both black) = P(B∩B) =$\frac{5}{20}\ x \frac{4}{19} = \frac{20}{380}$

P(B/2nd B) = $\frac{20}{380}\ \div\ \frac{95}{380} = \frac{20}{95} = \frac{4}{19}$

2. A box contains 4 red balls and 6 black balls two balls are randomly drawn one after the other without replacement. Find the probability that



(i) second ball is red given that the first ball is red

P(2nd red/1st red) = $\frac{3}{9} = \frac{1}{3}$

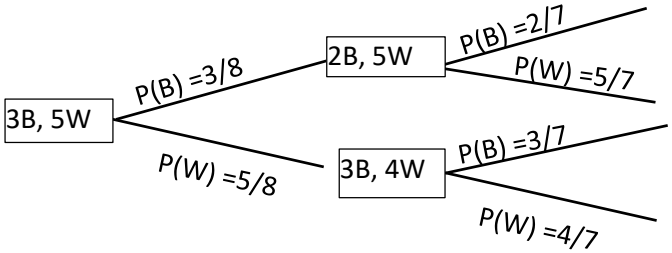
(ii) both balls are red

P(both red) =P(R∩R) = $\frac{4}{10} x \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$

(iii) both balls of different colour

P(both different colour) = P(R∩B) + P(B∩R) = $\frac{4}{10} x \frac{6}{9} + \frac{6}{10} x \frac{4}{9} = \frac{8}{15}$

3. A box contains 3 black balls and 5 white balls. Two balls are randomly drawn one after the other without replacement. find the probability



(i) second ball is white

P(2nd W) = P(B∩W) + P(W∩ W) = $\frac{3}{8}\ x\ \frac{5}{7} + \frac{5}{8}\ x\ \frac{4}{7} = \frac{5}{8}$

(ii) first ball is white given that the second ball is white = $\frac{4}{7}$

4. A box contains 3 red sweets, 8 blue sweets and 7 green sweets. Three sweets are randomly drawn one after the other without replacement. Find that

(i) all sweets are blue

P(all blue) = $\frac{8}{18}\ x \frac{7}{17}\ x \frac{6}{16}\ = \frac{7}{102}$

(ii) all sweets are red

P(all red) =$\frac{3}{18}\ x \frac{2}{17}\ x \frac{1}{16} = \frac{1}{816}$

(iii) one of each colour

P(each colour) = P(R∩B∩G) + P(R∩G∩B) + P(B∩R∩G) + P(B∩G∩R)+ P(G∩B∩R) + P(G∩R∩B)

$$=6\left(\frac{3}{18}\ x\frac{8}{17}\ x\frac{7}{16}\right)=\ \frac{7}{34}$$

5. A box contains 7 black sweets and 3 white sweets. Three sweets are randomly drawn one after the other with replacement. Find the probability that:
   (i) all three sweets are black
   P(all black) = P(B∩B∩B) = 0.7 x 0.7 x 0.7 = 0.343
   (ii) a white, black and a white sweet in that order are chosen
   =0.3 x 0.7 x 0.3= 0.063
   (iii) two white and one black sweets are drawn
   P(2W, 1B) = P(W∩W∩B) + P(W∩B∩W) + P(B∩W∩W)
   = 3(0.3 x 0.3 x 0.7) = 0.189
   (iv) at least one black sweet drawn
   P(at least 1B) = [P(W∩W∩B) + P(W∩B∩W) + P(B∩W∩W)]+ P(W∩B∩B) + P(B∩B∩W) + P(B∩W∩B) + P(B∩B∩B)
   = 0.189 +3(0.3 x 0.7 x 0.7) + 0.7 x 0.7 x 0.7
   = 0.973
6. A coin is tossed four times, find the probability of obtaining less than two heads.
   P(less than 2H) = P(H∩T∩T∩T) + P(T∩H∩T∩T) + P(T∩T∩H∩T)+ P(T∩T∩T∩H) + P(T∩T∩T∩T)
   $= 5\left(\frac{1}{2}\ x\frac{1}{2}\ x\frac{1}{2}\ x\frac{1}{2}\right)\ =\ \frac{5}{16}$
7. The probability that I am late for work is 0.05. Find the probability that on two consecutive mornings:
   (i) I am late for work twice
   = P(L∩L) = 0.05 x 0.05= 0.0025
   (ii) I am late for work once
   = P(L∩L') + P(L'∩L) = 2(0.05 x 0.95) = 0.095
8. A box A contains 3 red balls and 4 brown balls while box B contains 3 red balls and 2 brown balls. A box is drawn at random and one ball is randomly drawn from it. Find the probability that;
   (i) the ball is red
   P(R) = P(A∩R) + P(B ∩ R) = $\frac{1}{2}\ x\ \frac{3}{7}+\frac{1}{2}\ x\ \frac{3}{5}=\frac{3}{14}+\ \frac{3}{10}=\ \frac{18}{35}$
   (ii) the ball came from box A given that it is red
   P(A/R) = $\frac{\mathrm{P(A\cap R)}}{P(R)}=\ \frac{3}{14}\ x\ \frac{35}{18}=\ \frac{5}{12}$
9. A bag contains 10 white and 6 red balls two balls are randomly drawn one after the other without replacement. Find the probability that the second ball drawn is
   (i) red given that the first one was white
   = $\frac{6}{15}=\ 0.4$
   (ii) white
   P(2nd W) = P(W∩W) + P(R∩W) = $\frac{10}{16}\ x\ \frac{9}{15}+\frac{6}{16}\ x\ \frac{10}{15}=\ 0.675$
10. Box P contains 2 red balls and 2 blue balls while Box Q contains 2 red balls and 3 blue balls. A box is drawn at random and two balls are randomly drawn from it, one after the other without replacement. Find the probability that the balls are of different colour. = $\frac{19}{30}$.
    P(different colour) = P(P∩R∩B) + P(P∩B∩R) + P(Q∩R∩B)+ P(Q∩R∩B)

$$= \frac{1}{2} x \frac{2}{4}\ x \frac{2}{3} + \frac{1}{2} x \frac{2}{3}\ x \frac{2}{4} + \frac{1}{2} x \frac{2}{5}\ x \frac{3}{4} + \frac{1}{2} x \frac{3}{5}\ x \frac{2}{4}$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{3}{20} + \frac{3}{20} = \frac{10+10+9+9}{60} = \frac{19}{30}$$

11. A box contains 4 white balls and 1 black ball. A second box contains 1 white and 4 black balls. A ball is drawn at random from the first bag and put into the second bag, then a ball is taken from the second bag and put into the first bag. Find the probability that a white ball will be picked when a ball is selected from the first bag = $\frac{7}{10}$

Let A = 1st bag and B = 2nd bag

W_1 = a white ball drawn from A for the first time

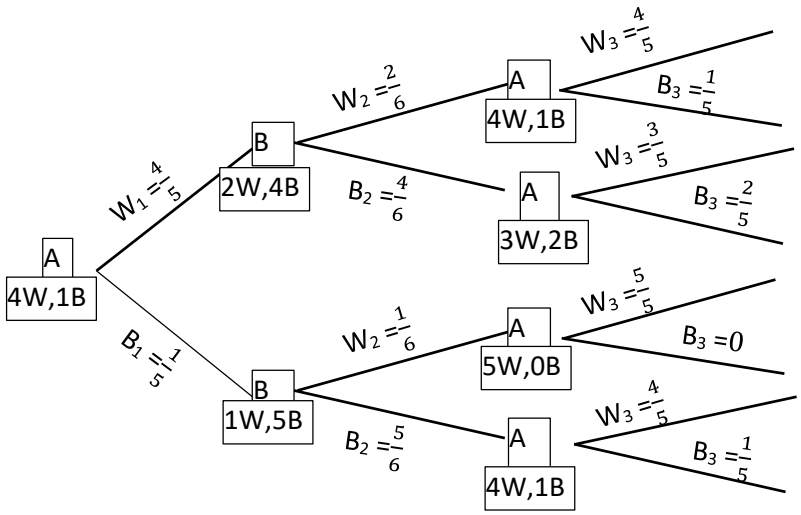
B_1 = a black ball draw from A for the first time

W_2 = a white ball drawn from A for the second time

B_2 = a black ball draw from A for the second time

W_3 = a white ball drawn from A for the second time

B_3 = a black ball draw from A for the second time



Let W = event that a white ball is then selected from Bag A

P(W) = P($W_1 \cap W_2 \cap W_3$) + P($W_1 \cap B_2 \cap W_3$)+ P($B_1 \cap W_2 \cap W_3$)+ P($B_1 \cap B_2 \cap W_3$)

$$= \frac{4}{5}\ x \frac{2}{6}\ x \frac{4}{5} + \frac{4}{5}\ x \frac{4}{6}\ x \frac{3}{5} + \frac{1}{5}\ x \frac{1}{6}\ x \frac{5}{5} + \frac{1}{5}\ x \frac{5}{6}\ x \frac{4}{5} = \frac{32}{150} + \frac{48}{150} + \frac{5}{150} + \frac{20}{150} = \frac{105}{150} = \frac{7}{10}$$

12. (a) A box contains 7 red balls and 6 blue balls. Three balls are selected at random without replacement. find the probability that:

(i) they are the same colour

P(same colour) = P(R∩R∩R) + P(B∩B∩B) = $\frac{7}{13}\ x \frac{6}{12}\ x \frac{5}{11} + \frac{6}{13}\ x \frac{5}{12}\ x \frac{4}{11} = 0.1923$

(ii) at most two are blue

P(at most 2B) = P(R∩R∩R) + P(R∩R∩B)+ P(B∩R∩R) + P(R∩B∩R)+ P(R∩B∩B) + P(B∩B∩R) + P(B∩R∩B)

= 1- P(B∩B∩B) = 1 - $\frac{6}{13}\ x \frac{5}{12}\ x \frac{4}{11}$ = 0.9301

(b) Two boxes P and Q contain white and brown cards. P contains 6 white and 4 brown. Q contains 2 white and 3 brown. A box is selected at random and a card is selected at random. Find the probability that

(i) a brown card is selected

$P(B) = P(P \cap B) + P(Q \cap B) = \frac{1}{2}\ x\ \frac{4}{10} + \frac{1}{2}\ x\ \frac{3}{5} = 0.5$

(ii) Box Q is selected given that the card is white

$P(W) = P(P \cap W) + P(Q \cap W) = \frac{1}{2}\ x\ \frac{6}{10} + \frac{1}{2}\ x\ \frac{2}{5} = 0.5$

$P(Q/W) = \frac{P(Q \cap W)}{P(W)} = \frac{0.2}{0.5} = 0.4$

13. A box contains two types of balls, red and black. When a ball is picked from the box, the probability that it is red is $\frac{7}{12}$. Two balls are picked at random from the box without replacement. find the probability that

(i) The second ball is black

$P(R) = \frac{7}{12}$; $P(B) = \frac{5}{12}$

$P(B) = P(R \cap B) + P(B \cap B) = \frac{7}{12}\ x\ \frac{5}{11} + \frac{5}{12}\ x\ \frac{4}{11} = \frac{5}{11}$

(ii) The first ball is red given that the second one is black

$P(1^{st}\ B/B) = \frac{P(R \cap B)}{P(B)} = \frac{35}{121} \div \frac{5}{11} = \frac{35}{121} x \frac{11}{5} = \frac{7}{11}$

14. A bag contains 30 white, 20 blue and 20 red balls. Three balls are selected at random without replacement. Find the Probability that the first ball is white and the third ball is also white

$P = P(W \cap B \cap W) + P(W \cap R \cap W) + P(W \cap W \cap W) = \frac{30}{70}\ x\ \frac{20}{69}\ x\ \frac{29}{68} + \frac{30}{70}\ x\ \frac{20}{69}\ x\ \frac{29}{68} + \frac{30}{70}\ x\ \frac{29}{69}\ x\ \frac{28}{68}$

$= 0.18$

15. A box A contains 4 white and 2 re ball. Box B contains 3 white and 3 red balls. A box is selected at random and two balls are picked one after the other without replacement.

(i) Find the probability that the two balls picked are red.

$P(R) = P(A \cap R_1 \cap R_2) + P(B \cap R_1 \cap R_2) = \frac{1}{2} x\ \frac{2}{6}\ x\ \frac{1}{5} + \frac{1}{2} x\ \frac{3}{6}\ x\ \frac{2}{5} = \frac{8}{60} = 0.1333$

(ii) Given that two white balls are picked, what is the probability that they are from box B

$P(W) = P(A \cap W_1 \cap W_2) + P(B \cap W_1 \cap W_2) = \frac{1}{2} x\ \frac{4}{6}\ x\ \frac{3}{5} + \frac{1}{2} x\ \frac{3}{6}\ x\ \frac{2}{5} = \frac{18}{60}$

$P(B/W) = \frac{P(B \cap W1 \cap W2)}{P(W)} = \frac{6}{60} \div \frac{18}{60} = \frac{6}{18} = \frac{1}{3}$

$= 0.3333$

Conditional probability using a tree diagram/ Baye's Rule

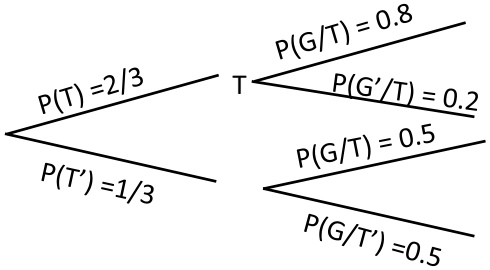
Example 32

During planting season a farmer treats $\frac{2}{3}$ of his seeds and $\frac{1}{3}$ of the seed are left untreated. The seeds which are treated have a probability of germinating of 0.8 while the untreated seeds have a probability of germinating of 0.5, find the probability that a seed selected at random

(i) will germinate

(ii) had been treated, given that it had germinated.

Solution



(i) P(G) = P(T∩ G) + P(T'∩G) = $\frac{2}{3}\ x\ 0.8 + \frac{1}{3}\ x\ 0.5 = 0.7$

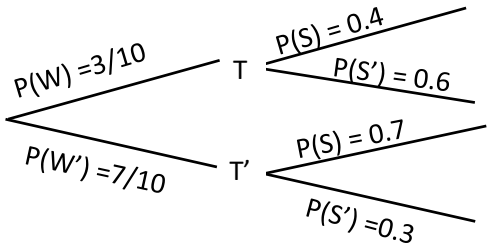
(ii) P(T/G) = $\frac{P(T \cap G)}{P(G)} = \frac{\frac{2}{3} x\ 0.8}{0.7}$ = 0.762

Example 33

The probability that a golfer hits the ball on the green if it is windy as he strikes the ball is 0.4 and the corresponding probability if it is not windy as he strikes the ball is 0.7. The probability that the wind blow as he strikes the ball is $\frac{3}{10}$. Find the probability that

(i) He hits the ball on the green

(ii) It was not windy, given that he does not hit the ball on the green



(i) P(S) = P(W∩S) + P(W'∩S) = 0.3 x 0.4 + 0.7 x 0.7 = 0.61

(ii) P(W'/S') = $\frac{P(W' \cap S')}{P(S')} = \frac{\frac{7}{10} x\ 0.3}{(1-0.61)}$= 0.5385

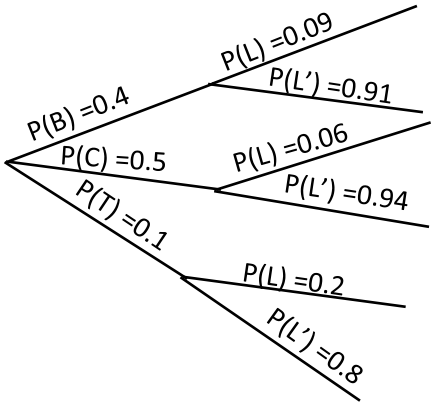
Example 33

When students were to go for a geography tour, the school hired three different types of vehicles, buses, coaster and taxis. Of the hiring's 40% were buses, 50% were coasters and 10% were taxis. For bus hired 9% arrive late, the corresponding percentages for a coaster and a taxi being 6% and 20% respectively. Find the probability that the next vehicle hired

(i) will be a bus and will not arrive late

(ii) will arrive late

(iii) will be a coaster given that it will arrive late



(i) P(B∩ L') = 0.4 x 0.91 = 0.364

(ii) P(L) = P(B∩L) + P(C∩L) + P(T∩L)
= 0.4 x 0.09 + 0.5 x 0.06 + 0.1 x 0.2
= 0.086

(iii) P(C/L) = $\frac{P(C \cap L)}{P(L)} = \frac{(0.5\ x\ 0.06)}{0.086}$ = 0.3488

Revision exercise G

1. 55% of the teachers at a certain school are male. 30% of the male teachers are science teachers and 5% of female teachers teach sciences. If a teacher is selected at random, what is the probability that
   (i) Teaches sciences = 0.1875
   (ii) She is a female given that she does not teach sciences. = 0.5262
2. The probability that a golfer hits the ball on the green if it is windy as he strikes the ball is 0.4 and the corresponding probability if it is not windy as he strikes the ball is 0.7. The probability that the wind blows as he strikes the ball is $\frac{3}{10}$. Find the probability that:
   (i) he hits the ball on the green= 0.61
   (ii) it was not windy, given that he does not hit the ball on the green = 0.5385
3. In a restaurant, 40% of the customer's order for chicken. If a customer orders, the probability that he will take juice is 0.6. If he does not order chicken, the probability that he will take juice is 0.3. Find the probability a customer picked at random will order
   (i) Chicken and juice = 0.24
   (ii) Juice = 0.42
4. In a factory, there are two different machines A and B. Items are produced from A and B with respective probabilities of 0.2 and 0.8. it was established that 5% and 8%produced by A and B respectively are defective. If one item is selected randomly, find the probability that
   (i) it is defective = 0.074
   (ii) it is produced by A given that t is defective = 0.1351
5. Data from electoral commission showed that in the previous electron, of all the Kampala parliamentary contestants, 70% were N.R.M, 20% were F.D.C and 10% were independents. 5% N.R.M contestants won elections, 95% of F.D.C contestants won elections and 25% of the independent contestants won elections. If a contestant is chosen at random, find the probability that the person
   (i) won election = 0.25
   (ii) an F. D.C won elections = 0.19
6. A student travels to school by route A or route B. The probability that she uses route A is 0.25. The probability that she is late to school if she uses route A is $\frac{2}{3}$ and the corresponding probability if she uses route B is $\frac{1}{3}$.
   (i) Find the probability that she will be late to school = $\frac{5}{12}$.
   (ii) Given that she is late, what is the probability that she used route B = $\frac{3}{5}$.
7. A student is to travel to school for an interview. The probability that he will be in time for the interview when he travels taxi and boda respectively are 0.1 and 0.2. The probability that he will travel by taxi and boda are 0.6 and 0.4 respectively.
   (i) find the probability that he will be on time = 0.14
   (ii) given that he is not on time, what is the probability that he travelled by boda = 0.372

8. Of the group of students studying A-level in a school, 56% are boys and 44% are girls. The probability that a boy of this group is studying chemistry is $\frac{1}{5}$ and the probability that a girl of this group is studying chemistry is $\frac{1}{11}$:
   (i) Find the probability that a student selected at random from this group is a girl studying chemistry. = $\frac{1}{25}$

(ii) Find the probability that a student selected at random from this group is not studying chemistry. = $\frac{106}{125}$

(iii) Find the probability that a chemistry student selected at random from this group is male = $\frac{14}{19}$

9. When a school wants to buy chalk, the school phones three suppliers A, B or C of the phone calls to them. 30% are to A, 10% to B and 60% to C. the percentage of occasions when the supplies deliver chalk after a phone call to them are 20% for A, 6% for B and 9% for C.
   (i) Find the probability that the suppler phoned will not deliver chalk on the day if phoning.=0.88
   (ii) Given that the school phones a supplier and the supplier can deliver chalk that, find the probability that the school phoned supplier B. = 0.05
10. In Kampala city, 30% of the people are F.D.C, 50% are N.R.M and 20% are independent. Records show that in previous elections, 65% of the F.D.C voted, 85% of the N.R.M voted and 50% of the independent voted. A person is randomly selected from the city.
   (i) Find the probability that the person voted= 0.72
   (ii) Given that the person didn't vote, determine the probability that is an F.D.C = 0.375
11. A shop stocks two brands of toothpaste, Colgate toothpaste and Fresh up toothpaste, and two sizes, large and small. Of the stock 70% is Colgate and 30% is Fresh up. Of the Colgate, 30% are small size and of Fresh up 40% are small size. Find the probability that;
   (i) A toothpaste chosen at random from the stalk will be of small size = 0.33
   (ii) Small toothpaste chosen at random from the stalk will be of Colgate. = $\frac{7}{11}$
12. At a bus park, 60% of the buses are of Teso coaches, 25% are Kakise buses and the rest are Y.Y buses. Of the Teso coaches 50% have TVs, while for the Kakise and Y.Y buses only 55 and 15 have TVs respectively. If a bus is selected at random from the park, determine the probability that
   (i) Has a TV = 0.314
   (ii) Kakise bus is selected that it has TV= 0.0398
13. On a certain day, fresh fish from lakes, Kyoga, Victoria, Albert and George were supplied to a market in ratio 30%, 40%, 20% and 10% respectively. Each lake had an estimated ratios of poisoned fish of 2%, 3%, 3% and 1% respectively. If a health inspector picked a fish at random
   (i) What is the probability that the fish was poisoned = 0.025
   (ii) Given that the fish was poisoned, what is the probability that it was from L. albert. = 0.24
14. The chance that a person picked from a Kampala street is employed is 30 in every 48. The probability that a person is a university graduate is employed is 0.6. Find
   (i) the probability that the person picked at random from the street is a university graduate and is employed = 0.375
   (ii) number of people that are not university graduates and are employed from a group of 120 people. = 30
15. A mobile phone dealer imports Nokia and Motorola phones. In a given consignment, 55% were Nokia and 45% were Motorola phones. The probability that a Nokia phone is defective is 4%. The probability that a Motorola phone is defective is 6%. A phone is picked at random from the consignment. Determine that it is
   (i) defective = 0.049
   (ii) a Motorola given that it is defective = 0.551

Solutions to revision exercise G

1. 55% of the teachers at a certain school are male. 30% of the male teachers are science teachers and 5% of female teachers teach sciences. If a teacher is selected at random, what is the probability that

   P(M) = 0.55, P(F) =0.45, P(S/M) = 0.3, P(S/F) =0.05, P(S'/F) =0.95

   (i) Teaches sciences

   P(S) = P(M∩S) + P(F∩S) = 0.55 x 0.3 + 0.45 x 0.05 = 0.1875

   (ii) She is a female given that she does not teach sciences.

   P(S') = 1 – P(S) = 1- 0.1875 = 0.8125

   P(F/S') = $\frac{P(F \cap S')}{P(S')} = \frac{0.45\ x\ 0.95}{0.8125} = 0.5262$

2. The probability that a golfer hits the ball on the green if it is windy as he strikes the ball is 0.4 and the corresponding probability if it is not windy as he strikes the ball is 0.7. The probability that the wind blows as he strikes the ball is $\frac{3}{10}$. Find the probability that:

   Summary

   P(S/W) = 0.4, P(S/W') = 0.7, P(W) = 0.3, P(W') = 0.7

   (i) he hits the ball on the green

   P(S) = P(S∩W) + P(S∩W) = 0.4 x 0.3 + 0.7 x 0.7= 0.61

   (ii) it was not windy, given that he does not hit the ball on the green

   P(S') = 1- 0.61 = 0.39

   P(W'/S') = $\frac{P(W' \cap S')}{P(S')} = \frac{0.7\ x\ 0.3}{0.39} = 0.5385$

3. In a restaurant, 40% of the customer's order for chicken. If a customer orders, the probability that he will take juice is 0.6. If he does not order chicken, the probability that he will take juice is 0.3. Find the probability a customer picked at random will order

   P(C) = 0.4, P(C') = 0.6, P(J/C) = 0.6, P(J/C') = 0.3

   (i) Chicken and juice

   P(C∩J) = 0.4 x 0.6 = 0.24

   (ii) Juice

   P(J) = P(C∩J) + P(C'∩J) = 0.24 + 0.6 x 0.3= 0.42

4. In a factory, there are two different machines A and B. Items are produced from A and B with respective probabilities of 0.2 and 0.8. it was established that 5% and 8%produced by A and B respectively are defective. If one item is selected randomly, find the probability that

   Summary

   P(A) = 0.2, P(B) = 0.8, (D/A) = 0.05, P(D/B) 0.08

   (i) it is defective

   P(D) = P(A∩D) + P(B∩B) = 0.2 x 0.05 + 0.8 x 0.08 =0.074

   (ii) it is produced by A given that t is defective

   P(A/D) = $\frac{P(A \cap D)}{P(D)} = \frac{0.2\ x\ 0.05}{0.074} = 0.1351$

5. Data from electoral commission showed that in the previous electron, of all the Kampala parliamentary contestants, 70% were N.R.M, 20% were F.D.C and 10% were independents. 5% N.R.M contestants won elections, 95% of F.D.C contestants won elections and 25% of the independent contestants won elections. If a contestant is chosen at random, find the probability that the person

   Summary

   P(N) = 0.7, P(F) = 0.2, P(I) = 0.1, P(W/N) = 0.05, P(W/F) = 0.95, P(W/I) = 0.25

   (i) won election

P(W) = P(N∩W) + P(F∩W) + P(I∩W) = 0.7 x 0.05 + 0.2 x 0.95 + 0.1 x 0.25= 0.25

(ii) an F. D.C won elections = 0.2 x 0.95 = 0.19

6. A student travels to school by route A or route B. The probability that she uses route A is 0.25. The probability that she is late to school if she uses route A is $\frac{2}{3}$ and the corresponding probability if she uses route B is $\frac{1}{3}$.

P(A) = 0.25, P(B) = 0.75, P(L/A) = $\frac{2}{3}$, P(L/B) = $\frac{1}{3}$

(i) Find the probability that she will be late to school.

P(L) = P(A∩L) + P(B∩L) = 0.25 x $\frac{2}{3}$ + 0.75 x $\frac{1}{3}$ = 0.417

(ii) Given that she is late, what is the probability that she used route B

P(L/B) = $\frac{\text{P(B∩L)}}{P(L)} = \frac{0.25}{0.417}$ = 0.6

7. A student is to travel to school for an interview. The probability that he will be on time for the interview when he travels taxi and boda respectively are 0.1 and 0.2. The probability that he will travel by taxi and boda are 0.6 and 0.4 respectively.

Summary

P(T) = 0.6, P(B) = 0.4, P(t/T) = 0.1, P(t'/T) = 0.9, P(t/B) = 0.2, P(t'/B) = 0.8

(i) find the probability that he will be on time

P(on time) = P(t) = P(T∩t) + P(B∩t) = 0.6 x 0.1 + 0.4 x 0.2= 0.14

(ii) given that he is not on time, what is the probability that he travelled by boda

Probability that he s not on time = 1- P(t) = 1 - 0.14 = 0.86

P(B/t') = $\frac{P(t'\cap B)}{P(t')} = \frac{0.4\ x\ 0.8}{0.86} = 0.372$

8. Of the group of students studying A-level in a school, 56% are boys and 44% are girls. The probability that a boy of this group is studying chemistry is $\frac{1}{5}$ and the probability that a girl of this group is studying chemistry is $\frac{1}{11}$:

Summary

P(B) = 0.56, P(G) = 0.44, P(C/B) = 0.2, P(C/G) = $\frac{1}{11}$

(i) Find the probability that a student selected at random from this group is a girl studying chemistry.

P(G∩C) = P(G)x P(G/C) = 0.44 x $\frac{1}{11}$ = 0.04

(ii) Find the probability that a student selected at random from this group is not studying chemistry.

P(C') = 1- P(C) = 1 – 0.152= 0.848

(iii) Find the probability that a chemistry student selected at random from this group is male = $\frac{14}{19}$

P(B/C) = $\frac{\text{P(B∩C)}}{P(C)} = \frac{0.56\ x\ 0.2}{0.152} = 0.73684$

9. When a school wants to buy chalk, the school phones three suppliers A, B or C of the phone calls to them. 30% are to A, 10% to B and 60% to C. the percentage of occasions when the supplies deliver chalk after a phone call to them are 20% for A, 6% for B and 9% for C.

Summary

P(A) = 0.3, P(B) = 0.1, P(C) = 0.6, P(D/A) = 0.2, P(D'/A) = 0.8, P(D/B) = 0.06, P(D'/B) = 0.94,

P(D/C) = 0.09, P(D'/C) = 0.91

(i) Find the probability that the suppler phoned will not deliver chalk on the day if phoning.
P(D') = P(D'∩A) + P(D'∩B) + P(D'∩C) = 0.3 x 0.8 + 0.1 x 0.94 + 0.6 x 0.91 =0.88

(ii) Given that the school phones a supplier and the supplier can deliver chalk that, find the probability that the school phoned supplier B.
P(D) = P(D∩A) + P(D∩B) + P(D∩C) = 0.3 x 0.2 + 0.1 x 0.06 + 0.6 x 0.09 = 0.12
Or = 1- P(D') = 1- 0.88 = 0.12
P(B/D) = $\frac{\mathrm{P(D\cap B)}}{P(D)} = \frac{0.1\ x\ 0.06}{0.12} = 0.05$

10. In Kampala city, 30% of the people are F.D.C, 50% are N.R.M and 20% are independent. Records show that in previous elections, 65% of the F.D.C voted, 85% of the N.R.M voted and 50% of the independent voted. A person is randomly selected from the city.
Summary
P(F) = 0.3, P(N) = 0.5, P(I) = 0.2, P(V/F) = 0.65, P(V/N) = 0.85, P(V/I) =0.5
(i) Find the probability that the person voted
P(V) = P(V∩F) + P(V∩N) + P(V∩I) = 0.3 x 0.65 + 0.5 x 0.85 + 0.2 x 0.5 =0.72
(ii) Given that the person didn't vote, determine the probability that is an F.D.C
P(V') = P(V'∩F) + P(V'∩N) + P(V'∩I) = 0.3 x 0.35 + 0.5 x 0.15 + 0.2 x 0.5= 0.28
P(v'/F) = $\frac{\mathrm{P(V'\cap F)}}{P(V')} = \frac{0.3\ x\ 0.35}{0.28} = 0.375$
11. A shop stocks two brands of toothpaste, Colgate toothpaste and Fresh up toothpaste, and two sizes, large and small. Of the stock 70% is Colgate and 30% is Fresh up. Of the Colgate, 30% are small size and of Fresh up 40% are small size. Find the probability that;
Summary
P(C) = 0.7 P(F) = 0.3, P(s/C) = 0.3, P(s/F) = 0.4
(i) A toothpaste chosen at random from the stalk will be of small size
P(s) = P(C∩s) + P(F∩s) = 0.7 x 0.3 + 0.3 x 0.4 = 0.33
(ii) Small toothpaste chosen at random from the stalk will be of Colgate.
P(C/s) = $\frac{P(C\cap s)}{P(s)} = \frac{0.21}{0.33} = \frac{7}{11}$
12. At a bus park, 60% of the buses are of Teso coaches, 25% are Kakise buses and the rest are Y.Y buses. Of the Teso coaches 50% have TVs, while for the Kakise and Y.Y buses only 55 and 15 have TVs respectively. If a bus is selected at random from the park, determine the probability that:

Let T, K, Y stand for Teso, Kakise and Y, Y buses

P(T) = 0.6 P(K) = 0.25 and P(Y) = 0.15

P(R/T) = 0.5, P(R/K) = 0.05, P(R/Y) = 0.01

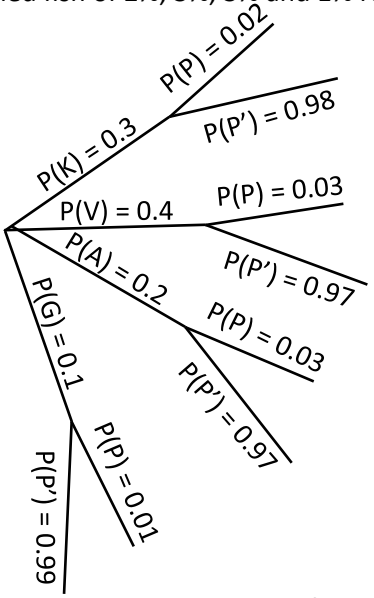
(i) Has a TV = 0.0315
P(t) = P(T∩t) + P(K∩t) + P(Y∩t)
= P(R/T).P(T) + P(R/K).P(K) . P(R/Y).P(Y)
= 0.6 x 0.5 + 0.25 x 0.05 + 0.15 x 0.01

= 0.314

(ii) Kakise bus is selected that it has TV

P(K/T) = $\frac{P(K \cap T)}{P(T)} = \frac{0.0125}{0.314}$ = 0.0398

13. On a certain day, fresh fish from lakes, Kyoga, Victoria, Albert and George were supplied to a market in ratio 30%, 40%, 20% and 10% respectively. Each lake had an estimated ratios of poisoned fish of 2%, 3%, 3% and 1% respectively. If a health inspector picked a fish at random



(i) What is the probability that the fish was poisoned

P(poisoned) = P(K∩P) + P(V∩P) + P(A∩P) + P(G∩P)

= 0.3 x 0.02 + 0.4 x 0.03 + 0.2 x 0.03 + 0.1 x 0.01

= 0.025

(ii) Given that the fish was poisoned, what is the probability that it was from L. albert. = 0.24

P(A/P) = $\frac{\mathrm{P(A \cap P}}{P(P)} = \frac{\mathrm{P(A \cap P)}}{P(P)} = \frac{0.006}{0.025}$ = 0.24

14. The chance that a person picked from a Kampala street is employed is 30 in every 48. The probability that a person is a university graduate is employed is 0.6. Find

P(E) = $\frac{30}{48} = 0.625$ and P(G/E) = 0.6

(i) the probability that the person picked at random from the street is a university graduate and is employed

P(G/E) = $\frac{P(G \cap E)}{P(E)}$

P(G∩E) = 0.625 x 0.6= 0.375

(ii) number of people that are not university graduates and are employed from a group of 120 people. = 30

From set theory

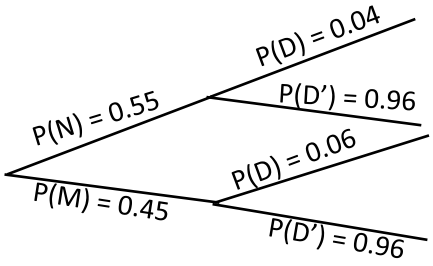
P(E) = P(G∩E) + P(G'∩E)

P(G'∩E) = 0.625 – 0.375 = 0.25

The number = 0.25 x 120 = 30

15. A mobile phone dealer imports Nokia and Motorola phones. In a given consignment, 55% were Nokia and 45% were Motorola phones. The probability that a Nokia phone is defective is 4%.

The probability that a Motorola phone is defective is 6%. A phone is picked at random from the consignment. Determine that it is



(i) defective = P(N∩D) + P(M∩D) = 0.55 x 0.04 + 0.45 x 0.06 = 0.049

(ii) a Motorola given that it is defective

P(M/D) = $\frac{P(M \cap D)}{P(D)} = \frac{0.45\ x\ 0.06}{0.049}$ = 0.551

Thank you

Dr. Bbosa Science

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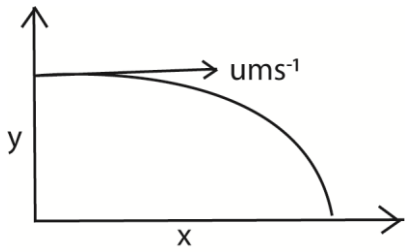
Projectile motion

This is the motion of a body which after being given an initial velocity moves freely under the influence of gravity.

Terms used in projectiles

1. ***Angle of projection:*** is the angle the initial velocity makes with the horizontal
2. ***Maximum height/ greatest height*** is the greatest height reached by projectile
3. ***Time of flight (T)*** is the time taken for projectile to complete motion. Note: the time of flight is twice the time to maximum height.
4. ***Range, R:*** is the horizontal distance covered by projectile
5. ***Maximum range (Rmax)*** is the greatest horizontal distance covered
6. ***Trajectory;*** is the path described by a projectile.

A. An object projected horizontally from a height above the ground.



Horizontal motion: Ux = u, a = 0

x = ut

Vertical motion: Uy = 0, a = $-9.8ms^{-2}$

$s = ut + \frac{1}{2} at^2$

$-y = -\frac{1}{2} gt^2$

V = u + at

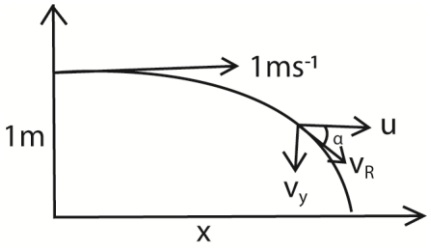
Vy = -gt

Example 1

A ball rolls of the edge of a table top 1m high above the floor with horizontal velocity 1ms-1. find:

(i) time taken to hit the floor
(ii) horizontal distance covered
(iii) the velocity when it hits the floor.

Solution



(i) vertical motion: u = 1ms^{-1}, θ = 0
y = 1m below the point of projection
y = ½ gt^2
-1 = $\frac{1}{2}\ x - 9.8\ t^2$

t = 0.4518s

(ii) x = ut = 1 x 0.4518 = 0.4518m

(iii) v_x = 1ms^{-1}

v_y = -gt = -9.8 x 0.4518 = -4.428ms^{-1}

$v_R = \sqrt{1^2 + (-4.428)^2}$ = 4.54ms^{-1}

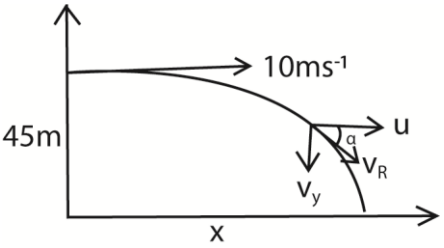
$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{4.428}{1}\right)$ = 77.3^0

Example 2

A ball is thrown forward horizontally from the top of a cliff with a velocity of 10ms^{-1}. The height of a cliff above the ground is 45m. Calculate

(i) time to reach the ground
(ii) distance from the cliff where the ball hits the ground
(iii) velocity and direction of the ball just before it hits the ground

Solution



(ii) vertical motion: u = 10ms^{-1}, θ = 0
y = 45m below the point of projection
y = ½ gt^2
-45 = $\frac{1}{2}\ x - 9.8\ t^2$

t = 3.03s

(ii) x = ut = 10 x 3.03 = 30.3m

(iii) v_x = 10ms^{-1}

v_y = -gt = -9.8 x 3.03 = -29.694ms^{-1}

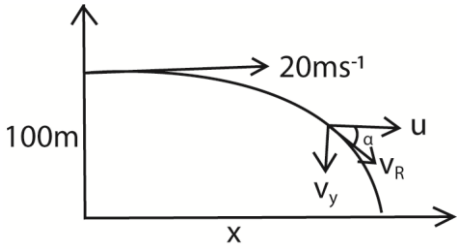
$v_R = \sqrt{10^2 + (-29.694)^2}$ = 31.33ms^{-1}

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{29.694}{10}\right)$ = 71.4^0

Example 3

An object is projected horizontally at a speed of 20ms^{-1} from a height of 100m. Find

(i) the time of flight



y = ½ gt^2
-100 = $\frac{1}{2}\ x - 9.8\ t^2$
t = 4.52ms^{-1}

(ii) the horizontal range
x = ut = 20 x 4.52 = 90.4m

(iii) its velocity on reaching the ground

$v_x = 20ms^{-1}$

$v_y = -gt = -9.8 \times 4.52 = -32.7ms^{-1}$

$v_R = \sqrt{20^2 + (-32.7)^2} = 38.33ms^{-1}$

$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{32.7}{20}\right) = 58.5^0$

Example 4

At time t = 0, a particle is projected with a velocity of 3ms-1 from a point with position vector (5i + 25j)m. Find the

(i) speed and direction of the particle when t = 2s

$v_x = u = 3ms^{-1}$

$v_y = gt = -9.8 \times 2 = -19.6ms^{-1}$

$v_R = \sqrt{3^2 + (-19.6)^2} = 19.83ms^{-1}$

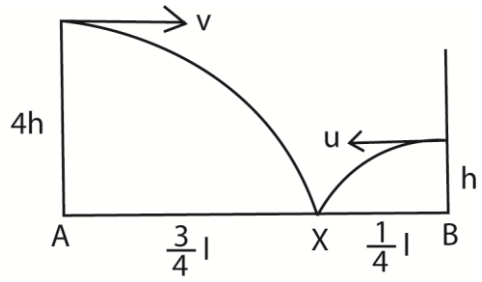
$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{19.6}{3}\right) = 81.3^0$

(ii) Position vector of the particle when t = 2s

$$P_{(t=t)} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} + \begin{pmatrix} ut \\ -\frac{1}{2}gt^2 \end{pmatrix} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} + \begin{pmatrix} 3\ x\ 2 \\ -\frac{1}{2}x9.8\ x\ 2^2 \end{pmatrix} = \begin{pmatrix} 11 \\ 5.4 \end{pmatrix} m$$

Example 5

A and B are two points on a level ground. A vertical tower of height 4h has its base at A and vertical tower of height h has its bas at B. When a stone is thrown horizontally with speed v from the top the taller tower towards the smaller tower, it lands at point X where AX = $\frac{3}{4}AB$. When a stone is thrown horizontally with speed u from the top of the smaller tower towards the taller tower, it also lands at the point A. Show that 3u = 2V



For A Vertical motion: $y = \frac{1}{2}gt^2$

$$4h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{8h}{g}}$$

Horizontal motion: x = vt

$$\frac{3}{4}l = vt$$

$$l = \frac{4}{3}vt = \frac{4}{3}v\sqrt{\frac{8h}{g}} \ldots . (i)$$

For B Vertical motion: $y = \frac{1}{2}gt^2$

$$h = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}}$$

Horizontal motion: x = ut

$$\frac{1}{4}l = ut$$

$$l = \frac{4}{3}ut = \frac{4}{3}v\sqrt{\frac{2h}{g}} \ldots . (ii)$$

Eqn (i) and (ii)

$\frac{4}{3}v\sqrt{\frac{8h}{g}} = \frac{4}{3}u\sqrt{\frac{2h}{g}}$;

$v = \frac{3}{2}u$

2v = 3u

Revision exercise 1

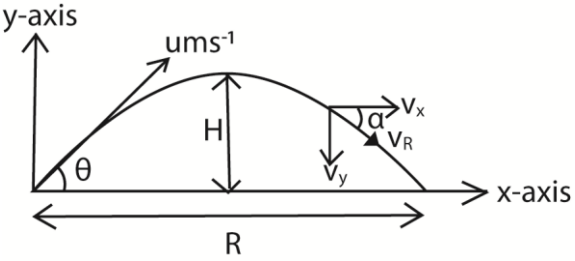
1. A pencil is accidentally knocked off the edge of a horizontal desktop. The height of the desk is 64.8cm and the pencil hits the floor a horizontal distance of 32.4cm from the edge of the desk. What was the speed of the pencil as it left the desk? [0.9ms^{-1}]
2. A particle is projected horizontally at 20ms^{-1} from a point 78.4m above a horizontal surface. Find the time taken for the particle to reach the surface and the horizontal distance travelled in that time. [4s, 80m]
3. A particle is projected horizontally with a speed of 2ms^{-1}
   (i) Find the horizontal and vertical displacements of the particle from the point of projection, $2\frac{6}{7}$s after projection [60m, 40m below]
   (ii) Find how far the particle is then from the point of projection. [72.1m]
4. A particle is projected horizontally from a point 2.5m above the horizontal surface. The partial hits the surface at a point which is horizontally 10m from the point of projection. Find the initial speed of the particle.[14ms^{-1}]
5. At time t= 0, a particle is projected with a velocity of 2ims^{-1} from a point with position vector (10i + 150j)m. Find the
   (i) speed and direction of the particle when t = 5s [49.04ms^{-1}, at 87.6^0]
   (ii) position vector of the particle when t = 1s $\left[\begin{pmatrix} 20 \\ 27.5 \end{pmatrix} m\right]$
6. At time t = 0, a particle is projected with a velocity of 5ims-1 from a point with position vector (20j)m. find the position vector of the particle when t = 2s. $\left[\begin{pmatrix} 10 \\ 0.4 \end{pmatrix} m\right]$.
7. A batsman strikes a ball horizontally when it is 1m above the ground. The ball is caught 10cm above the grounder by a fielder standing 12m from the batsman. Find the speed with which the batsman hits the ball. [28ms^{-1}]
8. A darts player throws a dart horizontally with a speed of 14ms^{-1}. The dart hits the board at a point 10cm below the level at which it is released. Find the horizontal distance travelled by the dart. [2m]
9. A tennis ball is served horizontally with initial speed of 21ms^{-1} from a height of 2.8m, by what distance does the ball clears a net 1m high situated 12m horizontally from the server? [20cm]
10. A fielder retrieves cricket ball and throws it horizontally with a speed of 28ms^{-1} to a wicket-keeper standing 12m away. If the fielder releases the ball at a height of 2m above level ground, find the height of the ball when it reaches the wicket-keeper. [110cm]
11. Initially a particle is at an origin and is projected with a velocity of ai ms^{-1}. After t second, the particle is at the point with position vector (30i – 21j)m. Find the value of t and a. $[1\frac{3}{7}, 21]$
12. Two vertical towers stand a horizontal ground level and are of height 40m and 30m. A ball is thrown horizontally from the top of the higher tower with a speed of 24.5ms^{-1} and just clears the short tower. Find the distance
    (i) between the two towers [35m]
    (ii) between the short and the point on the ground where the ball first lands. [35m]
13. The top of a vertical tower is 20m above ground level. When a ball is thrown horizontally from the top of this tower. By how much does the ball clears a vertical wall of height 13m situated 12m from the tower.[2m]
14. A stone is thrown horizontally with speed u from the edge of a vertical cliff of height h. The stone hits the ground at a point which is a distance d horizontally from the base of the cliff. Show that $2hu^2 = gd^2$.
15. A vertical tower stands with its base on a horizontal ground. Two particles A and B are both projected horizontally and in the same direction from the top of the tower with initial velocities

of 14ms^{-1} and 17.5ms^{-1} respectively. If A and B hit the ground at two points 10cm apart, find the height of the tower[40m]

16. O, A and B are three points with O on level ground and A and B respectively 3.6 and 40m vertically above O. A particle is projected horizontally from B with a speed of 21ms^{-1} and 2seconds later, a particle is projected horizontally from A with a speed of 70ms^{-1}. Show that the two particles reach the ground at the same distance from O, find this distance.
17. An aeroplane moving horizontally at 150ms-1 releases a bomb at height of 500m. The aeroplane hits the intended target. What was the horizontal distance of aeroplane from the target when the bomb was released? [1500m]
18. A projectile is fired horizontally from the top of a cliff 250m high. The projectile lands 144m from the bottom of the cliff. Find the
    (i) initial speed of the projectile [198ms-1]
    (ii) velocity of the projectile just before it hits the ground [210ms^{-1} at 19.5^0]

b. Object projected upwards from the ground at an angle to the horizontal

Suppose an object is project with velocity u at an angle θ from a horizontal ground. H and R are the maximum height reached and range respectively.



Horizontally; u_x = ucosθ, a = 0

v = u + at

v_x = ucosθ

x = ucosθt

v_y = s = ut + $\frac{1}{2}at^2$

x = ucosθt

Vertically; uy = usinθ, a = -9.8ms^{-2}

v = u + at

v_y = usinθ – gt

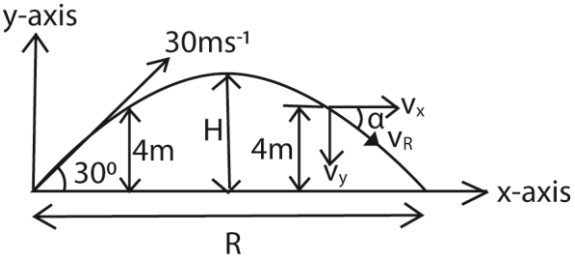
s = ut + $\frac{1}{2}at^2$

y = usinθt - $\frac{1}{2}gt^2$

Example 6

A particle is projected with a velocity of 30ms^{-1} at an angle of elevation of 30^0. Find

(i) the greatest height reached
(ii) the time of flight
(iii) the velocity and direction of motion at a height of 4m on its way upwards.



(i) (↑) v_y = usinθ, a = -9.8ms^{-2}

At maximum height, v_y = 0

From $v^2 = u^2 + 2as$

H = $\frac{(30\sin 30)^2}{2\ x\ 9.8} = 11{,}47m$

(ii) at time of flight, s = 0

s = ut + $\frac{1}{2}at^2$

0 = 30sin30T - $\frac{1}{2}x\ 9.8\ xT^2$

0 = (30sin30 - $\frac{1}{2}x\ 9.8\ T)T$

Either T = 0

or (30sin30 - $\frac{1}{2}x\ 9.8\ T$) = 0

T = 3.0612s

(→): u_x = ucosθ, a = 0

s = ut + $\frac{1}{2}at^2$

R = ucosθT

R = (30cos 30) x 3.0612 = 79.5329m

(↑) y = usinθt - $\frac{1}{2}gt^2$

4 = 30sin30t - $\frac{1}{2}x\ 9.8\ x\ t^2$

t = 0.30s or t = 2.76s

t = 0.30s is the correct time since it is smaller indicating that the body is moving upwards

u_x = ucosθ = 30cos30 = 25.98ms^{-1}

v_y = usinθ – gt

= 30sin30 – 9.8 x 0.30 = 12.06ms^{-1}

v = $\sqrt{(v_x)^2 + (v_y)^2}$

= $\sqrt{25.98^2 + 12.06^2}$ = 28.64ms^{-1}

direction α = $\tan^{-1}\left(\frac{12.06}{25.98}\right)$= 24.9^0

Example 7

A particle is projected from the origin at a velocity of (10i + 20j)ms^{-1}. Find the position and velocity vectors of the particle 3s after projection. (Take g = 10ms^{-2})

Solution

$P_{t=t} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$

$P_{(t=3)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} ucos\theta t \\ usin\theta t - \frac{1}{2}gt^2 \end{pmatrix}$

$P_{(t=3)} = \begin{pmatrix} 10\ x\ 3 \\ 20\ x\ 3 - \frac{1}{2}\ x9.8\ x\ 3^2 \end{pmatrix} = \begin{pmatrix} 30 \\ 15 \end{pmatrix} m$

$v_{(t=t)} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} ucos\theta \\ usin\theta - gt \end{pmatrix}$

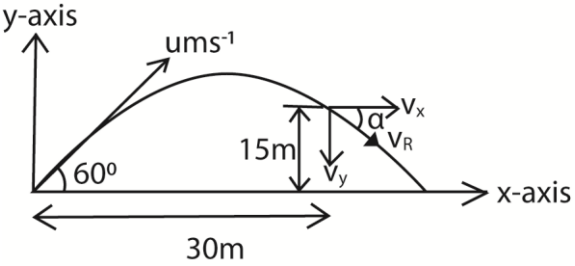
$v_{(t=3)} = \begin{pmatrix} 10 \\ 20 - 10x\ 3 \end{pmatrix}$

$= \begin{pmatrix} 10 \\ -10 \end{pmatrix} ms^{-1}$

Example 8

A projectile fired at an angle of 60^0 above the horizontal strikes a building 30m away at a point 15m above the point of projection. Find

(i) speed of projection (i) velocity when it strikes a building



(→) x = ucosθt

30 = utcos 60

t = $\frac{60}{u}$

(↑) y = $usin\theta t - \frac{1}{2}gt^2$

15 = usin60 x $\frac{60}{u}$ - $\frac{1}{2}\ x\ 9.8\ x\ \left(\frac{60}{u}\right)^2$

u = 21.86ms^{-1}

(ii) t = $\frac{60}{u} = \frac{60}{21.86}$ = 2.75s

u_x = ucosθ

= 21.86x cos 60 = 10.93ms^{-1}

u_y = usinθ – gt

= 21.86x sin 60 – 9.8 x 2.75

= -8.09ms^{-1}

v = $\sqrt{(v_x)^2 + (v_y)^2}$

= $\sqrt{10.93^2 + (-8.93)^2}$ = 13.60ms^{-1}

direction α = $\tan^{-1}\left(\frac{8.09}{10.93}\right)$= 36.58^0

Example 9

A football player projects a ball at a speed of $8ms^{-1}$ at an angle of 30^0 with the ground. The ball strikes the ground at a point which is level with the point of projection. After impact with the ground, the ball bounces and the horizontal component of the velocity of the ball remains the same but the vertical component is reversed in direction and halved in magnitude. The player running after the ball. Kicks it again at appoint which is at a horizontal distance 1.0m from the point where it bounced, so that the ball continues in the same direction. Find the

(a) horizontal distance between the point of projection and the point at which the ball first strikes the ground. (Take g = $10ms^{-2}$)
(b) (i) the time interval between the ball striking the ground and the player kicking it again.
(ii) the height of the ball above the ground when it is kicked again (take g = $10ms^{-2}$)

Solution

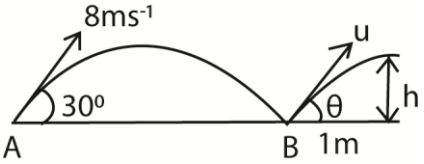
$y = usin\theta t - \frac{1}{2}gt^2$

At AB:

$0 = 8sin30 \times t - \frac{1}{2} \times 10 \times t^2$

t = 0.8s

$x = cos\theta t = 8cos30 \times 0.8 = 5.543m$



(b)(i) $x = ucos\theta t$

$1 = 8cos(30)t$

t = 0.1443s

(ii) $y = usin\theta t - \frac{1}{2}gt^2$

$h = 4sin30 \times 0.1443 - \frac{1}{2} \times 10 \times (0.1443)^2$

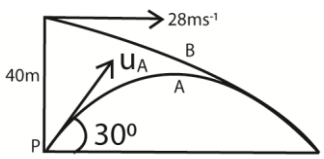
h = 0.185m

Example 10

Two objects A and B are projected simultaneously from different points. A is projected from the top of vertical cliff and A from the base. Particle B is projected horizontally with a speed $28ms^{-1}$ and A is projected at an angle θ above the horizontal. The height of the cliff is 40m and the particles hit the same point on the ground, find;

(a) time taken and the distance from P to where they hit
(b) speed and angle of projection of A

Solution



(a) For B: $y = \frac{1}{2}gt^2$

$-40 = \frac{1}{2} \times -9.8t^2$

$t = \frac{20}{7}s$

$x = ut = 28 \times \frac{20}{7} = 80m$

For A: $x = ucos\theta t$

$80 = u_Acos\theta \times \frac{20}{7}$

$u_Acos\theta = 28$ …….. (i)

$y = usin\theta t - \frac{1}{2}gt^2$

$0 = u_Asin\theta \times \frac{20}{7} - \frac{1}{2} \times -9.8 \times \left(\frac{20}{7}\right)^2$

$u_Asin\theta = 14$…… (ii)

(i) ÷ (ii)

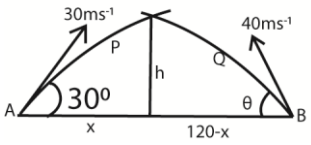
$\theta = \tan^{-1}\frac{14}{28} = 26.6^0$

$u_Asin26.6 = 14$

$u_A = 31.3ms^{-1}$

Example 11

A particle P is projected a point A with initial velocity 30ms-1at an angle of elevation 300 to the horizontal. At the same instant a particle Q is projected in the opposite direction with initial speed of 40ms-1 from a point at the same level with a and 120m from A. Given that the particles collide. Find (i) angle of projection of Q (ii) time when collision occur.



y = usinθt - $\frac{1}{2}gt^2$

For P

h = 30sin30 x t - $\frac{1}{2}\ x9.8t^2$.... (i)

For Q

h = 40sinθ x t - $\frac{1}{2}\ x9.8t^2$..... (ii)

(i) and (ii)

30sin30 x t = 40sinθ x t

θ = 24.5^0

For P

x – 30cos30t …… (iii)

For Q

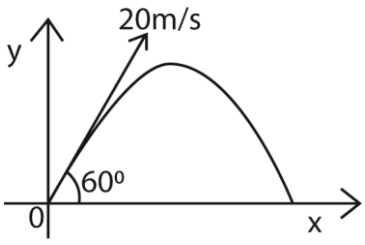
120 –x = 40cos24.5t … (iv)

(iii)and (iv)

t = 1.9s

Example 12

A particle is projected from a point O with speed20m/s at an angle 60^0 to the horizontal. Express in vector form its velocity v and its displacement, r, from O at any time t seconds . (05marks)



At time t = 0

V = 20cos60^0i + 20sin60^0j = 10i + 10$\sqrt{3}$j

But A = -gj

At any time t,

V = $\int adt$ = -gj$\int dt$

= -gtj + c

At t = 0

10i + 10$\sqrt{3}$j = 0 + cn

⇨ At time t

V= 10i + (10$\sqrt{3} - gt$)j = 10i + (10$\sqrt{3} - 9.8t$)j

Or

V = $\begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix}$

r = $\int vdt = \int \begin{pmatrix} 10 \\ 10\sqrt{3} - 9.8t \end{pmatrix} dt$ = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$+ c

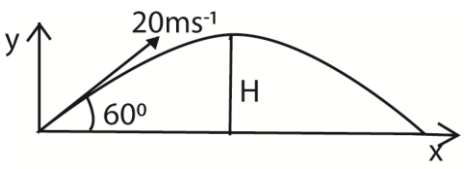
at t = 0, r = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

⇨ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + c$, c = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence at time t, r = $\begin{pmatrix} 10t \\ 10\sqrt{3}t - 4.9t^2 \end{pmatrix}$

Example 13

A particle is projected at an angle 600 to the horizontal with velocity of 20ms^{-1}. Calculate the greatest height the particle attains. [Use g = 10ms^{-2}]



Use $v^2 = u^2 + 2as$; at maximum height, v = 0

0 = $(20 \sin 60)^2 - 2gH$

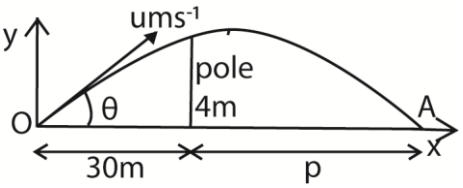
H = 15m

Examples 14

A particle is projected from ground level towards a vertical pole 4m high and 30m away from the point of projection. It just passes the pole in one second. Find

(i) its initial speed and angle of projection
(ii) the distance beyond the pole where the particle falls

Solution



(→) $x = u_x t$ but t = 1

$30 = u\cos\theta \times 1$

$30 = u\cos\theta$ ..... (i)

(↑) $y = u_y t - \frac{1}{2}gt^2$

$4 = u\sin\theta \times 1 - \frac{1}{2} \times 9.8 \times 1^2$

$4 = u\sin\theta - 4.9$

$8.9 = u\sin\theta$ ..... (ii)

(ii) ÷ (i)

$\theta = \tan^{-1}\left(\frac{8.9}{30}\right) = 16.5^0$

$30 = u\cos 16.5$

$u = 31.29 ms^{-1}$

(b) At point O and A, y=0

$y = u\sin\theta t - \frac{1}{2}gt^2$

$0 = 31.29\sin 16.5T - \frac{1}{2} \times 9.8 \times T^2$

T = 1.8136s

Range, R= $u\cos\theta T$

R = 31.29cos 16.6 x 1.8136 = 54.36m

p = 54.36 – 30 = 24.36m

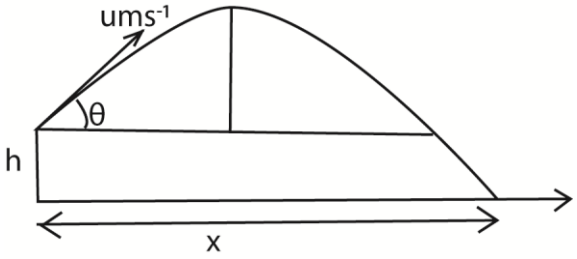
Revision exercise 2

1. A particle projected at an angle of 30^0 to the horizontal with velocity of $60ms^{-1}$. Calculate
   (a) time taken for the particle to reach maximum height [3s]
   (b) maximum height [45m]
   (c) horizontal range of the particle [312m]
2. A particle is projected from an origin O and has an initial velocity of $30\sqrt{2}$ ms^{-1} at angle 450 above the horizontal. Find the horizontal and vertical components of displacement 2s after projection. [60m, 40.4m] Hence find the distance of motion of the particle at that time [72.3m]
3. A particle projected from a point on the level ground has horizontal range of 240m and time of flight of 6s. find the magnitude and direction of velocity of projection [$50ms^{-1}$, 36.9^0]
4. A particle is projected with a velocity of $30ms^{-1}$ at an angle of 40^0 above the horizontal plane. Find
   (i) the time for which the particle is in air [3.9s]
   (ii) the horizontal distance it travels [22.9m]
5. A body is projected with a velocity of 200ms-1 at an angle of 300 above the horizontal. Calculate
   (i) time taken to reach the maximum height[10.2s]
   (ii) its velocity after 16s [$183ms^{-1}$ at 19.1^0]
6. A football is kicked from O on a level ground. 2s later the football just clears a vertical wall of height 2.4m. If O is 22m from the wall, find the velocity with which the ball is kicked. [$15.6ms^{-1}$ at 45^0 above the horizontal]

7. A particle is projected from a level ground in such away that its horizontal and vertical components of velocity are $20ms^{-1}$ and $10ms^{-1}$ respectively. find
   (a) maximum height of the particle [5.0m]
   (b) its horizontal distance from the point of projection when it returns to the ground. [40m]
   (c) the magnitude and direction of the velocity on landing [$22.4ms^{-1}$ at 26.6^0 below the horizontal]
8. A particle is projected with a speed of 25ms-1 at 300 above the horizontal. Find
   (a) time taken to reach the height point of the trajectory. [1.5s]
   (b) the magnitude and direction of velocity after 2.0s.[$22.9ms^{-1}$ at 19.1^0 below the horizontal]
9. A particle is projected from the origin at a velocity $(4i + 13j)ms^{-1}$. Find the position vector and distance of the particle in 2s after projection (take $g = 10ms^{-2}$) [(8i + 6j)m, 10m]
10. A particle is projected from the origin at a velocity of $(4i + 11j)ms^{-1}$. and passes a point P which has a position vector (8i + xj)m. Find the time taken for the particle to reach P from O and the value of x. [2s, 2.4m]
11. A Particle is projected from the origin at a velocity of $(7i + 5j)ms^{-1}$ and passes a point P which has position vector (xi – 30j)m. Find the time taken for the particle to reach P from O and the value of x [3s, 21]
12. A particle is projected from the origin at velocity of $(4i + 2j)ms^{-1}$. Find
   (a) the direction in which it is moving after 1s[63.43^0]
   (b) Two second later after launch of the first particle, a second particle is projected from the same point with a velocity $(8i – 26j)ms^{-1}$. Show that the two particles collide and find the time and position at which this occurs [t =4s, (16i – 72j)m]
13. A particle is projected at $84ms^{-1}$ to hit a point 360m away and on the same horizontal level at the point as the point of projection. Find the two possible angles of projection. [15^0, 75^0]
14. A golfer hits a golf ball at $30ms^{-1}$ and wishes it to land at a point 45m away, on the same horizontal level as the starting point. Find the two possible angles of projection. [4.7^0, 75.3^0]
15. A particle is projected from a horizontal ground and has an initial speed of $35ms^{-1}$. When the ball is travelling horizontally, it strikes a vertical wall. If the wall is 25m from the point of projection, find the two possible angles of projection. [11.8^0, 8.2^0]
16. A particle is projected from a point O and passes through a point A when the particle is travelling horizontally. If A is 10m horizontally and 8m vertically from, find the magnitude and direction of the velocity of projection. [$14.8ms^{-1}$, 58^0 above the horizontal]
17. A stone is projected at an angle of 20^0 to the horizontal and just clears a wall which is 10m high and 30m from the point of projection. Find the
   (a) speed of projection [$73.78ms^{-1}$]
   (b) angle which the stone makes with the horizontal as it clears the wall [16.9^0]
18. A particle is projected from a point on a horizontal plane and has an initial speed of 28ms-1. If the particle passes through a point above the plane, 40m horizontally and 20m vertically from the point of projection, find the possible angles of projection. [45^0, 71.6^0]
19. Two objects A and B are projected simultaneously from different points. A is projected from the top of a vertical cliff and B from the base. Particle A is projected horizontally with a speed 3u ms^{-1} and B is projected at an angle θ above the horizontal with speed 5u ms^{-1}. The height of the cliff is 56m and the particles collide after 2s, find
   (a) horizontal and vertical distances from the point of collision to the base of the cliff. [42m, 36.4m]
   (b) value of angle u and θ. [$7ms^{-1}$, 53.1^0]

c. Objects projected upwards from a point above the ground at an angle to the horizontal

Suppose an object is projected with velocity u at an angle θ from a height h.



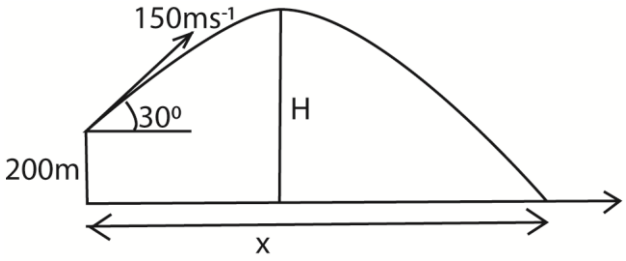
Horizontal: $u_x = u\cos\theta$, a = 0

Vertical: uy = usinθ, a = -g = -9.8

Example 15

A bullet is fired from a gun at a height of 200m with velocity150ms-1 at an angle of 300 to the horizontal. Find

(i) maximum height attained.



$v^2 = u^2 + 2as$

at maximum height, H, v =0

$0^2 = (150\sin30)^2 - 2 \times 9.8H$

H = 86.70m

(ii) Time taken for the bullet to hit the ground

y = usinθt - $\frac{1}{2}gt^2$

y = -200m since it's below the point of projection

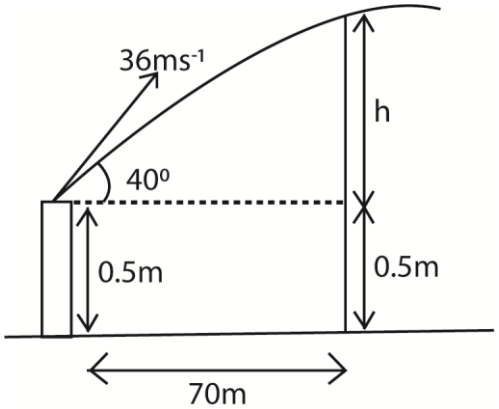
-200 = 150sin30t - $\frac{1}{2}x\ 9.8t^2$

t = 17.61s, or t = -2.32s

time taken = 17.61

Example 16

A particle I projected with a speed of 36ms^{-1} at an angle of 40^0 to the horizontal from a point 0.5m above the level ground. It just clears a wall which is 70 metres on the horizontal plane from a point of projection. Find the;



Vx = 36cos40^0

Vy = 36sin40^0

(a) (i) time taken for the particle to reach the wall.

Time take to clear the wall = time taken to cover a horizontal distance of 70m

Usin X= Vxt

t = $\frac{70}{36\sin40^0}$ = 2.5384s

(ii) height of the wall (08marks)

Using h = usinθt - $\frac{1}{2}gt^2$

=36sin40^0 x 2.5384 - $\frac{1}{2}\ x\ 9.8\ x\ (2.5384)^2$

= 27.1664 + 0.5 = 27.6664m

(a) Maximum height reached by the particle from the point of projection. (04marks)

From $v^2 = u^2 + 2as$

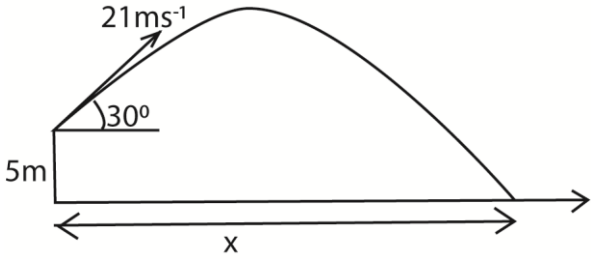
At maximum height vertical component of velocity is zero

⇨ $0 = (36\sin 40^2)^2 - 2\ x\ 9.8H$

H = $\frac{(36\sin 40^2)^2}{2\ x\ 9.8}$ =27.32m

Example 17

A particle is projected at an angle of 30^0 with speed of 21ms^{-1}. If the point of projection is 5m above the horizontal ground, find the horizontal distance that the particle travels before hitting the ground



y = usinθt - $\frac{1}{2}gt^2$

y = -5m since it's below the point of projection

-5 = 21sin30T - $\frac{1}{2}x\ 9.8T^2$

T = 2.54s

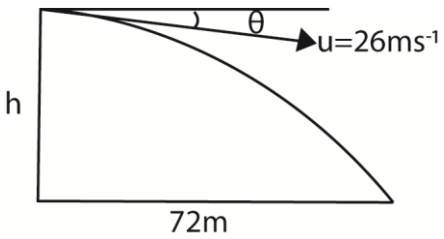
(→): x = ucosθ x T

= 21cos30 x 2.54 = 46.19m

Example 18

A stone is thrown from the edge of a vertical cliff and has initial velocity of 26ms^{-1} at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ below the horizontal. The stone hits the sea at a point level with the base of the cliff and 72m from it. Find the height of the cliff and the time for which the stone is in the air.

Take g = 10ms^{-2}.



y = xtanθ - $\frac{gx^2(1+\tan^2\theta)}{2u^2}$

h = 72 x$\left(-\frac{5}{12}\right)$ = $\frac{10\ x\ 72^2\left[1+\left(\left(-\frac{5}{12}\right)\right)^2\right]}{2\ x\ 26^2}$

= -75m

h= 75m below the point of projection

x = ucosθt

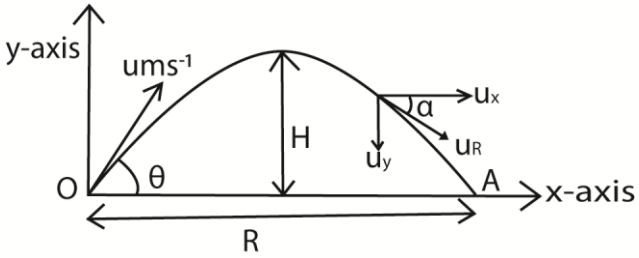
72 = 26$\left[cos\left(-\tan^{-1}\frac{5}{12}\right)\right]t$; t =3s

Revision exercise 3

1. A stone is thrown from the edge of a vertical cliff with velocity of 50ms^{-1} at an angle of $\tan^{-1}\frac{7}{24}$ above the horizontal. The stone strikes the sea at a point 240m from the foot of the cliff. Find the time for which the stone is in air and the height of the cliff. [5s, 52.5m]
2. A particle is projected with a velocity of 10ms-1 at an angle of 45^0 to the horizontal; it hits the ground at a point which is 3m below the point of projection. Find the time for which it is in the air and the horizontal distance covered the particle in this time. [1.76s, 12.42m]
3. A batsman hits a ball with velocity of 17ms^{-1} at an angle $\tan^{-1}\frac{3}{4}$ above the horizontal, the ball initially being60cm above the level ground. The ball is caught by a fielder standing 28m from the batsman. Find the time taken for the ball to reach the fielder and the height above the ground at which he takes the catch. [2s, 2m]
4. A stone is thrown from the edge of a vertical cliff 70m high at an angle of 300 below the horizontal. the stone hits the sea at a point level with the base of the cliff and 20m from it. Find the initial speed of the stone and the direction it is moving when it hits the sea. [6.69mms^{-1}, 81.2^0]
5. A vertical tower stands on a level ground. A stone is thrown from the top of the tower and has an initial velocity of 24.5ms^{-1} at an angle of $\tan^{-1}\frac{4}{3}$ above the horizontal. The stone strikes the ground at a point 73.5m from the foot of the tower. Find the time taken for the stone to reach the ground and the height of the tower. [5s, 24.5m]
6. A stone is thrown from the top of a vertical cliff, 100mabove sea level. The initial velocity of the stone is 13ms-1 at an angle of elevation of$\tan^{-1}\frac{5}{12}$. Find the time taken for the stone to reach the sea and its horizontal distance from the cliff at that time. Take g = 10ms^{-2}. [5s. 60m]
7. A golfer hits a golf ball with a velocity of 30ms^{-1} at an angle of $\tan^{-1}\frac{4}{3}$ above the horizontal. The ball lands on green 5m below the level from which it was struck. Find the horizontal distance travelled by the ball. Take g = 10ms^{-2}. [90m]
8. A pebble is thrown from the top of a cliff at a speed of 10ms^{-1} and at 30^0 above the horizontal. It hits the sea below the cliff 6s later. Find
   (a) the height of the cliff [150m]
   (b) the distance from the base of the cliff at which the pebble falls into the sea. [52m]
9. An arrow is fired from a point at a height 1.5m above the horizontal. It has a velocity of 12ms^{-1} at an angle 30^0 above the horizontal. The arrow hits the target at a height of 1m above the horizontal ground, find
   (i) time taken for the arrow to hit the target [1.3s]
   (ii) horizontal distance between where the arrow is fired and the target. [13.51m]
   (iii) speed of the arrow when the arrow hit the target [12.39ms^{-1}]

Standard equations of the projectile

Suppose an object is projected with velocity u at an angle θ from a horizontal ground



(a) Maximum height (greatest height), H
For vertical motion, at maximum height v = 0,
a = -g = -9,8ms^{-2}
$v^2 = u^2 + 2as$

$0 = (usin\theta)^2 - 2gH$

$H = \frac{(usin\theta)^2}{2g}$

(b) Time to reach maximum height
(↑): $v = u_y + at$, at maximum height, v = 0
$0 = usin\mu - gt$
$t = \frac{usin\theta}{g}$

(c) Time of flight, T
(↑): $s_y = u_y t - \frac{1}{2}gt^2$
At A $s_y = 0$

$0 = usin\theta T - \frac{1}{2}gT^2$

$T = \frac{2usin\theta}{g}$

(d) Range
(→) $x = ucos\theta t$
$R = ucos\theta T$
$= ucos\theta . \frac{2usin\theta}{g}$

$= \frac{2u^2 cos\theta sin\theta}{g}$

$R = \frac{2u^2 sin2\theta}{g}$

(e) Maximum range
for maximum range $sin2\theta = 1$
$2\theta = \sin^{-1} 1$
$2\theta = 90$

$R_{max} = \frac{2u^2 sin90}{g} = \frac{u^2}{g}$

(f) Equation of trajectory
A trajectory is expresses zin terms of horizontal distance x and vertical distance y
(→): $x = ucos\theta t$
$t = \frac{x}{ucos\theta}$ ..... (i)
(↑): $y = usin\theta t - \frac{1}{2}gt^2$ .... (ii)
putting (i) into (ii)
$y = usin\theta\left(\frac{x}{ucos\theta}\right) - \frac{1}{2}g\left(\frac{x}{ucos\theta}\right)^2$

$y = xtan\theta - \frac{gx^2}{2u^2 cos^2\theta}$
$= xtan\theta - \frac{gx^2 sec^2\theta}{2u^2}$
$y = xtan\theta - \frac{gx^2(1+ tan^2\theta)}{2u^2}$

Example 19

A ball is projected from the horizontal ground and has an initial velocity of 20ms^{-1} at an angle of elevation$\tan^{-1}\frac{7}{24}$. When the ball is travelling horizontally it strike a vertical wall. How high above the ground does the impact occur.

Projectile travel horizontally at maximum height

H = $\frac{(usin\theta)^2}{2g}$ = $\frac{20^2 sin^1\left(\tan^{-1}\frac{7}{24}.\right)}{2\ x\ 9.8}$ = 1.6m

Example 20

A particle is projected from a point on a horizontal ground at a speed of 84ms^{-1}. If the particle hits a point 300m away and on the same horizontal plane as the projection, find the

(i) angle of projection

R =$\frac{2u^2 sin2\theta}{g}$

360 = $\frac{2x\ 84^2\ x\ sin2\theta}{9.8}$; θ = 15^0 0r θ = 75^0

(ii) maximum height

H_1 = $\frac{(usin\theta)^2}{2g}$ = $\frac{84^2 sin^2(15)}{2\ x\ 9.8}$ = 24.1m

H_2 = $\frac{(usin\theta)^2}{2g}$ = $\frac{84^2 sin^2(75)}{2\ x\ 9.8}$ = 335.9m

(iii) times of flight

$T_1 = \frac{2usin\theta}{g}$ = $\frac{2x\ 84\ x\ sin15}{9.8}$ = 4.44s

$T_2 = \frac{2usin\theta}{g}$ = $\frac{2x\ 84\ x\ sin75}{9.8}$ = 16.56s

Example 21

A gun has its barrel set at an angle of elevation of 15^0. The gun fires a shell with initial speed of 210ms^{-1}. Find the

(a) horizontal range of the shell

R =$\frac{2u^2 sin2\theta}{g}$ = $\frac{2x\ 84^2\ x\ sin30}{9.8}$ = 2250m

(b) maximum range

R =$\frac{2u^2}{g}$ = R =$\frac{2\ x\ 84^2}{g}$= 4500m

Example 22

A stone thrown upwards at an angle θ to the horizontal with speed ums-1 just clears a vertical wall4m high and 10m from the point of projection when travelling horizontally. Find the angle of projection.

Solution Projectile travel horizontally at maximum height H = $\frac{(usin\theta)^2}{2g}$	4 = $\frac{(usin\theta)^2}{2g}$ 8g = u^{2}sin^{2}θ =…… (i) Also, x = ucosθt and t = $\frac{ucos\theta}{g}$ x = ucosθ$\left(\frac{ucos\theta}{g}\right)$	10g = u^{2}cosθsinθ ……(ii) (i)÷(ii) θ = $\tan^{-1}\frac{8}{10}$ = 38.7^0

Example 23

If the horizontal range of a particle with velocity u is r, show that the greatest height H is satisfied by the equation $16gH^2 - 8Hu^2 + gR^2 = 0$

$H = \frac{(usin\theta)^2}{2g}$

$sin^2\theta = \frac{2gH}{u^2}$ ..... (i)

$R = \frac{2u^2 sin2\theta}{g}$

$gR = u^2(2cos\theta sin\theta)$

$cos\theta = \frac{gR}{2u^2 sin\theta}$

$cos^2\theta = \frac{(gR)^2}{4u^4 sin^2\theta}$ ..... (ii)

(i) and (ii)

$cos^2\theta = \frac{(gR)^2}{4u^4\left(\frac{2gH}{u^2}\right)}$

$cos^2\theta = \frac{gR^2}{8u^2H}$

$Cos^2\theta + sin^2\theta = 1$

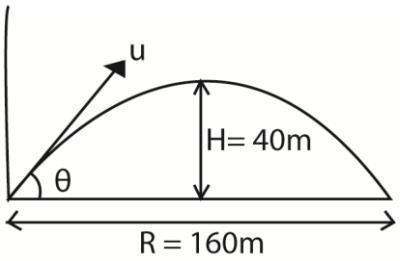
$\frac{2gH}{u^2} + \frac{gR^2}{8u^2H} = 1$

$16gH^2 - 8Hu^2 + gR^2 = 0$

Example 24

A ball is projected from point A and falls at point B which is in level with A at a distance of 160m from A. The greatest height of the ball attained is 40m. find the;

(a) angle and velocity at which the ball is projected (10marks)



Using $v^2 = u^2 + 2as$

$v^2 = u^2sin^2\theta - 2gh$

At maximum height $v_y = 0$

$0 = u^2sin^2\theta - 2gH$

$u^2 = \frac{2gH}{sin^2\theta}$ ..... (i)

Range, $R = \frac{2u^2 sin\theta cos\theta}{g}$ ..... (ii)

Eqn. (i) and eqn. (ii)

$R = 2\ x\ \frac{2gH}{sin^2\theta}.\frac{sin\theta cos\theta}{g} = 4Hcot\theta$

..... (iii)

Substituting for R and H in eqn. (iii)

$160 = 4 \times 40cot\theta$

$tan\ \theta = 1$

$\theta = tan^{-1}\ 1 = 45^0$

Substituting for θ

$u^2 = \frac{2gH}{sin^2\theta} = \frac{2\ x\ 40\ x\ 9.8}{sin^2\ 45^0}$

$u = \sqrt{\frac{2\ x\ 40\ x\ 9.8}{sin^2\ 45^0}} = 39.60ms^{-1}$

(b) time taken for the ball to attain the greatest height (02marks)

$t = \frac{usin\ \theta}{g} = \frac{39.60\ x\ \sin 45^0}{9.8} = 2.8573s$

Example 25

A boy throws a ball at an initial speed of $40ms^{-1}$ at an angle of elevation θ. Taking g = $10ms^{-2}$, show that the times of flight corresponding to a horizontal range of 80m are positive roots of the equation $T^4 - 64T^2 + 256$.

$R = \frac{2u^2 cos\theta sin\theta}{g}$

$80 = \frac{2\ x\ 40^2 cos\theta sin\theta}{10}$

$sin\theta cos\theta = 0.25$

$sin\theta = \frac{1}{4cos\theta}$ ..... (i)

$x = ucos\theta t$

$80 = 40cos\theta T$

$cos\ \theta = \frac{2}{T}$ ..... (ii)

but $sin^2\theta + cos^2\theta = 1$

$\left(\frac{1}{4x\frac{2}{T}}\right)^2 + \left(\frac{2}{T}\right)^2 = 1$

$\frac{T^2}{64} + \frac{4}{T^2} = 1$

$T^4 - 64T^2 + 256$

Example 26

A particle projected from a point O on a horizontal ground moves freely under gravity and it hits the ground again at A. Taking O as the origin, the equation of the path of the particle is $60y = 20\sqrt{3}x - x^2$ where x and y are measured in meters. Determine the

(a) initial speed and angle of projection

$60y = 20\sqrt{3}x - x^2$

$y = \frac{\sqrt{3}}{3}x - \frac{x^2}{60}$

Comparing with

$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30^0$

$\frac{gx^2(1+\tan^2\theta)}{2u^2} = \frac{x^2}{60}$

$\frac{g(1+\tan^2\theta)}{2u^2} = \frac{1}{60}$

$\frac{9.8\left(1+\frac{3}{9}\right)}{2u^2} = \frac{1}{60}$; $u = 19.8ms^{-1}$

(b) Distance OA

At A, $y = 0$

$0 = 20\sqrt{3}x - x^2$

$0 = (20\sqrt{3} - x)x$

$x = 20\sqrt{3}m$

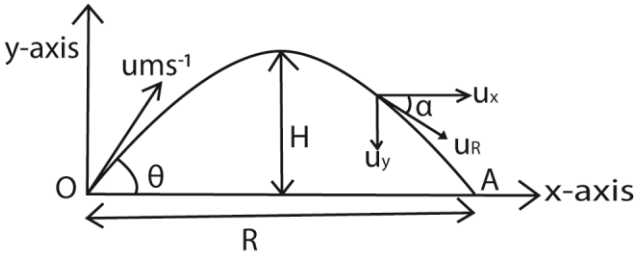
Example 27

A particle is projected from a point O on a level ground with initial speed $30ms^{-1}$ to pass through a point which is a horizontal distance 40m from O and a distance 10 vertically above the level of O.

(a) Show that there are two possible angles of projection

(b) If these angles are α and β, prove that $\tan(\alpha + \beta) = -4$, take $g = 10ms^{-2}$

Solution



$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$

$10 = 40\tan\theta - \frac{10\ x\ 40(1+\tan^2\theta)}{2\ x\ 30^2}$

$8\tan_2\theta - 36\tan\theta + 17 = 0$

since it's quadratic equation in tanθ;

it has two roots and hence two values of $\theta < 90$

(b) $\tan\alpha + \tan\beta = \frac{36}{8}$

$\tan\alpha \tan\beta = \frac{17}{8}$

$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \frac{\frac{36}{8}}{1 - \frac{17}{8}}$

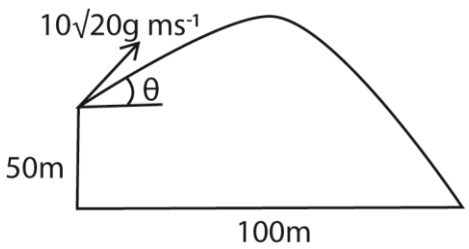
$= \frac{\frac{36}{8}}{\frac{8-17}{8}}$

$= \frac{36}{8}\ x\ \frac{8}{-9}$

$\tan(\alpha + \beta) = -4$

Example 28

A particle is projected with a speed $10\sqrt{2g}ms$-1from the top of a cliff 50m high. The particle hits the sea at a distance of 100m from the vertical through the point of projection. Show that there two possible directions of projection which are perpendicular. Determine the time taken from the point of projection in each case.



y = xtanθ $-\frac{gx^2(1+tan^2\theta)}{2u^2}$

-50 = 100tanθ $-\frac{g\ x\ 100^2(1+tan^2\theta)}{2x\ 100\ x\ 2g}$

$tan^2\theta$ -4tanθ – 1 = 0

$tan\theta_1 = 2+\sqrt{5}$ and $tan\theta_2 = 2-\sqrt{5}$

$tan\theta_1\ tan\theta_2 = (2+\sqrt{5})(2-\sqrt{5}) = -1$

Hence they are perpendicular

For horizontal motion

x = ucosθt

$\theta_1 = \tan^{-1}(2+\sqrt{5}) = 76.72^0$

$t_1 = \frac{100}{10\sqrt{2g}\cos(76.72^0)}$ = 9.83s

$\theta_2 = \tan^{-1}(2-\sqrt{5}) = -13.36^0$

$t_2 = \frac{100}{10\sqrt{2g}\cos(-13.38^0)}$ = 2.32s

Revision exercise 4

1. A golfer hits a ball with a velocity of 44.1ms-1 at an angle of $\sin^{-1}\frac{3}{5}$ above the horizontal. The ball lands on the green at a point which is level with the point of projection. Find the time for which the golf ball was in air. [5.4s]
2. a tennis ball is served horizontally from a point which is 2.5m vertically above a point A. The ball first strikes the horizontal ground through A at a distance 20m from A.
   (i) show that the ball is served with speed 28ms-1
   (ii) During its flight the ball passes over a net which is horizontal distance 12m from A. Find the vertical distance of the ball above the horizontal ground at the instant when it passes over the net. [1.6m]
3. An aircraft, at a height of 180m above horizontal ground and flying horizontally with speed of 70ms^{-1} releases emergency supplies. If these supplies are to land at a specific point, at what horizontal distance from this point must the aircraft release them? {take g = 10ms^{-2}) {420m]
4. A stone is projected from top of a vertical cliff of height of h and the stone attains a maximum height (h+ b) above the ground. The stone strikes a sea at a distance, a from the foot of the cliff. Prove that the angle of elevation θ of the stone is given by $a^2tan^2\theta$ -4abtanθ – 4bh = 0
5. At time t = 0 a particle is projected from a point O on a horizontal plane with speed 14ms^{-1} in a direction inclined at an angle $\tan^{-1}\frac{3}{4}$ above the horizontal. The particle just clears the top of a vertical wall, the base of which is 8m from O. Find
   (a) time at which the particle passes over the wall [$\frac{5}{7}s$]
   (b) height of the wall [3.5m]
6. A particle P is projected from a point O with a speed of 60ms-1 at an angle $\cos^{-1}\frac{4}{5}$ above the horizontal. Find
   (a) time the particle takes to reach the point Q whose horizontal displacement from O is 96m. [2s]
   (b) height of Q above O [52.4m]
   (c) speed of the particle 2s after projection [50.7ms^{-1}]
7. A particle P is projected from a point O with a speed50ms^{-1} at an angle $\sin^{-1}\frac{7}{25}$ above the horizontal. Find
   (a) height of P at the point where its horizontal displacement from O is 120m[4.375m]

(b) speed of P 2s after projection. [48.3ms^{-1}]

(c) times after projection at which P is moving at an angle of $\tan^{-1}\frac{1}{4}$ to the ground [0.204s, 2.65s]

8. A child throws a small ball from a height of 1.5m above level ground, aiming at a small target. The target is on top of a vertical pole of height 2m from the ground and horizontal displacement of the child from the pole is 6m. The initial velocity of the ball has magnitude ums^{-1} at an angle of elevation 40^0. The ball moves freely under gravity. (Take g = 10ms^{-2})
   (a) For u = 10, find the greatest height above the ground reached by the ball. [3.6m]
   (b) Calculate the value of u for which the ball hits the target[8.2]
9. A girl thrown a stone from a height of 1.5m above the ground with speed of 10ms^{-1} and hits a bottle standing on a wall 4m high and 5m from her. Take g = 10ms^{-1}.
   (a) Show that if α is the angle of projection of the stone as it leaves her hand then $1.25\tan^2\alpha - 5\tan\alpha + 3.75 = 0$
   (b) the horizontal component of the stone's velocity has to be6ms^{-1} for the bottle to be knocked off. By solving the above equation or otherwise, show that α has to be 45^0 for the bottle to be knocked off.
10. A basketball is released from player's hands with a speed of 8ms^{-1} at inclination of α^0 above the horizontal so as t land in the centre of the basket, which is 4m horizontally from the point of release and a vertical height of 0.5m above it. Take g =10ms^{-2}.
    (a) show that α satisfies the quadratic equation; $5\tan^2\alpha - 16\tan\alpha + 7 = 0$
    (b) Given that the player throws the ball at a large angle of projection find α for the ball to land in the basket. [70^0, 1.43s]

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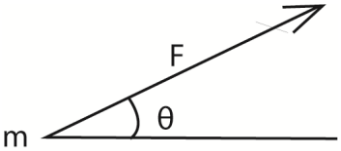
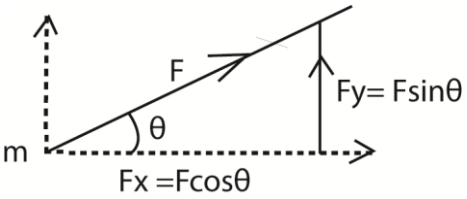
Resolution of forces

The component of a vector is the effective value of a vector along a particular direction. The component along any direction is the magnitude of a vector multiplied by the cosine of the angle between its direction and the direction of the component.

Suppose a force F pulls a body of mass m along a truck at an angle θ to the horizontal as shown below



The effective force that makes the body move along the horizontal is the component of F along the horizontal



cosθ =$\frac{F_x}{F}$;

$F_x = Fcos\theta$

sinθ =$\frac{F_y}{F}$;

$F_y = Fsin\theta$

Resultant $F_R = \sqrt{{F_x}^2 + {F_y}^2}$

Direction, α =$\tan^{-1}\left(\frac{F_y}{F_x}\right)$

Note

When a vector is inclined at an angle θ to the horizontal then;

- along the horizontal, the component of the vector is cosθ
- along the vertical, the component of the vector is sinθ

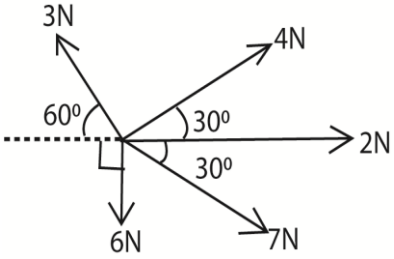
When a vector is inclined at an angle θ to the horizontal then;

- along the horizontal, the component of the vector is sinθ
- along the vertical, the component of the vector is cos θ

Example 1

Find the resultant of the system of forces below

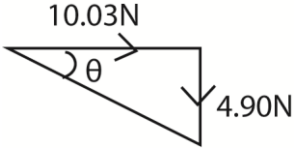
(a)



$F_R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}4cos30\\4sin30\end{pmatrix} + \begin{pmatrix}7cos30\\-7sin30\end{pmatrix} + \begin{pmatrix}-3cos60\\3sin60\end{pmatrix} + \begin{pmatrix}0\\-6\end{pmatrix}$

$= \begin{pmatrix}10.03\\-4.90\end{pmatrix}$

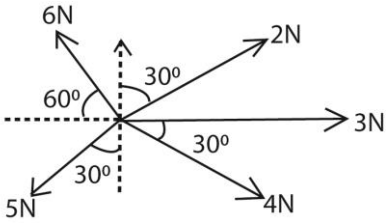
$F_R = \sqrt{(10.03)^2 + (-4.90)^2}$ = 11.16N



θ= $\tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\left(\frac{4.9}{10.03}\right)$ = 26.04⁰

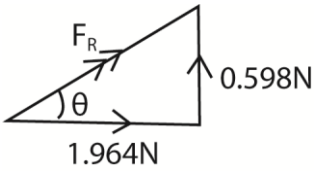
The resultant force is 11.16N at 26.04⁰ below the horizontal

(b)



$F_R = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}2sin30\\2cos30\end{pmatrix} + \begin{pmatrix}4cos30\\-4sin30\end{pmatrix} + \begin{pmatrix}-5sin30\\-5cos30\end{pmatrix} + \begin{pmatrix}-6\cos 60\\6\sin 60\end{pmatrix} = \begin{pmatrix}1.964\\0.598\end{pmatrix}$

$F_R = \sqrt{1.964^2 + 0.598^2}$ = 2.053N

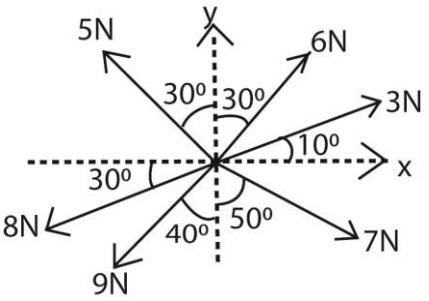


θ= $\tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\left(\frac{0.598}{1.964}\right)$ = 16.9⁰

The resultant force is 2.053N at 16.9⁰ below the horizontal

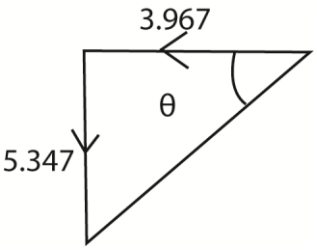
Example 2

Forces of 3N, 6N, 5N, 8N, 9N and 7N act on a particle as shown in the figure below. Find the resultant force.



$F_R = \begin{pmatrix}3cos10\\3sin10\end{pmatrix} + \begin{pmatrix}6sin30\\6cos30\end{pmatrix} + \begin{pmatrix}-5sin30\\5cos30\end{pmatrix} + \begin{pmatrix}-8cos30\\-8sin30\end{pmatrix} + \begin{pmatrix}-9sin40\\-9cos40\end{pmatrix} + \begin{pmatrix}7sin50\\-7cos50\end{pmatrix} = \begin{pmatrix}-3.967\\-5.347\end{pmatrix}$

$F_R = \sqrt{(-3.967)^2 + (-5.347)^2}$ =6.658N



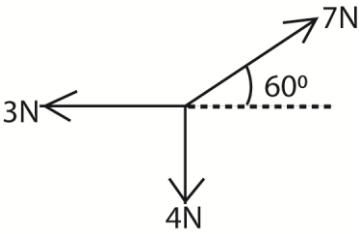
$\theta = \tan^{-1}\left(\frac{5.347}{3.967}\right) = 53.43^0$

Resultant force is 6.658N at 53.43^0 below horizontal

Example 3

A body of mass 1kg is acted on by the forces shown below. Find

(i) Magnitude of the resultant force
(ii) acceleration of the body
(iii) distance moved in 2s



$F_R = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 7\cos 60 \\ 7sin60 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2.06 \end{pmatrix}$

$F_R = \sqrt{(0.5)^2 + (2.06)^2}$ = 2.12N

but $F_R = ma$

2.12 =1a

a = 2.12ms^{-2}

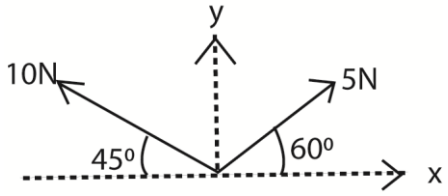
From s = ut + ½ at^2

u = 0, t = 2s, a =2.12ms^{-2}

s = 0 x 2 + ½ x 2.12 x 2^2 = 4.24m from the origin

Revision exercise

1. A force of 3N acts at 60^0 to a force of 5N. Find the magnitude and direction of their resultant. [7N at 21.8^0 to the 5N force]
2. A force of 3N at 90^0 to the force of 4N. Find the magnitude and direction of their resultant. [5N at 37^0 to the 4N force]
3. Two coplanar forces act on a point O as shown below



Calculate the magnitude and direction of the resultant force

[12.3N at 68.0 above the horizontal

4. The resultant of two forces pN and 3N is 7N. If the 3N is reversed, the resultant is $\sqrt{17}$N Find the value of p and the angle between the two forces.[$2\sqrt{6}N, 57.02^0$]

5. Forces of 2N, 3N, 5N and 5N act on a particle in the direction 030^0, 090^0, 120^0, 210^0, and 330^0 respectively. Find the magnitude and direction of the resultant force. [1.92N at 7.8^0 to the horizontal]
6. Forces of 7N, 2N, 2N and 5N act on a particle in the direction 060^0, 160^0, 200^0, and 315^0 respectively. Find the resultant force. [4.14N at 52.36^0 below the horizontal]
7. Find the magnitude and direction of the resultant of force 10N, 15N and 8N acting in the direction 030^0, 150^0 and 225^0. [12.1N at 55.6^0 below the positive x-axis]

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Resolutions of forces acting on a polygon

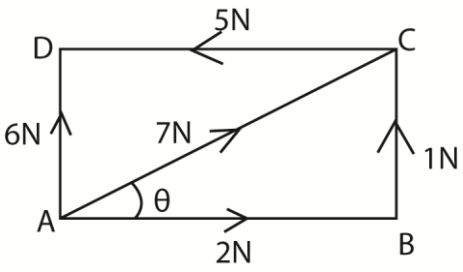
For any regular polygon

- all sides are equal
- all angles are equal
- an exterior angle $=\frac{360}{n}$ where n is the number of sides

Example 1

ABCD is a rectangle with AB= 4cm and BC = 3cm. Forces of magnitude 2N, 1N, 5N,6N and 7N act along AB, BC, CD, AD and AC respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine

(i) the magnitude of the resultant force

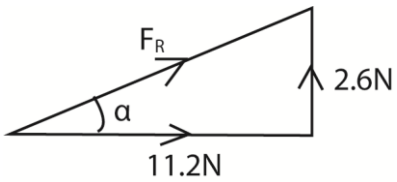


$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87$

R = $\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-5\\0\end{pmatrix} + \begin{pmatrix}0\\6\end{pmatrix} + \begin{pmatrix}7cos36.87\\7sin36.87\end{pmatrix} = \begin{pmatrix}2.6\\11.2\end{pmatrix}$

$R = \sqrt{2.6^2 + 11.2^2}$ = 11.498N

(ii) direction of the resultant with AB



Direction,$\alpha = \tan^{-1}\left(\frac{2.6}{11.2}\right)$

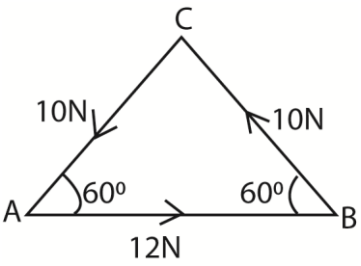
$=13.069^0$

Direction is 13.069^0 above AB

Example 2

ABC is an equilateral triangle. Forces of magnitude 12N, 10N and 10N act along AB, BC and CA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine

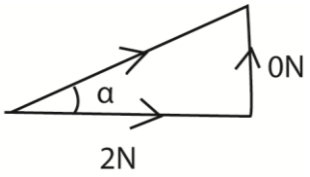
(i) the magnitude of the resultant force



$R = \begin{pmatrix}12\\0\end{pmatrix} + \begin{pmatrix}-10cos60\\10sin60\end{pmatrix} + \begin{pmatrix}-10cos60\\-10sin60\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$R = \sqrt{2^2 + 0^2}$ = 2N

(ii) Direction of the resultant with AB

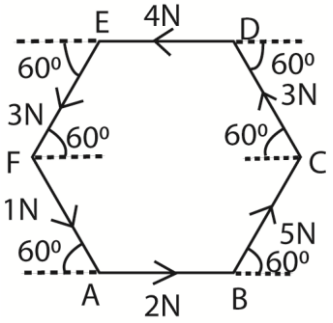


Direction $\alpha = \tan^{-1}\left(\frac{0}{2}\right) = 0^2$

Example 3

ABCDEF is a regular hexagon. Force of magnitude 2N, 5N, 3N, 4N, 3N and 1N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine

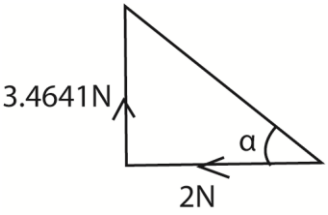
(i) the magnitude of the resultant force and



$R = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}5cos60\\5sin60\end{pmatrix} + \begin{pmatrix}-3cos60\\3sin60\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} + \begin{pmatrix}-3cos60\\-3sin60\end{pmatrix} + \begin{pmatrix}1\cos 60\\-1sin60\end{pmatrix} = \begin{pmatrix}-2\\3.4641\end{pmatrix}$

$R = \sqrt{(-2)^2 + 3.4641^2} = 4N$

(ii) direction of the resultant with AB.

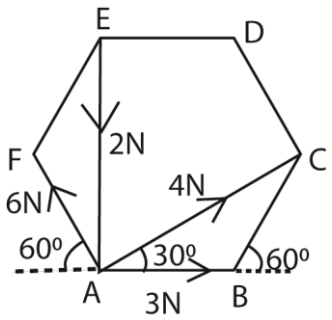


$\alpha = \tan^{-1}\left(\frac{3.461}{2}\right)$ = 60^0 to AB

Example 4

ABCDEF is a regular hexagon. Forces of magnitude 3N, 4N, 2N and 6N act along the line AB, AC, EA and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine

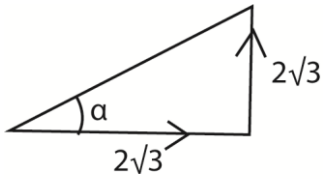
(i) the magnitude of the resultant force



$$R = \begin{pmatrix}3\\0\end{pmatrix} + \begin{pmatrix}4cos30\\4sin30\end{pmatrix} + \begin{pmatrix}-6cos60\\6sin60\end{pmatrix} + \begin{pmatrix}0\\-2\end{pmatrix} = \begin{pmatrix}2\sqrt{3}\\3\sqrt{3}\end{pmatrix}$$

$$R = \sqrt{\left(2\sqrt{3}\right)^2 + \left(3\sqrt{3}\right)^2} = 6.245\text{N}$$

(ii) direction of the resultant force



$$\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{2\sqrt{3}}\right) = 56.3^0$$

1. ABCD is a square. Forces of magnitude 6N, 4N and $2\sqrt{2}N$ act along AD, AB and AC respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
   [10N at 53.1^0 with AB]
2. ABCD is a square. Forces of magnitude 2N, 1N, $\sqrt{2}N$ and 4N act along AB, BC and AC and DA respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
   [5.13N at 33.7^0 with AB]
3. ABCD is a square. Three forces of magnitude 4N, 10N and $7N$ act along AB, AD and CA respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude [5.1388N]
4. In equilateral triangle PQR, three forces of magnitude 5N, 10N and 8N act along the side PQ, QR and PR respectively. Their direction are the order the letters. Find the magnitude of the resultant force. [16.1N]
5. ABCD is a square. Forces of magnitude $6\sqrt{3}N$ N, 2N and $4\sqrt{3}N$ act along AB, CB and CD respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
   [4N at 30^0 to AB]
6. ABCD is a rectangle with AB= 4cm and BC = 3cm. Forces of magnitude 3N, 1N, and 10N act along AB, DC and AC respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine the magnitude and direction of the resultant force. [13.4N at 26.6^0 with AB]
7. ABCD is a rectangle. Forces of magnitude 8N, 4N, 10N and 2N act along AB, CB, CD and AD respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine the magnitude and direction of the resultant force. [283N at 45^0 at AB]
8. In equilateral triangle ABC, forces of magnitude 10N each act along the side AB, BC and AC respectively. Their direction are the order the letters. Find the magnitude of the resultant force and the angle it makes with AB. [20N at $60^0$to AB]
9. In equilateral triangle ABC, forces of magnitude 5N, 9N and 7N act along the side AB, BC and CA respectively. Their direction are the order the letters. Find the magnitude of the resultant force and the angle it makes with AB. [$2\sqrt{3}$N at $30^0$to AB]

10. In equilateral triangle ABC, forces of magnitude 4N, 4N and 6N act along the side AB, BC and AC respectively. Their direction are the order the letters. Find the magnitude of the resultant force and the angle it makes with AB. [10N at 60^0 to AB]
11. ABCDEF is a regular hexagon. Forces of magnitude 2N, 5N, 3N, 4N, 3N and 1N act along the line AB, BC, CD,DE, EF and AF respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. [6N at 60^0 to AB]
12. ABCDEF is a regular hexagon. Forces of magnitude 8N, 7N, 6N, 4N, 7N, and 6N act along the line AB, BC, CD, DE, EF and FA respectively, in each case the direction of the force being given by the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force. [12.5N at 76^0 to AB]
13. PQRSTU is a regular hexagon. Forces of magnitude 4N, 5N, 2N, and 6N act along the line PQ, PR, PT and PU respectively, in each case the direction of the force being given by the order of the letters. Given that PQ is horizontal, determine the magnitude and direction of the resultant force. [11.065N at 61.2^0 to PQ]
14. ABCD is a square. Forces of magnitude $10N$ N, 9N, 8N and $5N$ act along AB, BC, CD and AD respectively in each case the direction of force being the order of the letters. Given that AB is horizontal, determine the magnitude and direction of the resultant force.
    [$2\sqrt{5}$N at 63.43^0 to AB]
15. ABCD is a rectangle with AB= 4cm and BC = 3cm. Forces of magnitude 3N, 10N, 4N, 6N and 5N act along AB, BC, CD, DA, and AC respectively. In each case the direction of the force being given by the order of the letters. Given that AB is horizontal determine the magnitude and direction of the resultant force. [7.62N at 66.8^0 with AB]

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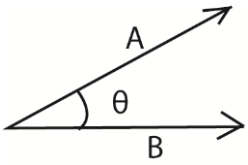
Resultant of forces

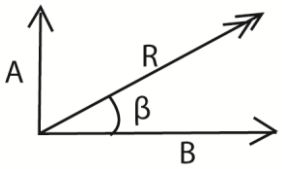
A force is anything which change a body's state of rest or uniform motion in a straight line. Examples are weight, tension, reaction, friction, resistance force.

Resultant of two forces

Consider two forces A and B inclined to each other at an angle θ



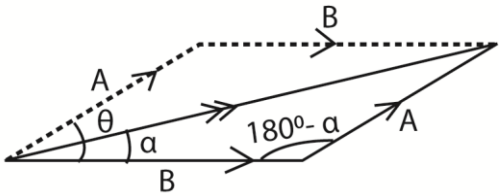
(i) θ is right angle (θ =90⁰)



Resultant, R = $\sqrt{A^2 + B^2}$

Direction of resultant, β =$\tan^{-1}\left(\frac{A}{B}\right)$

(ii) **θ is acute (0⁰ ≤ θ ≤ 90⁰)**

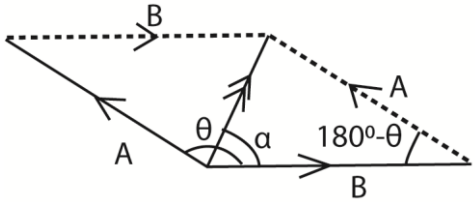


Direction of resultant, $\frac{\sin\alpha}{A} = \frac{\sin(180-\theta)}{R}$

$\alpha = \sin^{-1}\left(\frac{A\sin(180-\theta)}{R}\right)$

Resultant, R =$\sqrt{[A^2 + B^2 - 2ABcos\ (180-\theta)]}$

(iii) **θ is obtuse (90⁰ ≤ θ ≤ 180⁰)**



Direction of resultant, $\frac{\sin\alpha}{A} = \frac{\sin(180-\theta)}{R}$

$\alpha = \sin^{-1}\left(\frac{A\sin(180-\theta)}{R}\right)$

Resultant, R =$\sqrt{[A^2 + B^2 - 2ABcos\ (180-\theta)]}$

Example 1

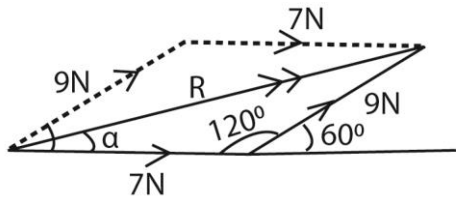
Two forces of magnitude 5N and 12N act on a particle with their direction inclined at 90^0. Find the magnitude and direction of the resultant

$R = \sqrt{5^2 + 12^2}$ = 13N α =$\tan^{-1}\left(\frac{5}{12}\right)$ = 22.6^0

The resultant = 13N at 22.6^0 to 12N force

Example 2

Forces of magnitude 7N and 9N act on a particle at an angle of 60^0 between them. Find the magnitude and direction of the resultant.



Direction of resultant, $\frac{\sin\alpha}{9} = \frac{\sin(180-\theta)}{13.89}$

$\alpha = \sin^{-1}\left(\frac{9\sin(180-60)}{13.89}\right)$ = 34.13^0

Resultant, R =$\sqrt{[A^2 + B^2 - 2ABcos\ (180-\theta)]}$

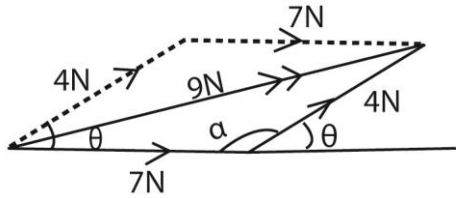
=$\sqrt{[7^2 + 9^2 - 2x7\ x\ 9cos\ (180-60)]}$

= 13.89N

Example 3

Find the angle between a force of 7N and 4N their resultant has a magnitude of 9N

Solution



$9^2 = 7^2 + 4^2 - 2 \times 7 \times 4 \times \cos\alpha$

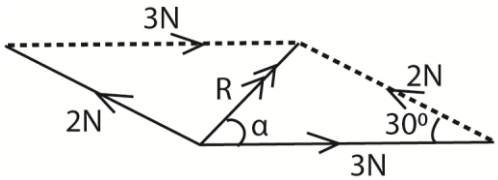
$\alpha = \cos^{-1}\left(-\frac{2}{7}\right)$ = 106.6^0

the angle θ between the forces = 180 – 106.6

= 73.4^0

Example 4

Forces of 3N and 2N act on a particle at an angle of 150^0 between them. Find the magnitude and direction of the resultant.



Direction of resultant, $\frac{\sin\alpha}{2} = \frac{\sin(30)}{1.61}$

$\alpha = \sin^{-1}\left(\frac{2\sin(180-60)}{1.61}\right)$ = 38.3^0

$R^2 = 2^2 + 3^2 - 2 \times 2 \times 2 \times \cos(30)$

R = 1.61N

Revision exercise

1. Two forces of magnitude 7N and 24N act on a particle with their direction at 90^0. Find the magnitude and direction of the resultant. [25N, 16.26^0 with 24N force]
2. Forces of 5N and 8N act on a particle at an angle of 50^0 between them. Find the magnitude and direction of the resultant. [11.9N at 19^0 with 8N force]
3. Forces of 4N and 6N act on a particle at angle 60^0 between them. Find the magnitude and the direction of the resultant. [5.29N, at 40.9^0 with 6N force]
4. Forces of 9N and 10N act on a particle at angle 40^0 between them. Find the magnitude and the direction of the resultant. [17.9N, at 18.9^0 with 10N force]
5. Forces of 12N and 10N act on a particle at angle 105^0 between them. Find the magnitude and the direction of the resultant. [13.5N, at 45.7^0 with 12N force]
6. Forces of 8N and 3N act on a particle at angle 160^0 between them. Find the magnitude and the direction of the resultant. [5.28N, at 11.2^0 with 8N force]
7. Find the angle between a force of 10N and 4N their resultant has a magnitude of 8N. [130.5^0]
8. The angle between a force αN and a force of 3N is 120^0. If the resultant of the two forces has magnitude 7N, find the value of α. [8N]
9. The angle between a force βN and a force of 8N is 45^0. If the resultant of the two forces has a magnitude 15N, find the value of β. [8.24N]

Thank you
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Simple harmonic motion

This is a periodic motion of a body whose acceleration is directly proportional to the displacement from a fixed point and is directed towards the fixed point.

$a \propto -x$ or;

$a = -\omega^2 x$ where ω is angular velocity, x, is displacement from fixed point

The negative sign means that the acceleration and displacement are always in opposite direction

Maximum acceleration

$a_{max} = -\omega^2 r$ where r is the maximum displacement or amplitude.

Force F

F = ma = $m\omega^2 x$

Maximum force, $F_{max} = m\omega^2 r$

Velocity in terms of displacement

Velocity of a body executing S.H.M can be expressed as a function of displacement x.

$a = -\omega^2 x$

$a = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$

But $\frac{dx}{dt} = v$

$a = v\frac{dv}{dx}$

$v\frac{dv}{dx} = -\omega^2 x$

$vdv = -\omega^2 x dx$

Integrating both sides

$\int v\,dv = -\omega^2 \int x dx$

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$ .....(i)

Where c is a constant of integration

At momentary rest v = 0

x = r (amplitude

$$\frac{0^2}{2} = -\frac{\omega^2 x^2}{2} + c$$

$c = \frac{\omega^2 r^2}{2}$

Substituting c in eqn. (i)

$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 r^2}{2}$

$v^2 = (\omega^2 r^2 - \omega^2 x^2)$

$\boldsymbol{v^2 = \omega^2(r^2 - x^2)}$

Velocity is maximum when x = 0

$v^2 = \omega^2 r^2$

$\boldsymbol{v_{max} = \omega r}$

Displacement at any time t

$\frac{dx}{dt} = v$

$\frac{dx}{dt} = \omega\sqrt{(r^2 - x^2)}$

$\int \frac{dx}{\sqrt{(r^2-x^2)}} = \int \omega dt$

$\sin^{-1}\frac{x}{r} = \omega t + \varepsilon$

x = rsin($\omega t + \varepsilon$)

When timing at the centre, t = 0, x = 0

x = rsinωt particle moves away from the centre

When timing at the amplitude, t = 0, x = r

x = rsinωt particle moves towards from the centre

Period

This is the time taken for one complete oscillation

T = $\frac{distance}{speed} = \frac{2\pi r}{v}$

But v = rω

T = $\frac{2\pi r}{r\omega}$

T = $\frac{2\pi}{\omega}$

Example 1

A particle moves in a straight line with simple harmonic motion about mean position O with a periodic time of $\frac{\pi}{2}$s. Find the magnitude of acceleration of the particle when 1m from O.

Solution

From $a = -\omega^2 x$ and $\omega = \frac{2\pi}{T}$

Negative ignored

$a = \left(\frac{2\pi}{\pi/2}\right)^2 x1 = 16ms^{-2}$

Example 2

A particle moves with S.H.M about a mean position O. When the particle is 25cm from O, its acceleration is 1ms^{-1} towards O. Find the

(i) Periodic time of motion
(ii) Magnitude of acceleration of the particle when 20cm from O

Solution

$a = -\omega^2 x$

$1 = \omega^2(0.25)$

$\omega^2 = 4$

$\omega = 2rads^{-1}$

T = $\frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$s

(ii) $a = -\omega^2 x$

$a = 2^2(0.2)$ = 0.8ms^{-2}

Example 3

A particle move with S.H.M of the periodic time $\frac{\pi}{2}$s and has a maximum speed of 3ms^{-1}. Find the maximum acceleration experienced by the particle

$\boldsymbol{v_{max} = \omega r}$

3 = $\left(\frac{2\pi}{\pi/_2}\right) r$; r = 0.75m

$a_{max} = -\omega^2 r$

$a_{max} = \left(\frac{2\pi}{\pi/_2}\right)^2 x\ 0.75$ = 12ms^{-2}

Example 4

A particle moves with S.H.M about a mean position O. the amplitude of the motion is 5m and the period is 8πs. Find the

(i) maximum speed of the particle (ii) speed of the particle when 3m from O

Solution

(i) $\boldsymbol{v_{max} = \omega r}$ = $\frac{2\pi}{8\pi} x5$ = 1.25ms^{-1}

(ii) $v^2 = \omega^2(r^2 - x^2)$

$v = \sqrt{\left(\frac{2\pi}{8\pi}\right)^2 (5^2 - 3^2)}$ =1ms^{-1}

Example 5

A body of mass 200g executes S.H.M with amplitude of 20mm. The maximum force which acts on it is 0.064N, calculate (a) its maximum velocity (ii) it period of oscillation

Solution

$F_{max} = m\omega^2 r$

0.064 = 0.2 $x\ \omega^2 x\ 0.02$

$\omega = 4rads^1$

$v_{max} = \omega r$ = 4 x 0.02 = 0.08ms^{-1}

(ii) T = $\frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ s

Example 6

A particle moves with S.H.M about O with a period of 2π seconds. It passes a point A with a velocity of 4ms^{-1} away from O. Given that OA = 4m, find

(i) The amplitude
(ii) The speed at B where OB = 3m

Solution

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1rads^{-1}$

From $v^2 = \omega^2(r^2 - x^2)$

$4^2 = 1^2(r^2 - 4^2)$

r = 5.66m

Amplitude = 5.66m

(ii) $v = \omega\sqrt{r^2 - x^2}$

$v = \omega\sqrt{5.66^2 - 3^2}$ = 4.8ms^{-1}

Example 7

A particle moving with S.H.M has velocities of 4ms^{-1} and 3ms^{-1} at distance of 3m and 4m respectively from equilibrium position. Find

(a) amplitude (b) period

$v^2 = \omega^2(r^2 - x^2)$

$4^2 = \omega^2(r^2 - 3^2)$

$16 = \omega^2(r^2 - 9)$ …….. (i)

Also,

$3^2 = \omega^2(r^2 - 4^2)$

$9 = \omega^2(r^2 - 16)$ ……..(ii)

(i) ÷ (ii) $\frac{16}{9} = \frac{\omega^2(r^2-9)}{\omega^2(r^2-16)}$

r = 5m

Amplitude = 5m

(b) eqn. (i)

$16 = \omega^2(5^2 - 9)$

ω=1rads^{-1}

Period,$T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\pi$s

Example 8

A particle moves with S.H.M about a mean position O. the particle is initially projected from O with speed $\frac{\pi}{6} ms^{-1}$ and just reaches a point A, 2m from O.

(a) Find how far the particle from O, 3 seconds after projection
(b) How many second after projection is the particle a distance of 1m from O
    (i) For the first time, (ii) second time (iii) third time

Solution

(a) At equilibrium position, $v_{max} = \omega r$

$\frac{\pi}{6} = \omega\, x2; \omega = \frac{\pi}{12} rads^{-1}$

x = rsinωt since particle moves away from O

x =2sin$\left(\frac{\pi}{12} x3\right)$= 1.414m

x = rsinωt

1 = rsin$\left(\frac{\pi}{12}\right)$t

$\left(\frac{\pi}{12}\right)$t =sin^{-1} 0.2= 30^0, 150^0, 210^0

T = 2s, 10s, 14s

Example 9

A particle is released from rest at point A. 1m from a second point O. the particle accelerates towards O and moves with S.H.M of period 12s and O is the centre of oscillation

(a) Find how far the particle is from O, 1s after release
(b) How many seconds after release is the particle at the midpoint of OA
    (i) For the first time (ii) second time

Solution

(a) x = rcosωt since particles moves towards centre

$\omega = \frac{2\pi}{12} = \frac{\pi}{6} rads^{-1}$

$x = 1cos\frac{\pi}{6}\, x\, 1 = \frac{\sqrt{3}}{2} m$

(b) x = rcosωt

$0.5 = 1cos\frac{\pi}{6}\, x\, t$

$\frac{\pi}{6} t = \cos^{-1} 0.5$ =60^0, 300^0

t = 2s, 10s

Example 10

A particle of mass 2kgmoving with S.H.M along the x-axis, is attracted towards the origin O by a force of 32x newton. Initially the particle is at x = 20. Find

(a) amplitude and period of oscillation
(b) velocity of the particle at any time t > 0
(c) speed when t = $\frac{\pi}{4}$s

Solution

(a) F = $m\omega^2x$
$32x = 2\omega^2x$
$\omega = 4rads^{-1}$
$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.571s$
$v^2 = \omega^2(r^2 - x^2)$
$0^2 = 4^2(r^2 - 20^2)$
r =20m

(b) x = rcosωt
$v = \frac{d}{dt}(rcos\omega t)$
v = -rωsinωt = -20 x 4sin4t
v = -80sin4t

(c) speed = $-80\sin\left(4\ x\frac{\pi}{4}\right) = 0ms^{-1}$

Example 11

A particle is initially released from rest at point A and performs S.H.M about mean position B. The particle just returns to A during each oscillation and AB = $2\sqrt{2}m$. If the particle passes through B with speed $\pi\sqrt{2}ms^{-1}$, find

(i) The time when the particle is first travelling with speed of πms^{-1}
(ii) How far from B the particle during this time

Solution

(i) At equilibrium position (B)

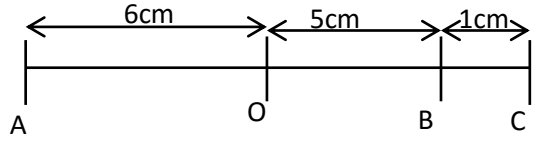
$v_{max} = \omega r$
$\pi\sqrt{2} = \omega\ x\ 2\sqrt{2}$
$\omega = \frac{\pi}{2}rads^{-1}$
$v^2 = \omega^2(r^2 - x^2)$
$\pi^2 = \left(\frac{\pi}{2}\right)^2\left(\left(2\sqrt{2}\right)^2 - x^2\right)$

x = 2m

(ii) x = rcosωt
$2 = 2\sqrt{2}cos\frac{\pi}{2}t$
$\frac{\pi}{2}t = \cos^{-1}\frac{1}{\sqrt{2}} = 45^0$
t = 0.5s

Example 12

The points A, O, B, C lie in the that order on a straight line AO= OOC = 6cm and OB = 5c. A particle perform S.H.M of period 3s and amplitude 6cm between A and C. find the time taken for the particle from A to B



Time for AO is half the period = 1.5s

B is 5cm from O

x = rcosωt

5 = 6cos$\frac{2\pi}{3}$t

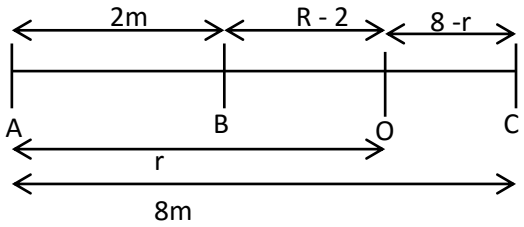
$\frac{2\pi}{3}$t = $\cos^{-1}\frac{5}{6}$ = 33.6^0

t = 0.28

Time for AB = 1.5 + 0.28 = 1.78s

Example 13

A particle passes through 3 points A, B and C in that order with velocity 0ms^{-1}, 2ms^{-1} and -1ms^{-1} respectively. The particle is moving with S.H.M in a straight line. What is the amplitude and period of the motion if AB = 2m and AC = 8m.



At B: v = 2ms^{-1}, x = r-2

Using $v^2 = \omega^2(r^2 - x^2)$

$2^2 = \omega^2(r^2 - (r-2)^2)$

$1 = \omega^2(r-1)$ ..... (i)

At C: v =-1ms^{-1}, x = 8-r

$(-1)^2 = \omega^2(r^2 - (8-r)^2)$

$1 = \omega^2(16r - 64)$ ..... (ii)

(i)÷ (ii) $\frac{1}{1} = \frac{\omega^2(r-1)}{\omega^2(16r-64)}$

r = 4.2

Amplitude = 4.2m

Using equation (i)

$1 = \omega^2(4.2 - 1)$

ω = 0.56rads^{-1}

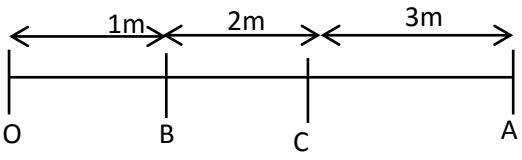
T = $\frac{2\pi}{\omega} = \frac{2\pi}{0.56} = 11.22$s

Example 14

A particle is performing S.H.M with centre O, amplitude 6m and period 2π. Points B and C lis between O and A with OB = 1m, OC = 3m and OA = 6m. find the least time taken while travelling from

(a) A to B (ii) A to C

Solution



$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

x = rcosωt

1 = 6cos (1x t)

t= 1.403s

(ii)

x = rcosωt

3 = 6cos (1x t)

t = 1.047s

Example 15

The velocity of a particle at any time is given by v(t) = -aωsinωt + bωcosωt.

(a) Find the expression for displacement x at any time that x = 0 when time t = 0

v(t) = -aωsinωt + bωcosωt.

But v(t) = $\frac{dx}{dt}$

⇨ $\frac{dx}{dt}$ =-aωsinωt + bωcosωt.

$\int dx = \int(-\text{a}\omega\text{sin}\omega\text{t} + \text{b}\omega\text{cos}\omega\text{t})dt$

x = $\frac{a\omega}{\omega}\text{cos}\omega\text{t} + \frac{b\omega}{\omega}\text{sin}\omega\text{t} + \text{c}$

Since a and b are expressed in terms of amplitude and phase angle ε

If at t =0, x = 0, this means that A = 0, hence a = b = 0

By substituting, 0= 0 + 0 + c

c = 0

hence : x = acosωt+ bsinωt

(b) Show that the motion of the particle is simple harmonic [x

$\frac{dx}{dt}$ =-aωsinωt + bωcosωt.

$\frac{d^2x}{dt^2}$ =-aω^2cosωt - bω^2sinωt

- $\omega^2(acos\omega t + bsin\omega t)$

But x = acosωt+ bsinωt

⇨ $\frac{d^2x}{dt^2}$ =- $\omega^2 x$, hence S.H.M

Example 16

A particle is moving with simple harmonic motion (SHM). When the particle is 15m from the equilibrium, its speed is 6ms^{-1}. When the particle is 13 m from equilibrium, its speed is 9ms^{-1}. Find the amplitude of the motion (05marks)

$v^2=\omega^2(A^2 - x^2)$

$6^2= \omega^2(A^2 - 15^2)$ ……. (i)

$9^2=\omega^2(A^2 - 13^2)$ …….. (ii)

Dividing (i) by (ii)

$\frac{36}{81} = \frac{A^2-225}{A^2-169}$

Amplitude A = 16.4256m

Revision exercise 1

1. A particle moving with S.H.M about a mean position O has velocities of 5ms^{-1} and 8ms^{-1} at distances of 16m and 12m respectively from O.
   (a) Amplitude [18.1m]
   (b) Period [10.63s]
2. A particle is describing S.H.M in a straight line directly towards a fixed point O. when it is a distance from O is 3m, its velocity is 25ms^{-1} and acceleration is 75ms^{-2}. Determine the
   (a) Period [$\left[\frac{2\pi}{5}\right]$ and amplitude [5.83m]
   (b) Time taken by the particle to reach O$\left[\frac{\pi}{10}\right]$
   (c) Velocity of the particle as it passes through O [29.15ms^{-1}]
3. A particle moving with simple harmonic motion about a mean position O has velocities of $3\sqrt{3}ms^{-1}$ and 3ms^{-1} at distances of 1m and 0.268m respectively. Find the amplitude of motion [2m]
4. A mass oscillates with S.H.M of period 1second and amplitude of oscillation is 5cm. Given that the particle begins from the centre of the motion, state the relationship between displacement x of the mass at any time t. Hence find the first two times when the is 3cm from its end position [x = rsinωt, 0.066s, 0.434s]
5. A particle moves in a straight line with S.H.M of period5s. the greatest speed is 4ms^{-1}, find the

(a) amplitude $\left[\frac{10}{2\pi}m\right]$.

(b) Speed when it is $\frac{6}{\pi}m$ from the centre. [3.2ms^{-1}]

6. A particle moves with S.H.M about mean position O with a periodic time $\frac{2\pi}{3}s$. When the particle is 0.8m from one extreme end, its speed is 3.6ms^{-1}. Find the amplitude of motion [1.3m]
7. A body of mass 0.30kg executes S.H.M with a period 2.5s and amplitude 0.04m. Determine
   (i) Maximum velocity of the body [0.101ms^{-1}]
   (ii) The maximum acceleration of the body [0.25ms^{-2}]
8. A particle moving with S.H.M about a mean position O has velocities of 1.6ms^{-1} and 1.2ms^{-1} at distances of 60cm and 80cm respectively from O. find
   (i) Amplitude [1m]
   (ii) Period [πs]
9. A particle moves with S.H.M in a straight line with amplitude 0.05m and period 12s. find
   (a) speed as it passes equilibrium position [0.026ms^{-1}]
   (b) maximum acceleration of the particle [0.014ms^{-2}]
10. A particle moves in straight line with S.H.M about mean position with periodic time $\frac{\pi}{2}s$ and amplitude 2m. Find the maximum speed of the particle [8ms^{-1}]
11. A body of mass 500g moves horizontally with S.H.M of periodic time $\frac{\pi}{2}s$ and amplitude 1m. Determine the magnitude of the greatest horizontal force experienced by the body during the motion. [8N]
12. A body of mass 100g moves horizontally with S.H.M about a mean position O. When the body is 50cm from O, the horizontal force on the body is of magnitude 5N, find the period of motion $\left[\frac{\pi}{2}s\right]$
13. A particle moves in a straight line with S.H.M about a mean position O with a periodic time $\frac{\pi}{4}s$ and amplitude 65cm. find how far the particle from O when its speed is 2ms^{-1} [60cm]
14. A particle moves in a straight line with S.H.M about a mean position O. The particle has zero velocity at a point which is 50cm from O and speed of 3ms^{-1} at O. Find
   (a) The maximum speed of the particle [3ms^{-1}]
   (b) The amplitude of motion[50cm]
   (c) The periodic time of motion[$\frac{\pi}{2}s$]
15. A particle moving with S.H.M about a mean position O has velocities 3ms^{-1} and 1.4ms^{-1} at distances of 2m and 2,4m respectively from point O. Find the
   (a) Amplitude of motion [2.5m]
   (b) Greatest speed attained by the particle [5ms^{-1}]
16. A particle is initially projected from a point A performs S.H.M about mean position A with periodic time of 3s and amplitude 50cm. find the
   (a) Maximum speed of projection [1.047ms^{-1}]
   (b) Speed of the particle 2s after projection [0.524ms^{-1}]
   (c) Distance of the particle from A 2s after projection[0.433m]
17. The head of piston moves with S.H.M of amplitude $\frac{\sqrt{3}}{10}$m about mean position O. How far from O the head of the piston when travelling with a speed equal to half of its maximum speed [15cm]
18. A particle is fastened to the midpoint of a stretched spring lying on a smooth horizontal surface. The particle is set in motion so that it moves with S.H.M about a mean position O. If one metre is the greatest distance the particle is from O during the motion. Find how far from O the particle is when it is travelling with speed equal to four fifth of its greatest speed. [60cm]

19. A particle performs S.H.M about mean position O with a periodic time of 3s and amplitude 6cm. Find time it takes the particle to travel from O to a point P, a distance of 3cm from O [0.25s]
20. A particle performs S.H.M about mean position O with a periodic time of 4s and amplitude 2cm. Find the time it takes the particle to travel from O to a point P, a distance of $\sqrt{2}$cm from O[0.5s]
21. A particle performs S.H.M about mean position O with a periodic time of 10s and amplitude 8cm. After passing through O, the particle moves through a point A which is 1cm from to a point B which is 2cm from O. find the time it takes the particle to move from A to B[0.186s]
22. A particle performs S.H.M about mean position O with a periodic time of 4.5s and amplitude 6cm. After passing through O, the particle moves through point P which is 3cm from O. Find the time that elapse before the particle passes through P [1.5s]
23. The points A, O, B, C lie in that order on a straight line with AO = OC = 4cm and OB = 2cm. A particle perform S.H.M of period 6s and amplitude 4cm between A and C. Find the time taken for the particle to travel from A to B [2s]

Thank you

Dr. Bbosa Science

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A-level math paper 2: Statistics

This a branch of mathematics dealing with collection, presentation, analysis and interpretation of data

Types of data

(a) Discrete data
Its information collected by counting and usually takes integral values that do not lie within a given range

(b) Continuous data
It is information that takes values within a given range

Discrete or ungrouped data

Measures of central tendency

These are values of the distribution that tend to locate the central values. They include mean, median and mode

(i) Mean or average of a sample

It is denoted by $\bar{x}$ and defined as $\bar{x} = \frac{\sum x}{n}$; where x is the variable given and n is the number of variable

If assumed mean (working mean) a is given then

$\bar{x} = A + \frac{\sum d}{n}$; where d = x – A

If the frequency, f, s given then $\bar{x} = \frac{\sum fx}{\sum f}$ or $\bar{x} = A + \frac{\sum fd}{\sum f}$

(ii) Mode

This is the value of the distribution that appears most

(iii) Median

This is the middle value of the distribution obtained after the values have been arranged either in ascending of descending order.

Median = $\left(\frac{N}{2}\right)^{th}$ value.

Example 1

Given the following sets of values

2, 1, 3, 4, 5, 6, 7, 8, 9, 10, 3, 4, 6, 8, 9, 6, 3, 2

(a) Form a frequency table of ungrouped data

x	f	fx	cf
1	1	1	1
2	2	4	3
3	3	9	6
4	2	8	8
5	1	5	9
6	4	24	13
7	2	14	15
8	2	16	17
9	2	18	19
10	1	10	20
	$\sum f = 20$	$\sum fx = 109$	

(b) Use the table to find the mean and mode

Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{109}{20} = 5.45$

Mode = 6 (the value that appear most

(c) Find the median value

Median = $\left(\frac{N}{2}\right)^{th}$ value = $\left(\frac{20}{2}\right)^{th}$ = 10th value from cumulative frequency, cf ; median = 6

Example 2

Given the information below

x	10	11	12	13	14	15	16	17	18
f	4	2	6	3	7	2	1	2	2

Find

(a) Mean
(b) modal value
(c) median

Solution

x	f	fx	Cf
10	4	40	4
11	2	22	6
12	6	72	12
13	3	39	15
14	7	98	22
15	2	30	24
16	1	16	25
17	2	34	27
18	2	36	29
	$\sum f = 29$	$\sum fx = 387$	

(a) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{387}{29} = 13.34$

(b) Modal value = 14 (appear most)

(c) Median = $\left(\frac{N}{2}\right)^{th}$ value = $\left(\frac{29}{2}\right)^{th}$ = 14.5th value from cumulative frequency, cf ; ***median = 13***

Example 3

The following are marks obtained by 30 students in a mathematic test marked out of 10

Mark (x)	0	1	2	3	4	5	6	7	8	9	10
Frequency (f)	2	2	3	2	3	4	5	2	1	3	3

(a) Find the mean using a working mean of 5
(b) Find the modal mark and he median

Solution

x	f	D= (x-5)	fd	fc
0	2	-5	-10	2
1	2	-4	-8	4
2	3	-3	-9	7
3	2	-2	-4	9
4	3	-1	-3	12
5	4	0	0	16
6	5	1	5	21
7	2	2	4	23
8	1	3	3	24
9	3	4	12	27
10	3	5	15	30
	$\sum f = 30$		$\sum fx = 5$	

(a) $\bar{x} = A + \frac{\sum fd}{\sum f} = 5 + \frac{5}{30} = 5.167$
(b) Modal value = 6
(c) Median = $\left(\frac{N}{2}\right)^{th}$ value = $\left(\frac{30}{2}\right)^{th}$ = 15th value from cumulative frequency, cf ; median = 5

Measures of dispersion

This is a measure of how the observations are spread out from the mean

(a) Range

It is the difference between the largest value and the smallest value.

Example 4

Find the range of the following set of values

2, 1, 3, 4, 5, 6, 7, 8, 9, 10, 3, 4, 6, 8, 9, 6, 3, 2

Range = 10 – 1 = 9

(b) Variance of a sample

The variance of x is denoted by Var(X) is defined as

Var(X) = $\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$

Var(X) = $\frac{\sum x^2}{n} - \bar{x}^2$

If the frequency is given then

Var(x) = $\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$

= $\frac{\sum fx^2}{\sum f} - \bar{x}^2$

(c) Standard deviation

S.d = $\sqrt{\text{Var(X)}}$

= $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

= $\sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$

Example 5

Find the variance and standard deviation for the following data; 5, 8, 12, 13, 15

Solution

x	x^2
5	25
8	64
12	144
13	169
15	225
$\sum x$ =53	$\sum x^2$ =627

Var(X) = $\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = \frac{627}{5} - \left(\frac{53}{5}\right)^2$

= 125.4 - 112.36 = 13.04

S.d = $\sqrt{\text{Var(X)}} = \sqrt{13.04}$ =3.611

Example 5

The frequency distribution table shows the marks of some student from a certain school

x	45	63	65	66	70	72	75	80	88
f	3	5	6	4	6	2	1	2	1

Calculate standard deviation

Solution

x	f	fx	fx^2
45	3	135	6075
63	5	315	19845
65	6	390	25350
66	4	264	17424
70	6	420	29400
72	2	144	10368
75	1	75	5625
80	2	160	12800
88	1	88	7744
	$\sum f$ =30	1991	$\sum fx^2$ =134631

S.d = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$

= $\sqrt{\frac{134631}{30} - \left(\frac{1991}{30}\right)^2}$

= $\sqrt{\frac{134631}{30} - \left(\frac{1991}{30}\right)^2}$

=$\sqrt{4487.7 - 4404.53}$

= 9.12

Using assumed mean to get variance and standard deviation

Var (X) = $\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$

s.d = $\sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$

Example 6

The frequency distribution table shows the heights of some students at a certain shool

Height	154	155	160	164	171	180
Frequency	4	6	8	5	4	3

Determine the variance and standard deviation of the data using a working mean of 160

Solution

x	f	d = x - A	fd	fd^2
154	4	-6	-24	144
155	6	-5	-30	150
160	8	0	0	0
164	5	4	20	80
171	4	11	44	484
180	3	20	60	1200
	$\sum f$ =30		$\sum fd$ =70	$\sum fd^2$ =2058

Var (X) = $\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$

= $\frac{2058}{30} - \left(\frac{70}{30}\right)^2$

= 63.156

s.d = $\sqrt{Var(x)}$ = $\sqrt{63.156}$ =7.95

Quartiles

A quartile is a value that divides given values into four equal parts

q_1 is the lower quartile and is defined by

$q_1 = \left(\frac{1}{4}N\right)^{th}$ value where N is the sum of all the variables

q_3 is the upper quartile and is defined by

$q_3 = \left(\frac{3}{4}N\right)^{th}$ value where N is the sum of all the variables

Percentiles

A percentile is a value that divides given values into 100 pats.

P_{10} is the 10^{th} percentile and is defined as

$P_{10} = \left(\frac{10}{100}N\right)^{th}$ value where N is the sum of all the variables

P_{90} is the 10^{th} percentile and is defined as

$P_{90} = \left(\frac{90}{100}N\right)^{th}$ value where N is the sum of all the variables

Deciles

A decile is a value that divides given values into 10 pats.

D_5 is the 5th decile and is defined as

$D_5 = \left(\frac{5}{10}N\right)^{th}$ value where N is the sum of all the variables

Example 7

The table below shows the marks obtained by 20 students in a mathematics test marked out of 20

Marks	10	11	12	13	14	15	16	17	18	19	20
Number of students	1	2	2	2	2	4	2	1	2	1	1

Find:

(a) Mean mark
(b) Standard deviation
(c) 60th percentile
(d) Interquartile range

Solution

x	f	cf	fx	fx^2
10	1	1	10	100
11	2	3	22	242
12	2	5	24	288
13	2	7	26	338
14	2	9	28	392
15	4	13	60	900
16	2	15	32	512
17	1	16	17	289
18	2	18	36	648
19	1	19	19	361
20	1	20	20	400
	$\sum f$ =20		$\sum fx$ =294	$\sum fx^2$ =4470

(a) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{294}{20} = 14.7$

(b) s.d = $\sqrt{\frac{\sum fd^2}{\sum f} - \bar{x}^2}$

$= \sqrt{\frac{4470}{20} - 14.7^2}$

= 2.722

(c) 60th percentile

$= \left(\frac{60}{100}\ x\ 20\right)^{th}$ value

= 12th value from cf =15

(d) $q_1 = \left(\frac{1}{4}\ x\ 20\right)^{th}$ value = 5th value from cf = 12

$Q_3 = \left(\frac{3}{4}\ x\ 20\right)^{th}$ value = 15th value from cf = 16

Interquartile range = 16 -12 = 4

Example 8

Given the following scores

8, 6, 8, 9, 10, 6, 4, 5, 6, 4, 4, 6, 8, 7, 10, 8, 6, 11, 12,8

(a) Form a frequency distribution table of ungrouped data.

(b) Find the standard deviation
(c) Calculate semi-quartile range
(d) Determine the range of 45th and 90th percentile.

Solution

x	f	cf	fx	fx^2
4	3	3	12	48
5	1	4	5	25
6	5	9	30	180
7	1	11	7	49
8	5	16	40	320
9	1	17	9	81
10	2	19	20	200
11	1	20	11	121
12	1	21	12	144
	$\sum f$ =20		$\sum fx$ =146	$\sum fx^2$ =1168

(b) s.d = $\sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$

$= \sqrt{\frac{1168}{20} - \left(\frac{146}{20}\right)^2}$ =2.26

(c) $q_1 = \left(\frac{1}{4}\ x\ 20\right)^{th}$ value

= 5th value from cf = 6

$q_3 = \left(\frac{3}{4}\ x\ 20\right)^{th}$ value

= 15th value, from cf = 8

Semi-interquartile range = $\frac{8-6}{2} = 1$

(d) 45th percentile = $\left(\frac{45}{10}\ x\ 20\right)^{th}$ value = 9th value from cf = 6

90th percentile = $\left(\frac{90}{100}\ x\ 20\right)^{th}$ value = 18th value from cf = 10

The range between the 90th percentile and 45th percentile = 10 – 6 = 4

Revision exercise 1

1. The data below represents the length of leaves in cm: 4.5 4.4, 6.2, 9.4, 8.2, 12.6, 10.0, 8.8, 3.8 and 13.6. find the;
   (a) Mean
   (b) Standard deviation
2. The concentration in m per litre of a trace element in 7 randomly chosen samples of water from spring wells were: 240.8, 237.3, 236.6, 2333.9 and 232.5. Determine the mean and the variance of the concentration of the trace elements per litre.
3. The table below shows the length of flowers from a certain plant to the nearest 0.5cm.

Length (cm)	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0
Number of flowers	4	9	11	8	10	7	2	3

   Find the:
   (a) Mean
   (b) Mode
   (c) The median
   (d) Standard deviation
4. The marks scored by 11 students in a test are:52, 61, 78, 49, 47, 79, 54, 58, 62, 73, 72
   Find;
   (a) Median,
   (b) Mean,
   (c) Interquartile range
   (d) Semi-quartile range

Solutions to revision exercise 1

1. The data below represents the length of leaves in cm: 4.5 4.4, 6.2, 9.4, 8.2, 12.6, 10.0, 8.8, 3.8 and 13.6. find the;
   (a) Mean
   (b) Standard deviation

   Solution

x	x^2
4.5	20.25
4.4	19.36
6.2	38.44
9.2	84.64
8.2	67.24
12	144
10	100
8.8	77.44
3.8	14.44
13.6	184.96
$\sum x$ =80.7	$\sum x^2$ =750.77

Mean, $\bar{x} = \frac{\sum x}{n} = \frac{80.7}{10} = 8.07$

s.d = $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{750.77}{10} - (8.07)^2}$ =3.155

2. The concentration in m per litre of a trace element in 7 randomly chosen samples of water from spring wells were: 240.8, 237.3, 236.6, 2333.9 and 232.5. Determine the mean and the variance of the concentration of the trace elements per litre.

   Solution

x	x^2
240.8	57984.64
237.3	56311.29
236.7	56026.89
234.2	54849.64
236.6	55979.56
233.9	54709.21
232.5	54056.25
$\sum x$ =1652	$\sum x^2$ =389917.5

Mean, $\bar{x} = \frac{\sum x}{n} = \frac{1652}{7} = 236$

Var(x) = $\frac{\sum x^2}{n} - \bar{x}^2$

$= \frac{389917.5}{7} - 236^2$

= 55,702.5 – 55,696

= 6.5

3. The table below shows the length of flowers from a certain plant to the nearest 0.5cm.

Length (cm)	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0
Number of flowers	4	9	11	8	10	7	2	3

Find the:
(a) Mean
(b) Mode
(c) The median
(d) Standard deviation

Solution

x	f	fx	fx^2	cf
7.5	4	30	225	4
8.0	9	72	576	13
8.5	11	93.5	794.75	24
9.0	8	72	648	32
9.5	10	95	902.5	42
10.0	7	70	700	49
10.5	2	21	220.5	51
11.0	3	33	363	54
	$\sum f$ =54	$\sum fx$ =486.5	$\sum fx^2$ =4429.75	

(a)Mean = $\frac{\sum fx}{\sum f}$

$= \frac{486.2}{54} = 9.0$

(b) mode = 8.5

(c) Median = $\left(\frac{54}{2}\right)^{th}$ value

= 27th value

From cf the 27th value = 9

(d) s.d = $\sqrt{\frac{\sum x^2}{\sum f} - \bar{x}^2}$ = $\sqrt{\frac{4429.75}{54} - \bar{x}^2}$ = 1

4. The marks scored by 11 students in a test are:52, 61, 78, 49, 47, 79, 54, 58, 62, 73, 72
   Find;
   (a) Median
   Arrange values in ascending order
   47, 49, 52, 54, 58, 61, 62, 72,73,78,79
   (a) Median =$middle\ number$ = 6th value = 61
   (b) Mean,
   $\sum x$ = 52+ 61+78 + 49 + 47 + 79 + 54 + 58 + 62 + 73 + 72 =685
   Mean = $\frac{\sum x}{n} = \frac{685}{11}$ = 62.273
   (b) Interquartile range
   $q_1 = \left(\frac{1}{4}\ x\ 11\right)^{th}$ value = 2.75th value = 52

   $q_3 = \left(\frac{3}{4}\ x\ 11\right)^{th}$ value = 8.25th value = 73

   Interquartile range = 73 -52 = 21

   (c) Semi-quartile range = $\frac{Interquartile\ range}{2} = \frac{21}{2} = 10.5$

Continuous or grouped data.

This is data whose scores or values are said to be continuous and take interval values

Example 9

Class	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 - 89
Number of students	4	5	7	3	6	4	1

Draw a frequency table

Terms used

(a) Class: these are limits of distribution. In the table above, the classes are: (20 – 29), (30 – 39), (40 – 49), (50 – 59), (60 – 69), (70 – 79), (80 – 89).

(b) Class mark (mark)
This is the mid-point value of the class. It is normally denoted by x. in the table above, the class marks are 24.5, 34.5, 44.5 …..

(c) Class boundary
These are continuous class limits. In the above table the first class boundary is (20-0.5) – (29 +0.5). In this case, the lower class boundary is 19.5 and upper class boundary is 29.5

For class interval 2.0 – 2.9, the class boundary is (2.0 -0.05) – (2.9 + 0.05) = 1.95-2.95.

(d) Class width or class interval
This is the width of each class boundary.
It is given by;
Class width = upper class boundary – lower class boundary
In the table above, class width = 29.5 – 19.5 = 10

Solution

Class	Frequency	Class width	Class mark, x	Class boundary
20 – 29	4	10	24.5	19.5 - 29.5
30 – 39	5	10	34.5	29.5 - 39.5
40 – 49	7	10	44.5	39.5 - 49.5
50 – 59	3	10	54.5	49.5 - 59.5
60 – 69	6	10	64.5	59.5 - 69.5
70 – 79	4	10	74.5	69.5 - 79.5
80 – 89	1	10	84.5	79.5 - 89.5

Example 10

The data below shows the heights in centimetres of 70 students.

Height (cm)	130 – 135	135 – 140	140 – 145	145 – 150	150 – 160	160 – 170	170 – 180
Number of students	10	12	8	9	11	15	5

Construct a frequency distribution table for the above data

Class	Class boundary	Class width	Class mark, x	Frequency, f
130 – 135	130 – 135	5	132.5	10
135 – 140	135 – 140	5	137.5	12
140 – 145	140 – 145	5	142.5	8
145 – 150	145 – 150	5	147.5	9
150 – 160	150 – 160	10	155	11
160 – 170	160 – 170	10	165	15
170 - 180	170 - 180	10	175	5

Example 11

Use the data below to construct a frequency distribution table

Marks	20-<30	30-<40	40-<50	50-<60	60-<70	70-<80	80-<90
Number of students	10	14	8	18	11	15	5

Solution

Class	Class boundary	Class width	Class mark, x	Frequency, f
20-<30	20 – 30	10	25	10
30-<40	30 – 40	10	35	14
40-<50	40 – 50	10	45	8
50-<60	50 – 60	10	55	18
60-<70	60 – 70	10	65	11
70-<80	70 – 80	10	75	15
80-<90	80 – 90	10	85	5

Example 12

The table below shows the ages of 35 people

Age	0-	5-	10-	15-	20-	30-	40-
Frequency	4	6	3	5	7	2	8

Draw a frequency table for the data

Solution

Class	Class boundary	Class width	Class mark, x	Frequency, f
0-	0 – 5	5	2.5	4
5-	5 – 10	5	7.5	6
10-	10 – 15	5	12.5	3
15-	15 – 20	5	17.5	5
20-	20 – 30	10	25	7
30-	30 – 40	10	35	2
40-	40 – 45	5	42.5	7

Note : the last class width is 5 since it is the most common

Example 13

The table below shows the marks of 40 students

Marks	-20	-30	-40	-50	-60	-65	-70
Frequency	8	4	7	10	2	2	7

Draw a frequency table

Solution

Class	Class boundary	Class width	Class mark, x	Frequency, f
-20	10 – 20	10	15	8
-30	20 – 30	10	25	4
-40	30 – 40	10	35	7
-50	40 – 50	10	45	10
-60	50 – 60	10	55	2
-65	60 – 65	5	62.5	2
-70	65- 70	5	67.5	7

Note: 1- the first class width is 10 because it is the most common

2- it is also acceptable for the first class to start from zero. i.e. (0 – 20)

Example 14

The data below shows the length in minutes made of different phone calls made by Airtel clients

Length (minutes)	<20	<30	<35	<40	<50	<60
Frequency	4	20	32	42	48	60

Construct a frequency table for the data.

Solution

Class	Class boundary	Class width	Class mark, x	Frequency, f
<20	10 – 20	10	15	4
<30	20 – 30	10	25	16
<35	30 – 35	5	32.5	12
<40	35 – 40	5	37.5	10
<50	40 – 50	10	45	6
<60	50 – 60	10	55	2

Measure of central tendency

(a) Mean or average of sample

The mean of grouped data is given by

$\bar{x}=\frac{\sum fx}{\sum f}$ where f = frequency, x is the mid-mark

(b) Median of grouped data

Median of grouped data is defined by

$$\text{Median} = L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right) C$$

Where L_b = lower class boundary of the median class

C= class width of the median class

f = frequency of the median class

$c.f_b$= cumulative frequency before that of the median class

Example 15

The ages of people in the town are as follows

Age	0-<5	<15	<30	<50	<70	<90
Number of people	44	81	105	147	158	160

Calculate the

(i) Mean
(ii) Median

Solution

Age	x	f	fx	cf	Class boundary
0-<5	2.5	44	110	44	0-<5
5-<15	10	37	370	81	5-<15
15-<30	22.5	24	540	105	15-<30
30-<50	40	42	1680	146	30-<50
50-<70	60	11	660	158	50-<70
70-<90	80	2	160	160	70-<90
		$\sum f$ =160	$\sum fx$ =3520		

(a) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{3520}{160}$ = 22years

(b) Median = $L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right) C$

$\frac{\sum f}{2} = \frac{160}{2} = 80$

Median class boundary is 5 -<15, f = 37 and C = 10

∴ Median = $5 + \left(\frac{80-44}{37}\right) x10$ = 14.73 years

(c) Mode of grouped data

Mode of grouped data with equal class width is defined as

Mode = $L_b + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) C$

Where

L_b = lower class boundary of modal class

C = Class width of the modal class

Δ_1 = Modal frequency (pre-modal frequency)

Δ_2 = Modal frequency (post modal frequency)

Example 16

The table below shows the weight of 250 students at The Science Foundation College

Weight (kg)	44.0 – 47.9	48.0 – 51.9	52.0 – 55.9	56.0 – 59.9	60.0 – 63.9	64.0 – 67.9	68.0 – 71.9	72.0 – 75.9
Frequency	3	17	50	45	46	57	23	9

Find

(i) Average weight

(ii) Median weight

(iii) Modal weight

Solution

Class	Class boundary	x	f	fx	cf
44.0 - 47.9	43.95 - 47.95	45.95	3	137.85	3
48.0 - 51.9	47.95 - 51.95	49.95	17	849.15	20
52.0 - 55.9	51.95 - 55.95	53.95	50	2697.5	70
56.0 -59.9	55.95 - 59.95	57.95	45	2607.75	115
60.0 - 63.9	59.95 - 63.95	61.95	46	2849.7	161
64.0 - 67.9	63.95 - 67.95	65.95	57	3759.15	218
68.0 - 71.9	67.95 - 71.95	69.95	23	1608.85	241
72.0 - 75.9	71.95 - 75.95	73.95	9	665.55	250
			250	15175.5	

(i) Mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{15175.5}{250} = 60.702kg$

(ii) Median = $L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right) C$

$\frac{\sum f}{2} = \frac{250}{2} = 125$

Median class boundary is 59.95 - 63.95, f = 46 and C = 4

∴ Median = $59.95 + \left(\frac{125-115}{46}\right) x\ 4$ = 60.82kg

(iii) Modal class boundary is 63.95 – 67.95, since 57 is the highest frequency and C = 4

$\Delta_1 = 57 - 46 = 11$ and $\Delta_2 = 57 - 23 = 34$

$Mode = 63.95 + \left(\frac{11}{11+34}\right) x\ 4 = 64.93kg$

(d) Mode of grouped data with unequal class width

Mode of grouped data with unequal class width defined as

Mode = $L_b + \left(\frac{\Delta_f.d_1}{\Delta_f.d_1 + \Delta_f.d_2}\right) C$

Modal class is determined from the highest frequency density

Frequency density = $\frac{frequency}{class\ width}$

Where

L_b = lower class boundary of modal class

C = Class width of the modal class

$\Delta_f.d_1$ = Modal frequency density (pre-modal frequency)

$\Delta_f.d_2$ = Modal frequency density (post modal frequency)

Example 17

Given the data below

Marks (x)	10-19	20-24	25-34	35-39	40-54	55-64	65-79
Frequency (f)	4	6	7	3	8	6	6

Find the mode

Solution

Class boundary	Class width	f	Frequency density
9.5 - 19.5	10	4	0.4
19.5 - 24.5	5	6	1.2
24.5 - 34.5	10	7	0.7
34.5 - 39.5	5	3	0.6
39.5 - 54.5	15	8	0.53
54.5 - 64.5	10	6	0.6
64.5 - 79.5	15	6	0.4
		$\sum f$ =40	

Mode = $L_b + \left(\frac{\Delta_f . d_1}{\Delta_f . d_1 + \Delta_f . d_2}\right) C$

$= 19.5 + \left(\frac{1.2-0.4}{(1.2-0.4)+(1.2-0.7)}\right) x\ 5$

= 22.58

Example 18

The table shows the weights (kg) of 150 patients who visited a certain health centre.

Weight (kg)	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60 - 69
Frequency (f	30	16	24	32	28	12	8

Calculate

(a) Mean
(b) Mode
(c) Median

Class	Class boundary	class mark (x)	f	fx	cf
0 - 9	0 - 9.5	4.5	30	135	30
10 - 19	9.5 - 19.5	14.5	16	232	46
20 - 29	19.5 - 29.5	24.5	24	588	70
30 - 39	29.5 -39.5	34.5	32	1104	102
40 - 49	39.5 - 49.5	44.5	28	1246	130
50 - 59	49.5 - 59.5	54.5	12	654	142
60 - 69	59.5 - 69.5	64.5	8	516	150
			$\sum f$ =150	$\sum fx$ =4475	

(a) Mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{4475}{150}$ = 29.83kg

(b) Mode = $L_b + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right) C$

Modal class boundary is 29.5 – 39.5, since 32 is the highest frequency and C = 10

$\Delta_1 = 32 - 24 = 8$ and $\Delta_2 = 32 - 28 = 4$

$Mode = 29.5 + \left(\frac{8}{8+4}\right) x\ 10 = 36.17kg$

(c) Median = $L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right) C$

$\frac{\sum f}{2} = \frac{150}{2} = 75$

Median class boundary is 29.5 -39.5, f = 32 and C = 10

∴ Median = $29.5 + \left(\frac{75-70}{32}\right) x\ 10$ = 30.06kg

Measure of dispersion of grouped data

(i) Variance of the sample

The variance of grouped data denoted by Var(x) is defined as

Var(x) = $\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2$

Or

Var(x) = $\frac{\sum fx^2}{\sum f} - \bar{x}^2$

Or

Var(x) = $\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$ where d = x – A and A = suggested mean

(ii) s.d = $\sqrt{Var(x)}$

Example 19

The table below shows the number of crimes committed by students

Number of crimes	5-<10	10-<20	20-<30	30-<50	50-<100
Number of students	10	15	25	40	26

Calculate the variance and standard deviation for the number of crimes committed

Solution

Number of crime	x	f	fx	fx^2
5-<10	7.5	10	75	562.5
10-<20	15	15	225	3375
20-<30	25	25	625	15625
30-<50	40	40	1600	64000
50-<100	75	25	1875	140625
		$\sum f$ =115	$\sum fx$ =4400	$\sum fx^2$ =224187.5

Var(x) = $\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2 = \frac{224187.5}{115} - \left(\frac{4400}{115}\right)^2 = 485.56$

s.d = $\sqrt{\text{Var(x)}} = \sqrt{485.56} = 22.04$

Example 20

The table below shows the weight of 250 students at a certain day school

Weight (kg)	44.0 – 47.9	48.0 – 51.9	52.0 – 55.9	56.0 – 59.9	60.0 – 63.9	64.0 – 67.9	68.0 – 71.9	72.0 – 75.9
Frequency	3	17	50	45	46	57	23	9

Using assumed mean of 57.95, find

(a) average weight
(b) variance
(c) standard deviation

Solution

weight	x	f	d = x - A	fd	fd^2
43.95-47.95	45.95	3	-12	-36	432
47.95-51.95	49.95	17	-8	-136	1088
51.95-55.95	53.95	50	-4	-200	800
55.95-59.95	57.95	45	0	0	0
59.95-63.95	61.95	46	4	184	736
63.95-67.95	65.95	57	8	456	3648
67.95-71.95	69.95	23	12	276	3312
71.95-75.95	73.95	9	16	144	2304
		$\sum f$ =250		$\sum fd$ =688	$\sum fd^2$ =12320

(a) $\bar{x} = A + \frac{\sum fd}{\sum f} = 57.95 + \frac{688}{250} = 60.702kg$

(b) Var(x) = $\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2$

$= \frac{12320}{250} - \left(\frac{688}{250}\right)^2 = 41.71kg$

(c) S.d = $\sqrt{Var(x)} = \sqrt{41.71} = 6.46kg$

Percentile and quartile of grouped data.

Percentile

This a value that divides a given distribution into 100 equal parts

The 60th percentile for instance is defined as

$P_{60} = L_b + \left(\frac{\frac{60}{100}\sum f - c.f_b}{f}\right)C$

Where

L_b = lower class boundary of the 60th class

C= class width

F = frequency of the 60th class

$c.f_b$ = cumulative frequency before that one of the 60th class

Quartiles

This a value that divides a given distribution into 4 equal parts

The lower quartile denoted q_1 for instance is defined as

$$q_1 = L_b + \left(\frac{\frac{1}{4}\sum f - c.f_b}{f}\right)C$$

Where

L_b = lower class boundary of the q_1 class

C= class width

f = frequency of the q_1 class

$c.f_b$ = cumulative frequency before that one of the q_1 class

The upper quartile denoted q_1 for instance is defined as

$$q_3 = L_b + \left(\frac{\frac{1}{4}\sum f - c.f_b}{f}\right)C$$

Where

L_b = lower class boundary of the q_3 class

C= class width

f = frequency of the q_3 class

$c.f_b$ = cumulative frequency before that one of the q_3 class

Interquartile range = $q_3 - q_1$

Semi-interquartile range = $\frac{q_3 - q_1}{2}$

Example 21

The following table shows the marks obtained by to students in a physics test marked out of 100

Marks (%)	20 -29	30-39	40-49	50-59	60-69	70-79	80-89	90-100
Number of students	4	6	2	5	7	8	5	2

Find

(a) Mean
(b) Standard deviation
(c) Median and mode
(d) Semi-interquartile range
(e) 40^{th} and 85^{th} percentile range

Solution

Class boundary	x	f	fx	fx2	cf
19.5-29.5	24.5	4	98	2401	4
29.5-39.5	34.5	6	207	7141.5	10
39.5-49.5	44.5	2	89	3960.5	12
49.5-59.5	54.5	5	272.5	14851.25	17
59.5-69.5	64.5	7	451.5	29121.75	24
69.5-79.5	74.5	8	596	44402	32
79.5-89.5	84.5	5	422.5	35701.25	37
89.5-99.5	94.5	3	283.5	26790.75	40
		$\sum f$ =40	$\sum fx$ =2420	$\sum fx^2$ =164370	

(i) Mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{2420}{40} = 60.5\%$

(ii) S.d = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{164370}{40} - \left(\frac{2420}{40}\right)^2}$ = 21.19%

(iii) **Median** = $L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right)C$

$\frac{\sum f}{2} = \frac{40}{2} = 20$

Median class boundary is 59.5-69.5, f = 7 and C = 10

∴ Median = $59.5 + \left(\frac{20-17}{7}\right) x\ 10$ = 63.786%

Mode = $L_b + \left(\frac{\Delta_1}{\Delta_1 + \Delta_2}\right)C$

Modal class boundary is 69.5-79.5, since 8 is the highest frequency and C = 10

$\Delta_1 = 8 - 7 = 1$ and $\Delta_2 = 8 - 5 = 3$

$Mode = 69.5 + \left(\frac{1}{1+3}\right) x\ 10 = 72\%$

(iv) $q_1 = L_b + \left(\frac{\frac{1}{4}\sum f - c.f_b}{f}\right)C$

$\frac{\sum f}{4} = \frac{40}{4} = 10$, L_b =29.5, f= 6, C = 10

$q_1 = 29.5 + \left(\frac{10-4}{6}\right)x10 = 39.5$

$q_3 = L_b + \left(\frac{\frac{3}{4}\sum f - c.f_b}{f}\right)C$

$\frac{3\sum f}{4} = \frac{3\ x\ 40}{4} = 30$, L_b =69.5, f= 8, C = 10

$q_1 = 69.5 + \left(\frac{30-24}{8}\right)x10 = 77\%$

Semi-quartile range = $\frac{q_3 - q_1}{2} = \frac{77-39.5}{2} = 18.75\%$

(v) $P_{40} = L_b + \left(\frac{\frac{40}{100}\sum f - c.f_b}{f}\right)C$

$\frac{40\sum f}{100} = \frac{40\ x\ 40}{100} = 16$, L_b =49.5, f= 5, C = 10

$P_{40} = 49.5 + \left(\frac{16-12}{5}\right)x10 = 57.5\%$

(vi) $P_{85} = L_b + \left(\frac{\frac{40}{100}\sum f - c.f_b}{f}\right)C$

$\frac{85\sum f}{100} = \frac{85\ x\ 40}{100} = 34$, L_b = 79.5, f= 5, C = 10

$P_{85} = 79.5 + \left(\frac{34-32}{5}\right)x10 = 83.5\%$

40^{th} and 85^{th} range = 83.5 – 57.5 = 26%

Example 22

Given the information in the table

Class	20-29	30-34	35-44	45-64	65-74	75-84
Frequency	5	5	12	20	10	8

Find

(a) Mean value
(b) Standard deviation
(c) Mode
(d) Median
(e) Interquartile range
(f) 90^{th} percentile

Solution

Class boundary	class width	x	f	f.d	fx	fx^2	cf
19.5-29.5	10	24.5	5	0.5	122.5	3001.25	5
29.5-34.5	5	32	5	1	160	5120	10
34.5-44.5	10	39.5	12	1.2	474	18723	22
44.5-64.5	20	54.5	20	1	1090	59405	42
64.5-74.5	10	69.5	10	1	695	48302.5	52
74.5-84.5	10	79.5	8	0.8	636	50562	60
			$\sum f$ =60		$\sum fx$ =3177.5	$\sum fx^2$ =185113.8	

(a) Mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{3177.5}{60} = 52.96$

(b) S.d = $\sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{185113.8}{60} - \left(\frac{3177.5}{60}\right)^2}$ = 16.75

(c) Mode = $L_b + \left(\frac{\Delta_f . d_1}{\Delta_f . d_1 + \Delta_f . d_2}\right) C$

=34.5 + $\left(\frac{1.2-1}{(1.2-1)+(1.2-1)}\right) 10$ = 39.5

(d) **Median** = $L_b + \left(\frac{\frac{\sum f}{2} - c.f_b}{f}\right) C$

$\frac{\sum f}{2} = \frac{60}{2} = 30$

Median = 44.5 + $\left(\frac{30-22}{20}\right) x\ 20$ = 552.5

(e) $q_1 = L_b + \left(\frac{\frac{1}{4}\sum f - c.f_b}{f}\right) C$

34.5 + $\left(\frac{\frac{1}{4} x\ 60-10}{20}\right) x\ 10$ = 38.67

$Q_3 = L_b + \left(\frac{\frac{3}{4}\sum f - c.f_b}{f}\right) C$

64.5 + $\left(\frac{\frac{3}{4} x\ 60-42}{10}\right) x\ 10$ = 67.5

Interquartile range = 67.5 – 38.67 = 28.83

(vii) $P_{90} = L_b + \left(\frac{\frac{90}{100}\sum f - c.f_b}{f}\right)C$

$= 74.4 + \left(\frac{\frac{90}{100}(60) - 52}{8}\right)x\ 10 = 77$

Graphs

(a) Grouped data with equal class width

(i) Histogram

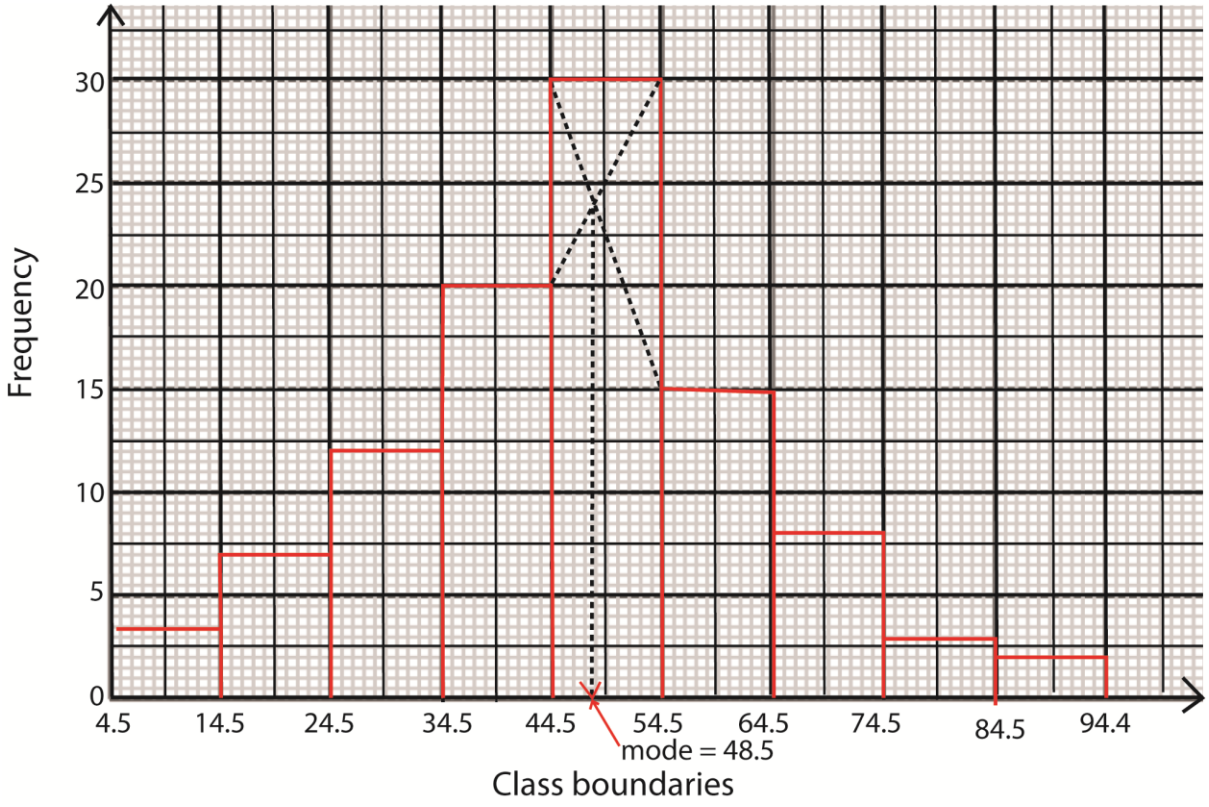
This is a graph consisting of vertical bars. It is a graph of frequency against class boundary. The area of the bar is equal to the frequency. Histogram is used to obtain the mode

Example 23

Given the data below

Marks	5-14	15-24	25-34	35-44	45-54	55-64	65-74	75-85	85-94
Frequency	3	7	12	20	30	15	8	3	2

Draw a histogram and use it to determine the mode



(ii) Cumulative frequency curve (Ogive)

This is a curve of cumulative frequency against class boundaries. An Ogive can be used to determine the median, quartiles, percentiles and deciles

Note: The values of cumulative frequency are plotted against the upper class boundaries and first value of the lower class boundary is plotted against cumulative frequency = 0

Example 24

Given the data below

Marks	20-29	30-39	40-49	50-59	60-69	70-79
Frequency	4	6	12	8	7	3

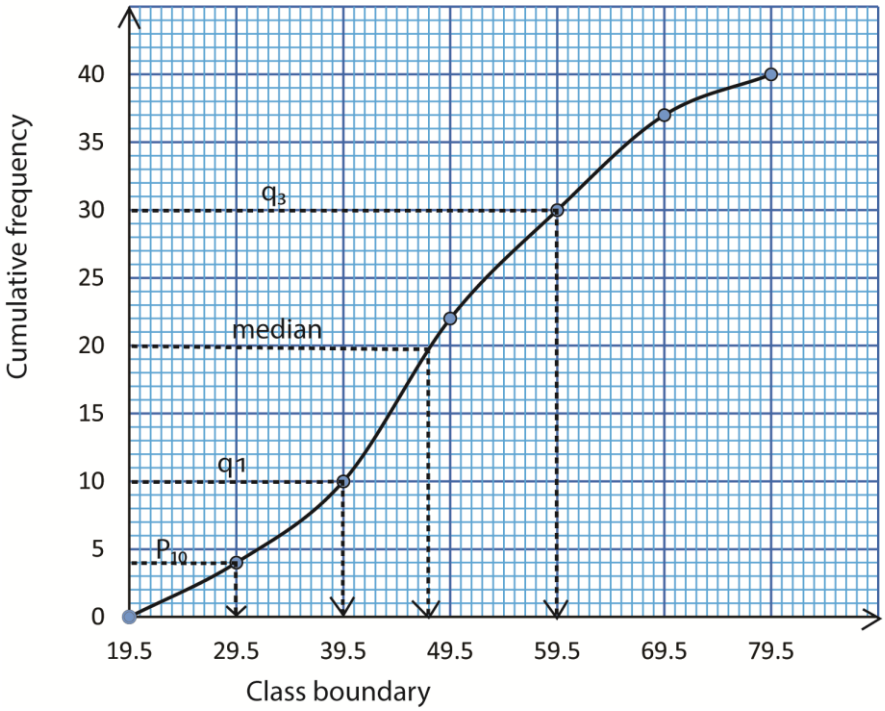
Draw an Orgive and use it to determine

(a) Median
(b) Interquartile range
(c) 10^{th} percentile

Solution

Class boundary	19.5 – 29.5	29.5–39.5	39.5–49.5	49.5 – 59.5	59.5 – 69.5	69.5 – 79.5
cf	4	10	22	30	37	40

An Orgive



(a) The median = $\left(\frac{N}{2}\right)^{th} = \left(\frac{40}{2}\right)^{th}$ = 20^{th} value from the graph = 48.5

(b) $q_1 = \left(\frac{N}{4}\right)^{th} = \left(\frac{40}{4}\right)^{th}$ = 10^{th} value ; from the graph q_1 = 39.5

$q_3 = \left(\frac{3N}{4}\right)^{th} = \left(\frac{3\ x\ 40}{4}\right)^{th}$ = 30^{th} value; from the graph q_3 = 59.5

Interquartile range = 59.5 – 39.5 = 20

(c) $P_{10} = \left(\frac{10N}{100}\right)^{th} = \left(\frac{10\ x\ 40}{100}\right)^{th}$ = 4^{th} value ; from the graph P_{10} = 29.5

Ungrouped data with unequal class width

(i) Histogram

This is a graph of frequency density against class boundary

Note that frequency density = $\frac{Frequency}{class\ width}$

Orgive

This is a graph of cumulative frequency against the class boundary

Example 25

The data shows the length in centimetres for different calendars produced by a printing press. A cumulative frequency distribution was formed

Length (cm)	<20	<30	<35	<40	<50	<60
Cumulative frequency	4	20	32	42	48	50

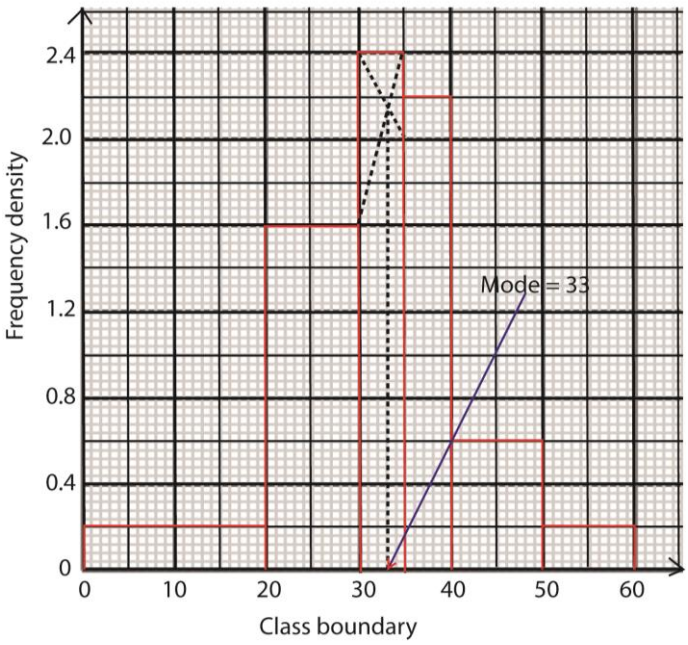
(a) Construct a frequency table.
(b) Find the mean length of the calendars
(c) Draw a histogram and use it to estimate the modal length
(d) Draw an Orgive and use it to estimate the median length.

Solution

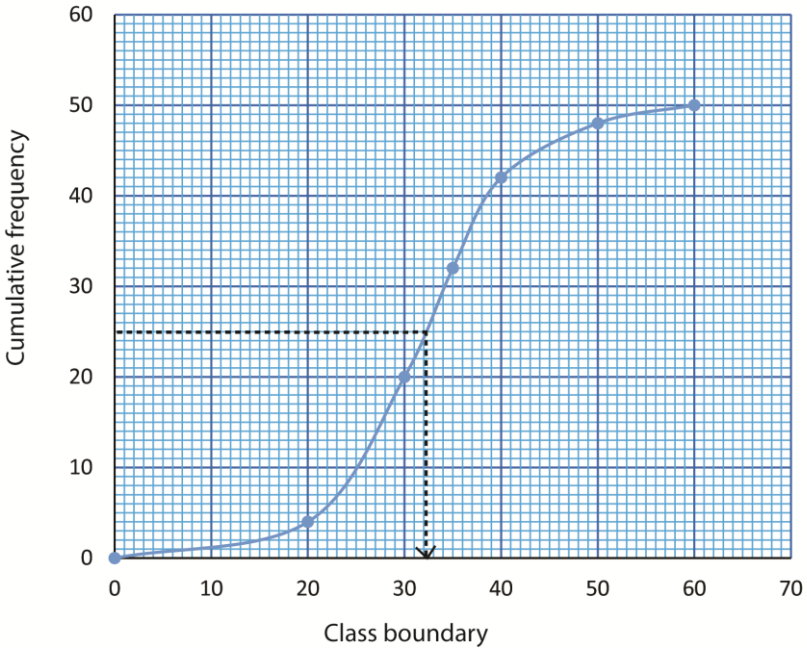
(a) Frequency table

Class boundary	x	f	fx	class width	frequency density	cf
0 - 20	10	4	40	20	0.2	4
20 - 30	25	16	400	10	1.6	20
30 - 35	32.5	12	390	5	2.4	32
35 - 40	37.5	10	375	5	2	42
40 - 50	45	6	270	10	0.6	48
50-60	55	2	110	10	0.2	50
		$\sum f$50	$\sum fx$ =1585			

(b) Mean $\bar{x} = \frac{\sum fx}{\sum f} = \frac{1585}{50} = 31.7$



An Orgive



Median length is the $\left(\frac{N}{2}\right)^{th} = \left(\frac{50}{2}\right)^{th}$ =25th value, from the graph = 32

Revision Exercise 2

UNEB 2002/2/7

The table below shows cumulative distribution of ages (in years of 400 student

Age(years)	<12	<13	<14	<15	<16	<17	<18	<19
Cumulative frequency	0	27	85	215	320	370	395	400

(a) Construct a cumulative frequency curve

(b) Use the curve to estimate

(i) Median age (Answer 14.9)

(ii) 20^{th} and 80^{th} percentile range (Ans. 2.1)

UNEB 2002/2/14

The table below shows the time taken by students to solve a mathematics problem

Time (mins)	5-9	10-14	15-19	20-24	25-29	30-34
Frequency	5	14	30	17	11	3

(a) Draw a histogram and use it to estimate the modal time. (ans. 17.3)

(b) Find the mean and standard deviation of solving the problem (Mean = 18.5mins, s.d = 5.9896 (4D))

UNEB 2004/2/14

The frequency distribution table shows the heights of s.6 students measured to the nearest cm;

Height	149-152	153-156	157-160	161-164	165-168	169-172	173-176
Frequency	5	17	20	25	15	6	2

(a) Calculate

(i) Mean height (Ans. 160.9cm)

(ii) Standard deviation (Ans. 5.5873)

(b) Draw a cumulative frequency curve and use it to estimate the median (Ans. 161cm) and range of height of the middle 60% of the candidates. (Ans. 10cm)

UNEB 2005/2/15

The table below the weights of some S.5 students from a certain school

Weight	50-53	54-57	58-61	62-65	66-69	70-73	74-77	78-81
Number of student	3	8	12	18	11	5	2	1

(a) Calculate

(i) Mean (63.1kg)

(ii) Standard deviation of students' weight (6kg)

(b) Draw a cumulative frequency curve and use it to estimate

(i) Median weight (63.1kg)

(ii) Number of students with weight between 58.9kg and 66.7kg (29students

UNEB 2006/2/12

The table below is the distribution of weights of a group of animals

Mass (kg)	Frequency
21-25	10
26-30	20
31-35	15
36-40	10
41-50	30
51-60	45
66-74	5

(a) Draw a cumulative frequency curve to estimate semi-quartile range (24kg)

(b) Find

(i) Mode (28.8333kg)

(ii) Standard deviation (11.772)

UNEB 2008/2/9

The table below shows the amount of money (in thousands of shillings) that was paid out as allowances to participants during a certain workshop

Amount (shs'000s)	110-114	115-119	120-129	130-134	135-144	145-159
Number of participants	13	20	32	17	16	12

(a) Draw a histogram and use it to estimate the modal allowance (shs. 11800)

(b) Calculate the:

(i) Median allowance (shs. 126,375/=)

(ii) Mean allowance (shs. 128,000/=)

UNEB 2009/2/11

The table below shows the income of 40 factory workers in millions of shillings per annum

1.0	1.1	1.0	1.2	5.4	1.6	2.0	2.5
2.1	2.2	1.3	1.7	1.8	2.4	3.0	2.2
2.7	3.5	4.0	4.4	3.9	5.0	5.4	5.3
4.4	3.7	3.6	3.9	5.2	5.1	5.7	1.5
1.6	1.9	3.4	4.3	2.6	3.8	5.3	4.0

(a) Form a frequency distribution table with class interval of 0.5millin shillings starting with the lowest limit of 1million shillings

(b) Calculate the

(i) Mean income (shs. 3,175,000)

(ii) Standard deviation (shs. 1,413,992.574)

(c) Draw a histogram to represent the above data. Use it to estimate the modal income
Modal income: (shs. 5, 200,000)

UNEB 2010/2/10

The table below shows the mars obtained by students in a physic test

Marks (%)	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	60-69	70-74
Frequency	9	12	10	17	13	25	18	14	8	8

(a) Draw a histogram and use it to estimate the modal mark. (52.5)

(b) Find the

(i) Mean mark (49.4627)

(ii) Standard deviation (12.424)

UNEB 2012/2/9

The table below shows the marks obtained in an examination by 200 candidates

Marks(%)	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89
Number of candidates	18	34	58	42	24	10	6	8

(a) Calculate the

(i) Mean mark (40.2%)

(ii) Modal mark (35.5%)

(b) Draw a cumulative frequency curve for the data. Hence estimate the lowest mark for a distinction one if the top 5% of the candidates qualify for the distinction. (75%)

UNEB 2013/2/1

A class performed on an experiment to estimate the diameter of a circular object. A sample of five students had the following results in centimetre. 3.13, 3.16, 2.94, 3.33 and 3.0.

Determine the sample;

(i) Mean (3.11)

(ii) Standard deviation (0.1356)(05marks)

UNEB 2017/2/12

The times taken for 55 students to have their lunch to the nearest minute are given in the table below

Time (minutes)	3 -4	5-9	10-19	20 – 29	30 – 44
Number of students	2	7	16	21	9

(a) Calculate the mean time for the student to have lunch. (mean=20.65) (04marks)

(b) (i) Draw a histogram for the given data

(ii) Use your histogram to estimate the modal time for the students to have lunch. (08marks) (modal time = 22 minutes)

UNEB 2018/2/9

1. The frequency distribution below shows the age of 240 students admitted to a certain University.

Age (years)	Number of student
18 - < 19	24
19 -< 20	70
20 -< 24	76
24 -< 26	48
26 -< 30	16
30 -< 32	6

(a) Calculate the mean age of the students. (mean =22.1458)(04mark)

(b) (i) Draw a histogram for the given data

(ii) Use the histogram to estimate the modal age (modal age = 19.58) (08mark)

UNEB 2019/2/1

The table shows the masses of bolts bought by a carpenter.

Mass (grams)	98	99	100	101	102	103	104
Number of bolts	8	11	14	20	17	6	4

Calculate the:

(a)median mass (101g)

(b) mean mass of the bolt(100.7625g) (05mark

UNEB 2019/2/10

The table below shows the marks obtained in a mathematic test by a group of student

marks	5 -<15	15-<25	25-<35	35-<45	45-<55	55-<65	65-<75	75-<100
Number of students	5	7	19	17	7	4	2	3

(a) Construct a cumulative frequency (O give) for the data (05 marks)

(b) Use your Ogive to find the

(i) Range between the 10^{th} and 70^{th} percentiles (26)

(ii) Probability that a student selected at random scored below 50 marks. 0.8125) (07 marks)

UNEB 2020/2/9

The table below shows the marks obtained by 100 students in a mathematics test

Marks	20-<40	40-<50	50-<55	55-<60	60-<70	70-<90	90-<100
Number of students	5	15	10	15	25	25	5

(a) Calculate the mean mark (63.125)

(b) Construct a cumulative frequency curve (Ogive) and use it to find the

(i) Median mark (61.5)

(ii) Range of the middle 40% of the mark (15)

Thank You

Dr. Bbosa Science

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Trapezium rule

It is used for estimating an integral area under a curve of continuous function over a given interval [a, b]

if y = f(x)

A = $\int_a^b y dx$

Using several strips between x = a and x = b of equal width, trapezium rule can be used to determine the area.

A $\approx \frac{1}{2}h[(first + last\ ordinates) + 2(sum\ of\ the\ middle\ ordinates)]$

where h = $\frac{b-a}{subintervals}$

Note

(i) sub-intervals, subdivision and strips are the same
(ii) subinterval = ordinates
(iii) when dealing with a trigonometric function, calculators must be in radian mode
(iv) when the final answer is required to a specific number of d.p'c, the working's should be done at least a d.p higher but the final answer rounded to the required d.p's

Example 1

Use the trapezium rule with four-intervals to estimate $\int_{0.2}^{1.0} \left(\frac{2x+1}{x^2+x}\right) dx$. Correct to two decimal places.

Let y = $\left(\frac{2x+1}{x^2+x}\right)$

h = $\frac{1.0-0.2}{4}$ = 0.2

x	y = $\frac{2x+1}{x^2+x}$	
0.2	5.8333	
0.4		3.2143
0.6		2.2917
0.8		1.8056
1.0	1.5000	
Sum	7.3333	7.3116

$\int_{0.2}^{1.0} \left(\frac{2x+1}{x^2+x}\right) dx = \frac{1}{2}\ x\ 0.2(7.3333 + 7.3116\ )$

= 2.1955

= 2.20 (2D)

Example 2

Use the trapezium rule with seven coordinates to estimate

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}}$dx correct to 2 decimal places (05marks)

Solution

For 7 ordinates, there are 6 subintervals

Width, h = $\frac{b-a}{subinterval} = \frac{3-0}{6} = 0.5$

Let y = $\sqrt{(1.2)^x - 1}$

x	y	
0	0	
0.5		0.309
1		0.447
1.5		0.561
2		0.663
2.5		0.760
3	0.853	
Sum	0.853	2.74

Using the trapezium rule

$\int_0^3 [(1.2)^x - 1]^{\frac{1}{2}} dx = \frac{0.5}{2}[0.853 + 2(2.74)]$ = 1.58

Example 3

(a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places.

$h = \frac{\frac{1}{2}-0}{5} = \frac{\pi}{10}$

x	y	
0	0	
$\frac{\pi}{10}$		0.6232
$\frac{2\pi}{10}$		1.2161
$\frac{3\pi}{10}$		1.7515
$\frac{4\pi}{10}$		2.2077
$\frac{\pi}{2}$	2.5708	
Sum	2.5708	5.7985

$\int_0^{\frac{1}{2}}(x + sinx)dx = \frac{1}{2}\ x\ \frac{\pi}{10}(2.5708 + 2\ x\ 5.7985)$

= 2.225

(b)(i) Evaluate $\int_0^{\frac{1}{2}}(x + sinx)dx$, correct to three decimal places

$\int_0^{\frac{1}{2}}(x + sinx)dx = \left|\frac{x^2}{2} - \cos x\right|_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(\frac{\pi^2}{4} - 0\right) - (\cos\frac{\pi}{2} - \cos 0)$

$= \frac{\pi^2}{8} + 1$

= 2.234

(ii) Calculate the error in your estimation in (a) above

Error = $|2.234 - 1.225|$ = 0.009

(iii) Suggest how the error may be reduced (06marks)

Increasing on number of strips or subintervals

Example 4

A student used the trapezium rule with five sub-intervals to estimate

$\int_2^3 \frac{x}{(x^2-3)} dx$ correct to **three** decimal places

Determine;

(a) The value the student obtained (06marks)

h = $\frac{3-2}{5}$ =0.2

X	y_1, y_6	y_2 ……y_5
2.0	2.0	
2.2		1.1956
2.4		0.8696
2.6		0.6915
2.8		0.5785
3	0.5	
Sum	2.5	3.3352

$\int_2^3 \frac{x}{(x^2-3)} dx = \frac{1}{2}\ x\ 0.2[2.5 + 2(3.3352)]$

= 0.91704 = 0.917 (3D)

(b) The actual value of the integral (03marks)

$\int_2^3 \frac{x}{(x^2-3)} dx = \left[\frac{1}{2} Inx^2 - 3\right]_2^3$

$= \frac{1}{2}(In\ 6 - In\ 1)$

= 0.896

(c) (i) the error the student made in the estimate

Error = $|0.896 - 0.917|$ = 0.021

(ii) how the student can reduce the error(03marks)

Increasing on the number of sub-intervals or ordinates or reducing the width of h

Example 5

Use trapezium rule with 4 subintervals to estimate to 3 decimal places $\int_0^{\frac{\pi}{2}} \cos x dx$

Solution

h = $\frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$

x	f(x) = cos x	
0	1.0000	
$\frac{\pi}{8}$		0.9239
$\frac{2\pi}{8}$		0.7071
$\frac{3\pi}{8}$		0.3827
$\frac{4\pi}{8}$	0.0000	
sum	1.0000	2.0137

$$\int_0^{\frac{\pi}{2}} \cos x dx = \frac{1}{2} x \frac{\pi}{8}[1 + 2\ x\ 2.0137]$$

=0.987

Example 6

Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x} dx$ correct to 3dp

Solution

h = $\frac{3-0}{7-1} = 0.5$

x	f(x) = cos x	
0	1.0000	
0.5		0.6667
1.0		0.5000
1.5		0.4000
2.0		0.3333
2.5		0.2757
3.0	0.2500	
sum	1.2500	2.157

$$\int_0^3 \frac{1}{1+x} dx = \frac{1}{2} x\ 0.5[1.25 + 2\ x\ 2.1857]$$

= 1.405

Example 7

(a) Use the trapezium rule to estimate the integral value of $\int_2^3 \frac{x}{1+x^2} dx$ using five subinterval and correct to 3d.p.

(b) (i) find the exact value of $\int_2^3 \frac{x}{1+x^2} dx$

(ii) suggest how the error may be reduced.

(a) h = $\frac{3-2}{5} = 0.2$

x	f(x) = $\frac{x}{1+x^2}$	
2.0	0.40000	
2.2		0.37671
2.4		0.35503
2.6		0.33505
2.8		0.31674
3.0	0.30000	
sum	0.70000	1.3353

$$\int_2^3 \frac{x}{1+x^2} dx = \frac{1}{2} x\ 0.2[0.7 + 2\ x\ 1.38353]$$

= 0.3467

(b)(i) $\int_2^3 \frac{x}{1+x^2} dx = \left[\frac{1}{2} In(1+x^2)\right]_2^3 = \frac{1}{2}(In10\ In5)$ = 0.3466

(ii) error = $|exact\ value - \ approximate\ value| = |0.3466 - 0.3467|$ = 0.0001

(iii) the error can be reduced by reducing h or increasing the number of sub-intervals.

Example 8

(a) Use trapezium rule to estimate the integral value of $\int_0^1 x^2 e^x\, dx$

(b) (i) find exact value of $\int_0^1 x^2 e^x\, dx$

(ii) determine the percentage error in your estimation

(a) h = $\frac{1-0}{5} = 0.2$

x	f(x) = x^2e^x	
0	0	
0.2		0.0489
0.4		0.2387
0.6.		0.6560
0.8		1.4243
1.0	2.7183	
sum	2.7183	2.3679

$\int_0^1 x^2 e^x\, dx = \frac{1}{2} x\ 0.2[2.713 + 2\ x\ 2.3679]$

=0.74541≈ 0.745

(b)(i) $\int_0^1 x^2 e^x\, dx = [x^2e^x - 2xe^x + 2e^x]_0^1$

= 0.718

(ii) error = $|0.718 - 0.745|$= 0.027

Percentage error = $\frac{error}{exact\ value}\ x\ 100\%$

= $\frac{0.027}{0.718}\ x\ 100$ = 3.8%

Revision Exercise

1. (a) Use trapezium rule with six strips to estimate $\int_0^\pi xsinxdx$ [3.069]
   (b) Determine the percentage error in your determination. [2.3%]
2. Use the trapezium rule to estimate the approximate value of $\int_0^1 \frac{1}{1+ x^2 dx}$ using 6 ordinates and correct to 3 decimal places. [0.784]
3. (a) Use trapezium rule with six strips to estimate $\int_2^4 \frac{10}{2x+1} dx$ correct 4dp. [2.9418]
   (b) Determine the percentage error in your estimation and suggest how this error may be reduce. [0.098%]
4. (a) Use trapezium rule to estimate the area of y = 3x between x-axis, x = 1 and x = 2, using five subintervals. Give your answer correct to four significant figures. [5.483]
   (b) Find the exact value of $\int_1^2 3^x dx$ [5.461]
   (c) Find the exact percentage error in calculations (a) and (b) above. [0.4028%]
5. Use trapezium rule with 7 ordinates to estimate $\int_0^3 \frac{1}{1+x} dx$, correct to 3 decimal places [1.405]
6. Use the trapezium rule with 6 ordinates to evaluate $\int_0^1 e^{-x^2}$ correct to 2 decimal place. [0.74]
7. Use the trapezium rule with 6 ordinates to estimate $\int_1^2 \frac{Inx}{x} dx$.Give your answer correct to 3 decimal places [0.237]
8. Find the approximate value to one decimal place of $\int_0^1 \frac{dx}{1+x}$, using the trapezium rule with five strips. [0.7]
9. (a) Use trapezium rule with five subintervals to estimate $\int_0^{\frac{\pi}{3}} \tan x\, dx$ correct to 3dp. [0.704]

(b) (i) Find the exact value of $\int_0^{\frac{\pi}{3}} \tan x \, dx$ to 3 d.p. [0.693]

(ii) Calculate the percentage error in your estimation in (a) above [1.587%]

(iii) Suggest how the percentage error in (b)(ii) may be reduced.

10. Use the trapezium rule with four subdivisions to estimate $\int_0^{\frac{\pi}{2}} \frac{1}{1+sinx} dx$. Give your answer correct to three decimal places. [1.013]
11. Find the approximate value of $\int_0^2 \frac{1}{1+x^2} dx$ using trapezium rule with 6 ordinates. Give your answer to 3 decimal places (05marks)[1.105]
12. Use the trapezium rule with five subintervals to estimate $\int_2^4 \frac{5}{(x+1)} dx$. Give your answer correct to 3 decimal places (05marks)[2.559]
13. A student used the trapezium rule with five sub-intervals to estimate $\int_2^3 \frac{x}{(x^2-3)} dx$ correct to **three** decimal places

    Determine;

    (a) The value the student obtained (06marks) [0.917]
    (b) The actual value of the integral (03marks) [0.896]
    (c) (i) the error the student made in the estimate [0.021]
    (ii) how the student can reduce the error(03marks)
14. (a) Use the trapezium rule with 6-ordinated to estimate the value of $\int_0^{\frac{1}{2}} (x + sinx) dx$, correct to three decimal places, [2.225]

    (b)(i) Evaluate $\int_0^{\frac{1}{2}} (x + sinx) dx$, correct to three decimal places [2.234]

    (ii) Calculate the error in your estimation in (a) above [0.009]

    (iii) suggest how the error may be reduced (06marks)

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Variable acceleration

This occurs when the rate of change of velocity is not constant.

Differential calculus

Let r = displacement, v = velocity and a = acceleration and are functions of time, t.

velocity, v = $\frac{dr}{dt}$ while acceleration, a = $\frac{dv}{dt} = \frac{d^2r}{dt^2}$

Differentiation; displacement => velocity => acceleration

Example 1

A particle moves along a straight line such that after t seconds, its displacement from a fixed point is r metres where r = $8t^2i - t^4j$. Find

(a) velocity after t seconds (b) velocity after 1s (c) speed after 1 s

Solution

$v_{(t=t)} = \frac{dr}{dt} = \frac{d(8t^2i - t^4j)}{dt}$

= $(16ti - 4t^3j)ms^{-1}$

(b) when t = 1

$v_{(t=1)}$= $(16i - 4j)ms^{-1}$

(c) speed = $|v_{(t=1)}| = \sqrt{16^2 - (-4)^2}$

= $16.49ms^{-2}$

Example 2

A particle move along a straight line such that after t seconds its displacement from a fixed point is s in metres where s = 2sinti + 3costj. Find

(a) acceleration after t seconds (b) acceleration after $\frac{\pi}{2}s$ (c) magnitude of acceleration after $\frac{\pi}{2}s$

(a) $v_{(t=t)} = \frac{dr}{dt} = \frac{d(2\text{sinti} + 3\text{costj})}{dt}$

= $(2\text{costi} - 3\text{sintj})ms^{-1}$

$a_{(t=t)} = \frac{dv}{dt} = \frac{d(2\text{costi} - 3\text{sintj})}{dt}$

= $(-2\text{sinti} - 3\text{costj})ms^{-2}$

(b) when t = $\frac{\pi}{2}$s

$a_{(t=\frac{\pi}{2})}$ = $(-2\sin\left(\frac{\pi}{2}\right)i - 3\cos\left(\frac{\pi}{2}\right)j) = -2i\ ms^{-2}$

(c) $\left|a_{(t=\frac{\pi}{2})}\right| = \sqrt{(-2)^2 - 0^2}$ = $2ms^{-2}$

Example 3

A particle moves along a straight line such that after t seconds its displacement from a fixed point is s where $s = \begin{pmatrix} \sin 2t \\ t+1 \\ cost + \sin t \end{pmatrix}$. Find

(a) velocity when t = $\frac{\pi}{2}s$ (b) speed when t= $\frac{\pi}{2}s$ (c) acceleration when t =$\frac{\pi}{2}s$

Solution

$v_{(t=t)} = \frac{dr}{dt} = \frac{d[\sin 2ti + (t+1)j + (cost+sint)k]}{dt}$

=2cos2ti + j + (sint –cost)k

When t = $\frac{\pi}{2}s$

$v_{(t=\frac{\pi}{2})} = 2\cos\left(\frac{2\pi}{2}\right)i + j + (\sin\frac{\pi}{2} - \cos\frac{\pi}{2})k$

= (-2i + j –k)ms^{-1}

(b) $\left|v_{(t=\frac{\pi}{2})}\right| = \sqrt{(-2)^2 + 1^2 + (-1)^2}$

= 2.45ms^{-1}

(c) $a_{(t=t)} = \frac{dv}{dt} = \frac{d[2cos2ti + j + (sint - cost)k]}{dt}$

= -4sin2ti + (cost + sint)k

$a_{(t=\frac{\pi}{2})} = -4sin2\left(\frac{\pi}{2}\right)i + [\cos\frac{\pi}{2} + \sin\frac{\pi}{2}]k$

a = kms^{-2}

Example 4

A particle moves in x-y plane such that its position at any time t is given by r= (3t^2-1)i + (4t^3 + t -1)j. Find (a) speed after time t = 2 (b) magnitude of acceleration after t = 2s

$v_{(t=t)} = \frac{dr}{dt} = \frac{d[(3t^2-1)i + (4t^3+t-1)]}{dt}$

= 6ti + (12t^2 + 1)j ms^{-1}

when t = 2s

$v_{(t=2)} = [6\ x\ 2i + (12\ x\ 2^2 + 1)j]ms^{-1}$

= [12i + 49j] ms^{-1}

speed = $\left|v_{(t=2)}\right| = \sqrt{12^2 + 49^2}$ = 50.45ms^{-1}

(b) $a_{(t=t)} = \frac{dv}{dt} = \frac{d[6ti + (12t^2+1)j]}{dt}$

= (6i + 24tj)ms^{-2}

when t = 2s

$a_{(t=2)}$= 6i =+ (24 x 2)j = (6i + 48i)ms^{-2}

$\left|a_{(t=2)}\right| = \sqrt{6^2 + 48^2}$ = 48ms^{-2}

Revision exercise 1

1. The position vector of a particle at any time (t) is given by r(t) = [(t^2 + 4t)I + (3t – t^3)j]m. Find the speed of the particle at t =3seconds. [26ms^{-1}]
2. The displacement of a particles after t seconds is given by r = t^{3}i + 9tj. Find the speed when t = 2s. [15ms^{-1}]
3. The displacement of a particle after t seconds is given by s = 2$\sqrt{3}$ sinti + 8costj. find the speed when t = $\frac{\pi}{6}s$. [5ms^{-1}]
4. The displacement of a particle after t seconds is given by r = 8t^{3}i + 2t^{2}j. Find
   (i) acceleration when t = 1s [(6i+ 4j)ms^{-2}]
   (ii) magnitude of the acceleration when t = 1s [7.21ms^{-2}]

5. The velocity of a particle after t seconds is given by v = sin2ti – costj. Find
   (i) acceleration when t = $\frac{\pi}{6}s$ [(-2i + j)ms^{-2}]
   (ii) magnitude of acceleration when t = $\frac{\pi}{6}s$ [2.24ms^{-2}]
6. The velocity of a particle after t seconds is given by v = 2t^{2}i + 6j. find the magnitude of acceleration when t = 3. [12ms^{-2}]
7. A particle of mass 6kg moves such that its displacement s = $\begin{pmatrix} t^2-5 \\ t^2-3t+2 \end{pmatrix}$m. Find the
   (a) velocity after time t [2ti + (2t – 3)j ms^{-1}]
   (b) speed of the particle at t = 2s [15ms^{-1}]
   (c) acceleration and hence determine the force acting on the particle [2i + 2j)ms^{-2}, (12i + 12j)N]
8. A particle of mass 2kg moves such that its displacement s = $\begin{pmatrix} t^2-4t-5 \\ t^2-4t+3 \end{pmatrix}$m. Find
   (a) speed of the particle at t= 2s [0ms^{-1}]
   (b) force acting on the particle [(4i + 4j)N]
9. A particle of mass 4kg moves such that its displacement s = (t^3 –t^2 – 4t + 3)I + (t^3 -2t^2 + 3t -7)j. Find the
   (i) the speed of the particle at t= 4s [50.21ms^{-1}]
   (ii) magnitude of the force acting on the particle when t = $\frac{2}{3}s$ [8N]
10. A particle of mass 0.5kgmoves such that its displacement r =$\begin{pmatrix} 4sin2t \\ 2cost-1 \end{pmatrix}$m. Find the
    (a) velocity of the particle after t = $\frac{\pi}{6}s$ [(4i – j)ms^{-1}]
    (b) force acting on the particle at any time t [(-8sin2ti –costj)N]
11. A particle of mass 2kg moves such that its displacement r = (2 – cos3t)I + (6sin2t)j. Find the
    (a) velocity of the particle at t = $\frac{\pi}{6}s$[(3i + 6j)ms^{-1}]
    (b) force acting on the particle at t = πs[-18N]
12. A particle moves such that its displacement s = (2sint +sin2t)I + (4cost + cos2t)j. Find the
    (a) velocity of the particle at t = $\frac{\pi}{3}s$ [-3$\sqrt{3}jms^{-1}$
    (b) acceleration of the particle at t= $\frac{\pi}{2}s$ [(-2i + 4j)ms^{-2}]

Integral calculus

If r, v or a are function of time t;

velocity, v = $\int adt + c$ and displacement, r = $\int vdt + c$

integration; acceleration => velocity => displacement

Example 5

The velocity of the particle v = 3t^{2}i + 10tj. Given that the displacement is 4i- 4j at t = 0. Find the distance of the body from the origin when t = 2s.

r = $\int vdt + c$

$r_{(t=t)} = \int(3t^2i + 10tj)dt + C$

$r_{(t=t)} = t^3i + 5t^2j + C$

$At\ t = 0, r = 4i - 4j$

4i -4j = 0^{3}i + 5x 0^{2}j +C

c = 4i -4j

$r_{(t=t)} = [(t^3 + 4)i + (5t^2 - 4)j]$

when t = 2s

$r_{(t=2)} = [(2^3 + 4)i + (5x2^2 - 4)j]$ = (12i + 16j)m

$|r_{(t=2)}| = \sqrt{12^2 + 16^2}$ = 20m

Example 6

A particle is accelerated from rest at the origin with an acceleration of (2t + 4)ms^{-2}. Find

(a) velocity attained after t = 2s (b) distance travelled at t = 1s

Solution

v = $\int adt + c$

v = $\int (2t + 4)dt + c$

$v_{(t=t)} = 2t^2 + 4t + c$

At t = 0, v = 0

0 = 2(0)^2 + 4(0) + c

c =0

$v_{(t=t)} = 2t^2 + 4t$

$v_{(t=2)} = 2x2^2 + 4x2$ = 12ms^{-1}

r = $\int vdt + c$

$r_{(t=t)} = \int (2t^2 + 4t)dt + c$

$r_{(t=t)} = \frac{t^3}{3} + 2t^2 + c$

At t = 0, r = 0

$0 = \frac{0^3}{3} + 2x\ 0^2 + c$

c = 0

$r_{(t=t)} = \frac{t^3}{3} + 2t^2$

$r_{(t=1)} = \frac{1^3}{3} + 2x1^2$ = 2.33m

Example 7

A particle is accelerated from rest at the origin with acceleration of (4t + 2)I – 3j. Find

(i) velocity attained after t = 3s (ii) speed after 3s

Solution

v = $\int adt + c$

v = $\int \{(4t + 2)i - 3j\}dt + c$

$v_{(t=t)} = (2t^2 + 2t)i - 3tj + c$

t = 0, v = 0

0 =$(2\ x\ 0^2 + 2\ x\ 0)i - 3\ x\ 0j + c$

c = 0

$v_{(t=t)} = (2t^2 + 2t)i - 3tj$

$v_{(t=3)} = (2\ x\ 3^2 + 2\ x\ 3)i - 3x\ 3j$

=24i -9j

Speed = $\left|v_{(t=3)}\right| = \sqrt{24^2 + (-9)^2}$ = 25.63ms^{-1}

Example 8

A particle starts from origin (0, 0). Its acceleration in ms^{-2} at time t seconds is given by a = 6ti -4j.

Find its speed after t =2s

v = $\int adt + c$

v = $\int (6ti - 4j)dt + c$

$v_{(t=t)}$ = 3t^{2}i -4tj + c

At t = 0, v= 0

0 = 3 x 0^{2}i -4 x 0j + c

c = 0

$v_{(t=t)}$ =3t^{2}i -4tj

$v_{(t=2)}$ =3 x 2^{2}i -4 x 2tj

=12i- 8j

$v_{(t=2)} = \sqrt{12^2 + (-8)^2} = 14.42ms^{-2}$

Example 9

An object of mass 5kg is initially at rest at a point whose position vector is -2i+ j. If it is acted on by a force F = 2i + 3j -4k. Find

(i) acceleration

F = ma

$a_{(t=t)} = \frac{1}{5}(2\text{i} + 3\text{j} - 4\text{k})ms^{-2}$

(ii) speed after t= 3s

v = $\int adt + c$

$v_{(t=t)} = \frac{1}{5}\int(2\text{i} + 3\text{j} - 4\text{k}) + c$

$= \frac{1}{5}(2\text{ti} + 3\text{tj} - 4\text{tk}) + c$

At t = 0, v = 0

$0 = \frac{1}{5}(2 \text{ x } 0\text{i} + 3 \text{ x } 0\text{j} - 4 \text{ x } 0\text{k}) + c$

c =0

$v_{(t=t)} = \frac{1}{5}(2\text{ti} + 3\text{tj} - 4\text{tk})$

$v_{(t=3)} = \frac{1}{5}(2 \text{ x } 3\text{i} + 3 \text{ x } 3\text{j} - 4 \text{ x } 3\text{k})$

$= \frac{1}{5}(6\text{i} + 9\text{j} - 12\text{tk})$

$|v_{(t=3)}| = \frac{1}{5}\sqrt{6^2 + 9^2 + (-12)^2}$ = 3.23ms^{-1}

(iii) its distance from the origin after 3 seconds

r = $\int vdt + c$

$r_{(t=t)} = \frac{1}{5}\int(2\text{ti} + 3\text{tj} - 4\text{tk}) + c$

$r_{(t=t)} = \frac{1}{5}(\text{t}^2\text{i} + 1.5\text{t}^2\text{j} - 2\text{t}^2\text{k}) + c$

At t = 0, r = -2i+ j

-2i+ j = $\frac{1}{5}(0^2\text{i} + 1.5 \text{ x } 0^2\text{j} - 2 \text{ x } 0^2\text{k}) + c$

c = -2i+ j

$r_{(t=t)} = \frac{1}{5}\{(\text{t}^2 - 10)\text{i} + (1.5\text{t}^2 + 5)\text{j} - 2\text{t}^2\text{k}\}$

$r_{(t=3)} = \frac{1}{5}\big((3^2 - 10)\text{i} + (1.5 \text{ x } 3^2 + 5)\text{j} - 18\text{k}\big)$

$= \frac{1}{5}(-\text{i} + 18.5\text{j} - 2 \text{ x } 3^2\text{k})$

$|r_{(t=3)}| = \sqrt{(-1)^2 + (18.5)^2 + 18^2}$=5.166m

Example 10

A particle starts from rest at point (2, 0, 0) and moves such that its acceleration in ms-2 at time t seconds is given by a = [16cos4ti + 8sin2tj + (sint -2sin2t)k]ms^{-2}. Find the

(a) speed when t = $\frac{\pi}{4}$.

a = [16cos4ti + 8sin2tj + (sint – 2sin2t)k]ms^{-2}

v = $\int adt$

= $\int[16\text{cos4ti} + 8\text{sin2tj} + (\text{sint} - 2\text{sin2t})\text{k}]\text{dt}$

= [4sin4ti – 4cos2tj + (-cos t + cos2t)k]+ c

At t = 0

0 = [4sin0i – 4cos0j + (-cos 0 + cos0)k]+ c

0 = -4j + c

c = 4j

⇨ v = [4sin4ti + (-4cos2t + 4)j + (-cos t + cos2t)k

At t = $\frac{\pi}{4}$

⇨ v = [4sinπi + (-4cos$\frac{\pi}{2}$ + 4)j + (-cos $\frac{\pi}{4}$ + cos$\frac{\pi}{2}$)k

= 4j – cos $\frac{\pi}{4}$ k

$|v| = \sqrt{4^2 + \left(-\cos\frac{\pi}{4}\right)^2}$ = $\sqrt{16 + \frac{2}{4}}$ =$\sqrt{16.5}$ = 4.062ms^{-1}

(b) distance from the origin when t = $\frac{\pi}{4}$

s = $\int vdt$

=$\int[4\text{sin4ti} + (-4\text{cos2t} + 4)\text{j} + (-\text{cos t} + \text{cos2t})\text{k}$

=-cos4ti +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$ + c

At t = 0, s = 2i

By substitution

2i = -cos0 i +(-2sin0 + 4(0))j + (-sin0 + $\frac{1}{2}sin2\ (0))k$ + c

2i = -I + c

c = 3i

⇨ s = (-cos4t + 3)i +(-2sin2t + 4t)j + (-sint + $\frac{1}{2}sin2t)k$

At t = $\frac{\pi}{4}$

⇨ s = (-cosπ + 3)i +(-2sin$\frac{\pi}{2}$t + π)j + (-sin$\frac{\pi}{4}$ + $\frac{1}{2}sin\frac{\pi}{2}$)$k$

= 4i + (π -2)j + $\left(-\frac{\sqrt{2}}{2}+\frac{1}{2}\right)k$

= 4i + 1.416j – 0.207k

$|s|=\sqrt{4^2+\ (1.416)^2+\ (-0.207)^2}$

= 4.24828

= 4.248 (3D)

Vector approach of finding work and power

Work done by a variable force is a dot product

W= F.d or $W_{(t=t)} = \int Fvdt$

Since power, P = F.v

Example 11

A particle of mass 4kg starts frm rest at the origin. It is acted on by a force F = (2ti + 3t^{2}j + 5k)N

Find the work done by the force after 3 second

Solution

F = ma

$a_{(t=t)} = \frac{1}{4}(2\text{ti}\ +\ 3\text{t}^2\text{j}\ +\ 5\text{k})$

v = $\int adt$ + c

$v_{(t=t)} = \frac{1}{4}\int(2\text{ti}\ +\ 3\text{t}^2\text{j}\ +\ 5\text{k})\,dt + c$

$v_{(t=t)} = \frac{1}{4}(\text{t}^2\text{i}\ +\ \text{t}^3\text{j}\ +\ 5\text{tk}) + c$

At t = 0, v = 0

$0 = \frac{1}{4}(0^2\text{i}\ +\ 0^3\text{j}\ +\ 5\text{ x }0\text{k}) + c$

c = 0

$v_{(t=t)} = \frac{1}{4}(\text{t}^2\text{i}\ +\ \text{t}^3\text{j}\ +\ 5\text{tk})$

$v_{(t=3)} = \frac{1}{4}(3^2\text{i}\ +\ 3^3\text{j}\ +\ 5\text{x }3\text{k})$

$v_{(t=t)} = \frac{1}{4}(9\text{i}\ +\ 27\text{j}\ +\ 15\text{k})$

$\left|v_{(t=t)}\right| = \frac{1}{4}\sqrt{9^2+27^2+15^2}$

$= \frac{1}{4}\sqrt{1{,}025}ms^{-1}$

$W_{(t=0\ and\ t=3)} = \frac{1}{2}m({v_{t=3}}^2 - {v_{t=0}}^2)$

$= \frac{1}{2}$ x 4 $\left(\frac{1}{16}\ x\ 1{,}025\ -0\right)$ = 129.375J

Alternatively

$P_{(t=t)} = F.v$

$P_{(t=t)} = \begin{pmatrix} 2t \\ 3t^2 \\ t \end{pmatrix}.\frac{1}{4}\begin{pmatrix} t^2 \\ t^3 \\ 5t \end{pmatrix} = \frac{1}{4}(2t^3 + 3t^5 + 5t^2)$

$W_{(t=0\ and\ t=3)} = \int_0^3 F.vdt$

$= \frac{1}{4}\int_0^3 (2t^3 + 3t^5 + 5t^2)\ dt$

$= \frac{1}{4}\left[\frac{1}{2}t^4 + \frac{1}{2}t^6 + \frac{25}{2}t^2\right]_0^3$

$= \frac{1}{4}\left\{\left(\frac{1}{2}x\ 3^4 + \frac{1}{2}\ x\ 3^6 + \frac{25}{2}x\ 3^2\right) - 0\right\}$

= 129.375J

Example 12

A particle of mass 3kg moves along a straight line such that after t seconds its position vector is s metres where s = $\begin{pmatrix} 25t + 4t^2 \\ 50 + 2t^2 \\ 15 + 3t^2 \end{pmatrix}$. Find

(a) Magnitude of force

$v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix} 25t + 4t^2 \\ 50 + 2t^2 \\ 15 + 3t^2 \end{pmatrix} = \begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} ms^{-1}$

$a_{(t=t)} = \frac{dv}{dt} = \frac{d}{dt}\begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} ms^{-2}$

$F = 3\begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix}$

$|F| = \sqrt{12^2 + 12^2 + 18^2}$

= 32.31N

(b) Power when t = 2s

$P_{(t=t)} = F.v$

$v_{t=2} = \begin{pmatrix} 25 + 8\ x\ 2 \\ 4\ x\ 2 \\ 6\ x\ 2 \end{pmatrix} = \begin{pmatrix} 41 \\ 8 \\ 12 \end{pmatrix} ms^{-1}$

$P_{(t=2)} = \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix}.\begin{pmatrix} 41 \\ 8 \\ 12 \end{pmatrix} = 1{,}296ms^2$

(c) Work done on the particle between t = 1s and t=2s

$W_{(t=1\ and\ t=2)} = \frac{1}{2}m({v_{t=2}}^2 - {v_{t=1}}^2)$

$v_{(t=1)} = \begin{pmatrix} 25 + 8\ x1 \\ 4\ x\ 1 \\ 6\ x\ 1 \end{pmatrix} = \begin{pmatrix} 33 \\ 4 \\ 6 \end{pmatrix} ms^{-1}$

$|v_{(t=1)}| = \sqrt{33^2 + 4^2 + 6^2}$ = $\sqrt{1141}$ms^{-1}

$|v_{(t=1)}| = \sqrt{41^2 + 8^2 + 12^2}$ = $\sqrt{1889}$ms^{-1}

$W_{(t=1\ and\ t=2)} = \frac{1}{2}m({v_{t=2}}^2 - {v_{t=1}}^2)$

= $\frac{1}{2}$ x 3 $(1889 - 1141)$ = 1122J

Alternatively

$P_{(t=1)} = F.v$

$= \begin{pmatrix} 24 \\ 12 \\ 18 \end{pmatrix}.\begin{pmatrix} 25 + 8t \\ 4t \\ 6t \end{pmatrix} = (600 + 348t)$

$W_{(t=1\ and\ t=2)} = \int_1^2 F.vdt$

$= \int_1^2 (600 + 348t)dt$

$[600t + 174t^2]_1^2$

$= (12000 + 696) - (6000 + 174)$

= 1122J

Example 13

A particle of mass 10kg starts from rest at a point A with position vector (4i + 3j + 2k)m. It is acted on by a constant force, F = (8i + 4j + 6k)N causing it to accelerate to B after 4s. Find the

(a) Magnitude of acceleration

F= ma

$a_{(t=t)} = \frac{1}{10}((8\text{i} + 4\text{j} + 6\text{k}))ms^{-2}$

$|a_{(t=t)}| = \frac{1}{10}\sqrt{8^2 + 4^2 + 6^2}$

= 1.077ms^{-2}

(b) velocity at any time t

$v_{(t=t)} = \int a dt + c$

$= \frac{1}{10}\int(8\text{i} + 4\text{j} + 6\text{k})\, dt$

$v_{(t=t)} = \frac{1}{10}(8ti + 4tj + 6tk) + c$

At t = 0; v=0

$0 = \frac{1}{10}(8\ x\ 0i + 4\ x\ 0j + 6\ x\ 0k) + c$

c = 0

$v_{(t=t)} = \frac{1}{10}(8ti + 4tj + 6tk)$

(c) position vector point B

$r_{(t=t)} = \int v dt + c$

$r_{(t=t)} = \frac{1}{10}\int(8ti + 4tj + 6tk)\, dt + c$

$r_{(t=t)} = \frac{1}{10}(4t^2 i + 2t^2 j + 3t^2 k) + c$

At t = 0, OA = (4i + 3j + 2k)

(2i + 3j + 2k) $= \frac{1}{10}(4\ x\ 0^2 i + 2x\ 0^2 j + 3\ x\ 0^2 k) + c$

c = (4i + 3j + 2k)

$r_{(t=t)} = \frac{1}{10}(4t^2 i + 2t^2 j + 3t^2 k)$ +(4i + 3j + 2k)

$r_{(t=t)} = \frac{1}{10}\{(4t^2 + 40)i + (2t^2 + 30)j + (3t^2 + 2)k\}$

(d) the displacement vector AB

$OB_{(t=4)} = \frac{1}{10}\begin{pmatrix} 4\ x\ 4^2 + 40 \\ 2\ x\ 4^2 + 30 \\ 3\ x\ 4^2 + 20 \end{pmatrix} = \begin{pmatrix} 10.4 \\ 3.2 \\ 6.8 \end{pmatrix} m$

$\overline{AB}$ = OB – OA = $\begin{pmatrix} 10.4 \\ 3.2 \\ 6.8 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6.4 \\ 3.2 \\ 4.8 \end{pmatrix} m$

(e) work done by the force F after 4s.

$W_{(t=4)} = F.r = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}.\begin{pmatrix} 6.4 \\ 3.2 \\ 4.8 \end{pmatrix} = 92.8J$

Alternative 1

$v_{(t=4)} = \frac{1}{10}\begin{pmatrix} 8\ x4 \\ 4\ x\ 4 \\ 6\ x\ 4 \end{pmatrix} = \begin{pmatrix} 3.2 \\ 1.6 \\ 2.4 \end{pmatrix} ms^{-1}$

$v_{(t=4)} = \sqrt{3.2^2 + 1.6^2 + 2.4^2} = \sqrt{18.56} ms^{-1}$

$W_{(t=0\ and\ t=4)} = \frac{1}{2}m\left(v_{(t=4)}{}^2 - v_{(t=0)}{}^2\right)$

$= \frac{1}{2}x\ 10(18.56 - 0)$ =92.8J

Alternative 2

$P_{(t=t)} = F.v$

$P_{(t=t)} = \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix}.\frac{1}{10}\begin{pmatrix} 8t \\ 4t \\ 6t \end{pmatrix} = 11.6t$

$W_{(t=0\ and\ t=4)} = \int F.v dt = \int_0^4 11.6t dt$

$=[5.8t^2]_0^4$

= [5.8 x 4^2 – 0]

= 92.8J

Example 14

A particle move along a curve such that after t seconds its position vector is r where r = $\begin{pmatrix} t+1 \\ \frac{10}{3}t^3 - 6 \\ 4 - \frac{3}{2}t^2 \end{pmatrix}$.

The particle is acted on by a force F = (2ti + t^2j -2tk)N. Find

(a) Power at any time t.

$$v_{(t=t)} = \frac{dr}{dt} = \frac{d}{dt}\begin{pmatrix} t+1 \\ \frac{10}{3}t^3 - 6 \\ 4 - \frac{3}{2}t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10t^2 \\ -3t \end{pmatrix} ms^{-1}$$

$$P_{(t=t)} = F.v = \begin{pmatrix} 2t \\ t^2 \\ -2t \end{pmatrix}.\begin{pmatrix} 1 \\ 10t^2 \\ -3t \end{pmatrix} = 2t + 10t^4 + 6t^2$$

(b) work done by the force in the interval t = 1s and t = 3s

$$W_{(t=1\ and\ t=3)} = \int_1^3 F.vdt = \int_1^3 2t + 10t^4 + 6t^2 dt$$

$$= [t^2 + 2t^3 + 2t^5]_1^3$$

$W = (9 + 54 + 486) - (1 + 2 + 2)$ = 544J

Example 15

A particle of mass 2kg has a displacement vector s = $\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix}$m and a force F= $\begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix} N$. Find

(a) work done at t = 2s

W = F.v

$$W_{(t=t)} = \begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}.\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix} = (6t^3 + 56t^4)J$$

$W_{(t=2)} = (6\ x\ 2^3 + 56\ x\ 2^4)$ = 944J

(b) Power when t = 4s

$$v_{(t=t)} = \frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix} 2t^2 \\ 4t \\ 8t^3 \end{pmatrix} = \begin{pmatrix} 4t \\ 4 \\ 24t^2 \end{pmatrix} ms^{-1}$$

$$P_{(t=t)} = F.v = \begin{pmatrix} 3t \\ 4t^3 \\ 5t \end{pmatrix}.\begin{pmatrix} 4t \\ 4 \\ 24t^2 \end{pmatrix} = (12t^2 + 136t^3)W$$

$P_{(t=4)} = (12\ x\ 4^2 + 136\ x\ 4^3)W$ = 8,896W

Example 16

1. A force F = (2t i + j – 3t k)N acts on a particle of mass 2kg. The particle is initially at a point (0,0,0) and moving with a velocity (i + 2j-k)ms^{-1}. Determine the:
   (a) Magnitude of the acceleration of the particle after 2 seconds (04marks)

   F = (2t i + j – 3t k) = $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$N

a = $\frac{F}{m} = \frac{1}{2}\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix} = \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} ms^{-1}$

At t = 2s

$\underline{a}$=2i + 0.5j – 3k

$|\underline{a}| = \sqrt{2^2 + 0.5^2 + (-3)^2}$ = 3.64ms^{-2}

(b) Velocity of the particle after 2seconds (04marks)

$\underline{v} = \int \underline{a}dt = \int \begin{pmatrix} t \\ 0.5 \\ -1.5t \end{pmatrix} dt = \begin{pmatrix} 0.5t^2 \\ 0.5t \\ -7.5t^2 \end{pmatrix}$ + C

At t = 0 initial velocity = (i + 2j –k)

$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$ => C = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore \underline{v} = \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix}$

At t = 2s

$\underline{v} = \begin{pmatrix} 0.5(2)^2 + 1 \\ 0.5(2) + 2 \\ -7.5(2)^2 - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} ms^{-1}$

(c) Displacement of the particle after 2 seconds (04marks)

$\underline{r} = \int \underline{v}dt \int \begin{pmatrix} 0.5t^2 + 1 \\ 0.5t + 2 \\ -7.5t^2 - 1 \end{pmatrix} dt = \begin{pmatrix} \frac{t^3}{6} + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix}$ + C

At t = 0; $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + C$

C = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} t^3 + t \\ \frac{t^2}{4} + 2t \\ -t - \frac{t^3}{4} \end{pmatrix}$

At t = 2s

$\underline{r} = \begin{pmatrix} \frac{2^3}{6} + 2 \\ \frac{2^2}{4} + 2x2 \\ -2 - \frac{2^3}{4} \end{pmatrix} = \begin{pmatrix} \frac{10}{3} \\ 5 \\ -4 \end{pmatrix} m$

Example 17

1. A particle of mass 4kg starts from rest at point (2i- 3j + k)m. it moves with acceleration a= (4i + 2j -3k)ms^{-2} when a constant force F acts on it.
   Find the:
   (a) Force F (02marks)
   F = ma
   = $4\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix}$N or F = (16i + 8j -12k)N
   (b) Velocity at any time t. (04marks)
   $v = \int adt$
   v = at + c
   = $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} t + c$
   At time t = 0, v = u = c = $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
   Substituting for c
   v = $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} t = \begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} ms^{-1}$
   or v = (4t i + 2t j -3t k)ms^{-1}
   (c) Work done by the force F after 6 seconds (06marks)
   Work done = force (F) x distance ($\underline{r}$)

$$\underline{r} = \int v dt = \int \begin{pmatrix} 4t \\ 2t \\ -3t \end{pmatrix} dt = \begin{pmatrix} 2t^2 \\ t^2 \\ -1.5t^2 \end{pmatrix} + c$$

$$At\ t = 0, \underline{r} = c = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$Hence\ \underline{r} = \begin{pmatrix} 2t^2 \\ t^2 \\ -1.5t^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

After time t= 6seconds

$$\underline{r} = \begin{pmatrix} 2(6)^2 \\ (6)^2 \\ -1.5(6)^2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$$

Work done = force (F) x distance ($\underline{r}$)

$$= \begin{pmatrix} 16 \\ 8 \\ -12 \end{pmatrix} x \begin{pmatrix} 74 \\ 33 \\ -53 \end{pmatrix}$$

= 1184 + 264 + 636 = 2,084J

Revision exercise

1. A particle of mass 3kg is acted on by a force F = (24t3i + (36t -16)j + 12k)N at time t. at time t = 0, the particle is at the pint with position vector (3, -1, 4) and moving with velocity (16i + 15j -8k)ms^{-1}. Determine the
   (i) acceleration of the particle at time t= 2s. [28.4253ms^{-2}]
   (ii) speed of the particle at t= 2s [42.9534ms^{-1}]
   (iii) distance of the particle from the origin at t = 2s [56.5155m]
2. A particle of mass 4kg is acted on by a force F = (6i -36t^{2}j + 54tk)N at time t. At time t = 0, the particle is the point with position vector (i -5j –k) and it velocity is (3i + 3j)ms^{-1}. Determine the
   (a) position vector of the particle a time t = 1s. $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$m.
   (b) Distance of the particle from the origin a time t = 1s [6.1644m]

3. The acceleration f a particle is 6ti + 2j. Given that the velocity is (4i –j)ms^{-1} and displacement is (2i + 3j)m when t = 1s. Find the displacement when t =3s [(30i +5j)m]
4. If the velocity of a particle is 4cos2ti + 2sin2tj, given the displacement is 6i – 2j when t = $\frac{\pi}{4}s$. Find the distance of the body from the origin when t = πs. [5m]
5. If the acceleration of a particle is 9sin3ti + 2costj and the body id initially at rest. Find its velocity when t = $\frac{\pi}{6}s$. [(3i +j)ms^{-1}]
6. If the acceleration of a particle is 6sin6ti + 9cos3tj, given that the velocity is (i + 3j)ms^{-1} and displacement is (5i +2j)m when t = $\frac{\pi}{6}s$. Find the displacement when t = $\frac{\pi}{6}s$. [(5i +3j)m]
7. If the acceleration of the particle is 6ti + 6j -2k, given that the velocity is (3i + 6j -3k)ms^{-1} and displacement is (2i + 5j -2k)m when t =1s. Find
   (a) velocity when t= 2s [(12i + 12tj -5k)ms^{-1}]
   (b) displacement when t = 3s [(28i + 29j – 12k)m
8. If the acceleration f a particle is 2i +6j + 12t^{2}k, given that the velocity is (3i + k)ms^{-1} and displacement is (-i + k)m when t = 1s. Find the
   (i) velocity when t = 2s [(-i + 3j + 5k)
   (ii) displacement when t = 2s [(-3i + 8j + 19k)m
9. If the acceleration of a particle is (6ti – 2k)ms^{-2}, given that the velocity is (i + 12j -4k)ms^{-1} and the displacement is (3i + 6j)m when t = 2s, Find the
   (i) velocity when t = 4s [37i + 12j -8k)ms^{-1}
   (ii) displacement when t = 3s [11i + 18j -5k)m

10. The velocity of a particle $v = 4t^3i + 6tj - 3t^2k$. Given that the displacement is $14i + 6j - 3k$ at t = 1s. Find the
    (i) acceleration when t = 5s [$(108i + 6j - 18k)ms^{-2}$
    (ii) displacement when t = 0 [$(13i + 3j - 2k)m$]
11. The velocity of a particle $v = (3t^2i - 10t)i + 2j - 6tk$. Given that the displacement is $-9i + 3j - 13k$ at t = 2s. Find the
    (i) acceleration when t = 5s [$(20i - 6k)ms^{-2}$]
    (ii) displacement when t = 3s [$15i + 5j - 28k)m$]
12. The particle starts from rest at the origin moving with velocity of $v = \begin{pmatrix} 2cos2t + 11 \\ 3sin3t \\ 4 \end{pmatrix}$. Find the
    (i) speed when t = $\frac{\pi}{6}s$. [$13ms^{-1}$]
    (ii) displacement when t = $\frac{\pi}{2}s$. [$(1.5\pi i + j - 2\pi k)m$]
    (iii) acceleration when t = πs [$9jms^{-2}$]

Thank you
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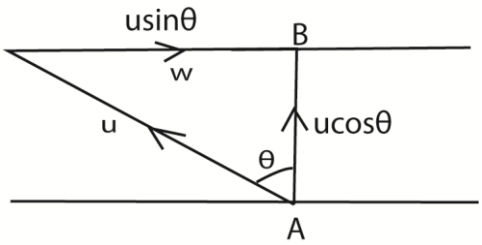
Vector mechanics

Crossing a river

There are two cases to consider when crossing a river

Case I: Shortest route

If the water s not still and boatman wishes to cross **directly opposite** to the standing point. In order to cross from point A to point B directly opposite A (perpendicularly), then the course set by the boat must be upstream of the river.



u = speed of the boat in still water

w= speed of running water

At point B: w = usinθ

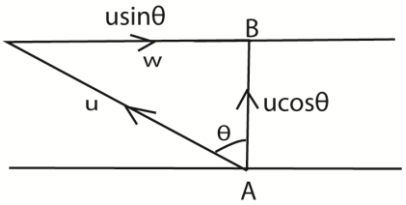
$\theta = \sin^{-1}\left(\frac{w}{u}\right)$

θ = is the direction to the vertical but the direction to the ban is $(90 - \theta)^0$

Time taken = $\frac{AB}{ucos\theta}$

Example 1

A man who can swim at 6km/h in still water would like to swim between two directly opposite point on the river banks of the river 300m flowing at 3km/h. Find the time taken to do this.



AB = 0.3km

$\theta = \sin^{-1}\left(\frac{w}{u}\right) = \sin^{-1}\left(\frac{3}{6}\right) = 30^0$

Time taken = $\frac{AB}{ucos\theta} = \frac{0.3}{6cos30}$

time = 0.058hrs = 3.46minute

He must swim at 300 to AB in order to cross directly and it takes him 3.46minutes.

Alternatively

Using Pythagoras theorem

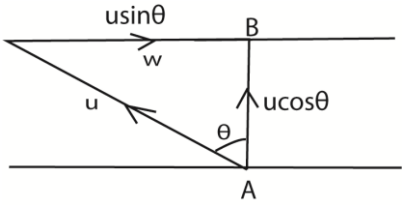
$6^2 = 3^2 + V_{AB}^2$

$V_{AB} = \sqrt{36 - 9} = 5.1962$km/h

Time = $\frac{AB}{V_{AB}} = \frac{0.3}{5.1963} = 0.058$hrs

Example 2

Two points A and B are on opposite banks of a river flowing at $\frac{5}{6}ms^{-1}$. A man who can swim at $\frac{25}{18}ms^{-1}$ in still water would like to swim directly from A to B. Find the width of the river if he takes 2minutes to cross the river.



$\theta = \sin^{-1}\left(\frac{w}{u}\right) = \sin^{-1}\left(\frac{5/6}{25/18}\right) = 36.87^0$

Time taken = $\frac{AB}{ucos\theta} = \frac{0.3}{\frac{25}{18}cos36.87}$

AB = 133.333m

Alternatively: using Pythagoras theorem

$\left(\frac{25}{18}\right)^2 = \left(\frac{5}{6}\right)^2 + V_{AB}^{\ 2}$

$V_{AB} = 1.1111m$

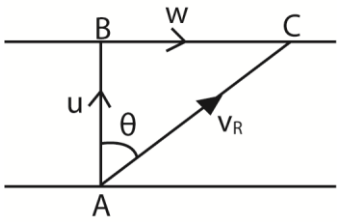
Time taken = $\frac{AB}{V_{AB}}$

2 x 60 = $\frac{AB}{1.1111}$

AB = 133.333m

Case II: the shortest time taken/ as quickly as possible

If the boatman wishes to cross the river as quickly as possible, then he should steer his boat directly from A to B as shown. The river pushes the boat down stream



Time taken to cross the river, t = $\frac{AB}{u}$

Distance covered downstream = wt

Or

Distance downstream = $w\frac{AB}{u}$

tanθ = $\frac{w}{u}$ or θ = $\tan^{-1}\frac{w}{u}$

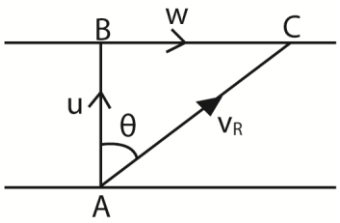
The resultant velocity downstream V_R

$V_R = \sqrt{w^2 + u^2}$

Example 3

A man who can swim at 2ms^{-1} in still water wishes to swim across a river 120m wide as quickly as possible. If the river flowsat0.5ms-1, find the time the man takes to cross far downstream he travels.

Solution



u = 2ms^{-1}, w = 0.5ms^{-1}, AB = 120m

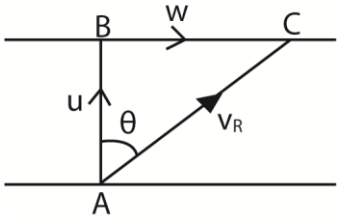
t = $\frac{AB}{u} = \frac{120}{2} = 60s$

Distance = wt = 0.5 x 60 = 30m

Example 4

A boat can travel at 3.5ms^{-1} in still water. A river is 80m wide and the current flows at 2ms^{-1}, calculate

(a) the shortest time to cross the river and the distance downstream the oat is carried.

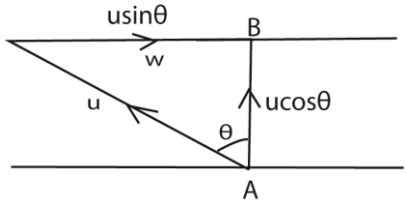


u = 3.5ms^{-1}, w = 2ms^{-1}, AB = 80m

t = $\frac{AB}{u} = \frac{80}{3.5} = 22.95s$

Distance, BC = wt = 2 x 22.95 = 45.8m

(b) the course that must be set to point exactly opposite the starting point and time taken for crossing



course, θ = $\sin^{-1}\frac{w}{u} = \sin^{-1}\frac{2}{3.5} = 34.8^0$

Time for crossing = $\frac{80}{3.5cos34.8}$= 27.8s

u = 3.5ms^{-1}, w = 2ms^{-1}, AB = 80m

Revision exercise 1

1. A man who can row at 0.9ms^{-1} in still water wishes to cross a river of width 1000m as quickly as possible. If the current flows at a rate of 0.3ms^{-1}. Find the time taken for journey. Determine the direction in which he should point the boat and position of the boat where he lands. [111.11s, 71.57^0 to the bank, 333.33 downstream]
2. A man swims at 5kmh^{-1} in still water. Find the time it takes the man to swim across the river250m wide, flowing at 3kmh^{-1}, if he swims so as to cross the river
   (a) the shortest route [225s]
   (b) in the quickest time[180s]
3. A boy can swim in still water at 1ms^{-1}, he swims across the river flowing at 0.6ms^{-1} which is 300m wide, find the time he takes
   (a) if he travels the shortest possible distance [375s]
   (b) if he travels as quickly as possible and the distance downstream, [300s, 180m]
4. A boy wishes to swim across a river 100m wide as quickly as possible. The river flows at 3kmh^{-1} and the boy can swim at 4kmh^{-1} in still water. Find the time that the boy takes to cross the river and how far downstream he travels. [90s, 75m]

Relative motion

It is composed of

(a) relative velocity
(b) Relative path

(a) Relative velocity

This is the velocity a body would have as seen by an observer on another body. Suppose A and B are two moving bodies, the velocity of A relative to B is the velocity of A as it appears to observer on B

It is denoted by $V_{AB} = V_A - V_B$

Note that $V_{AB} \neq V_{BA}$ since $V_{BA} = V_B - V_A$.

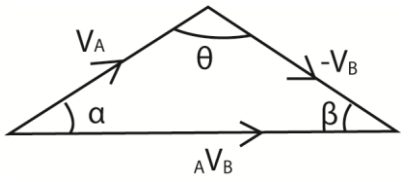
Numerical calculations

There are two methods used in calculations

- Geometric method and
- Vector method

(i) Geometric method.

If V_A and V_B are not given in vector form and the velocity of A relative to B is required, then we can reverse the velocity of B such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.



$V_{AB}^2 = V_A^2 + V_B^2 - 2V_A \times V_B \cos\theta$

and

$$\frac{V_{AB}}{sin\theta} = \frac{V_B}{sin\alpha} = \frac{V_A}{sin\beta}$$

(ii) Vector method

Find components of velocity for each separately

$\therefore V_{AB} = V_A - V_B$

Example 5

Particle A is moving due north at 30ms^{-1} and particle B is moving due south at 20ms^{-1}. Find the velocity of A relative to B.

Solution

$\uparrow V_A = 30ms^{-1}$ and $\downarrow V_B = 20ms^{-1}$

$V_{AB} = V_A - V_B$

$$V_{AB} = \begin{pmatrix} 0 \\ 30 \end{pmatrix} - \begin{pmatrix} 0 \\ -20 \end{pmatrix} = \begin{pmatrix} 0 \\ 50 \end{pmatrix}$$

$|V_{AB}| = \sqrt{0^2 + 50^2} = 50ms^{-1}$ due north

Example 6

A particle A has a velocity (4i + 6j -5k)ms^{-1} while B has a velocity of (-10i – 2j + 6k)ms^{-1}. Find the velocity of A relative to B.

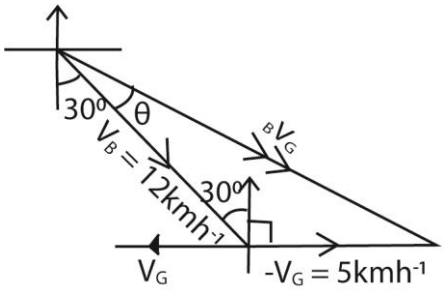
$V_{AB} = V_A - V_B$

$$V_{AB} = \begin{pmatrix} 4 \\ 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -10 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ 8 \\ -11 \end{pmatrix} ms^{-1}$$

Example 7

A girl walks at 5kmh^{-1} due west and a boy runs 12kmh^{-1} at a bearing of 150^0. Find the velocity of the boy relative to the girl

Method I (geometrical)



$V_{BG}^2 = V_B^2 + V_G^2 - 2V_B \times V_G \cos 120^0$

$V_{BG} = \sqrt{5^2 + 12^2 - 2\ x\ 5\ x\ 12\ cos120^0}$ = 15.13ms^{-1}

$\frac{5}{sin\theta} = \frac{15.13}{\sin 120}$

θ = 16.63^0

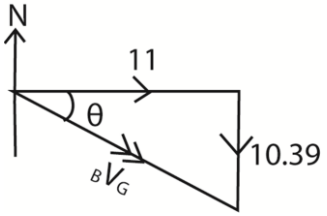
The relative velocity is 15.13ms^{-1} at S46.63^{0}E

$V_{BG} = V_B - V_G$

$= \begin{pmatrix} 12sin30 \\ -12cos30 \end{pmatrix} - \begin{pmatrix} -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ -10.39 \end{pmatrix}$

$|V_{BG}| = \sqrt{11^2 + (-10.39)^2}$

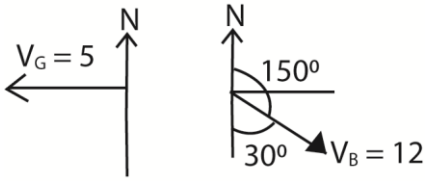
= 15.13ms^{-1}



$\theta = \tan^{-1}\frac{10.39}{11}$ = 43.40

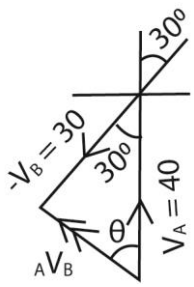
The relative velocity is 15.13ms^{-1} at S46.63^{0}E

Method II (Vector)



Example 8

Plane A is flying due north at 40kmh^{-1} while plane B is flying in the direction N30^{0}E at 30kmh^{-1}. Find the velocity of A relative B



$V_{AB}^2 = V_A^2 + V_B^2 - 2V_A \times V_B \cos 30^0$

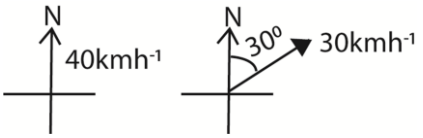
$V_{BG} = \sqrt{40^2 + 30^2 - 2\ x\ 40\ x\ 30\ cos30^0}$

= 20.53kmh^{-1}

$\frac{30}{sin\theta} = \frac{20.53}{\sin 30}$; θ = 46.94^0

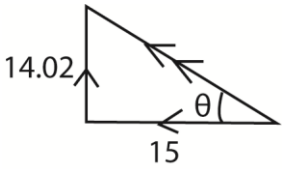
The relative velocity is 20.53kmh^{-1} at N46.9^{0}W

Method II (vectors)



$V_{AB} = V_A - V_B = \begin{pmatrix} 0 \\ 40 \end{pmatrix} - \begin{pmatrix} 30sin30 \\ -30cos30 \end{pmatrix} = \begin{pmatrix} -15 \\ 14.02 \end{pmatrix}$

$|V_{BG}| = \sqrt{(-15)^2 + (14.02)^2}$ = 20.53kmh^{-1}



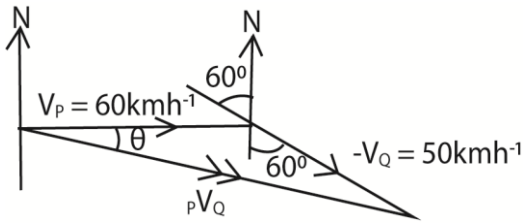
$\theta = \tan^{-1}\frac{14.02}{15} = 43.1^0$

The relative velocity is 20.53kmh^{-1} at N46.9^{0}W

Example 9

Ship P is steering 60kmh^{-1} due east while ship Q is steering in the direction N600W at 50kmh^{-1}. Find the velocity of P relative to Q.

Method I (Geometrical)



$V_{PQ}^2 = V_P^2 + V_Q^2 - 2V_P \times V_Q \cos 150^0$

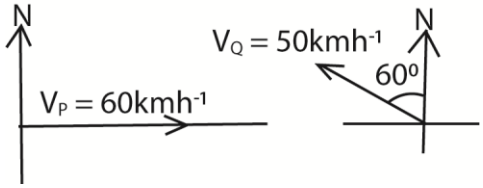
$V_{BG} = \sqrt{60^2 + 50^2 - 2\ x\ 60\ x\ 50\ cos 150^0}$

= 106.28kmh^{-1}

$\frac{50}{sin\theta} = \frac{106.28}{\sin 30}$; θ = 13.6^0

The relative velocity is 106.28kmh^{-1} at S76.4^{0}E

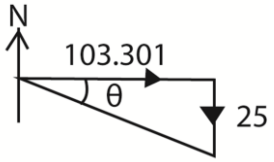
Method II (Vector)



$V_{PQ} = V_P - V_Q$

$= \begin{pmatrix}60\\0\end{pmatrix} - \begin{pmatrix}-50sin60\\50cos60\end{pmatrix} = \begin{pmatrix}103.301\\-25\end{pmatrix}$

$|V_{BG}| = \sqrt{(103.301)^2 + (-25)^2}$ =106.3kmh^{-1}



θ =$\tan^{-1}\frac{25}{103.301}$ = 13.6^0

Direction S(90^0 – 13.6^0)E

The relative velocity is 106.28kmh^{-1} at S76.4^{0}E

Finding true velocity

Example 10

To a cyclist riding due north at 40kmh^{-1}, a steady wind appears to blow from N60^{0}E at 50kmh^{-1}. Find the true velocity of the wind

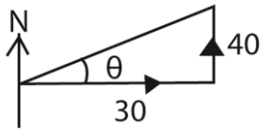
Solution

$V_{WC} = V_W - V_C$

$\begin{pmatrix}30\\0\end{pmatrix} = V_W - \begin{pmatrix}0\\40\end{pmatrix}$

$V_W = \frac{30}{40}$

$|V_W| = \sqrt{(30)^2 + (40)^2}$ = 50kmh^{-1}



θ =$\tan^{-1}\frac{40}{30}$ = 53.13^0

Direction N (90 – 15.13)^{0}E = N36.87^{0}E

Example 11

To a motorist travelling due north at 40kmh^{-1}, a steady wind appears to blow from N60^{0}E at 50kmh^{-1}.

(a) find the true velocity of the wind

$V_{WM} = V_W - V_M$

$\begin{pmatrix} 50sin60 \\ -50cos60 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$

$V_W = \frac{-43.5}{15}$

$|V_W| = \sqrt{(-43.5)^2 + (15)^2}$ = 46kmh^{-1}

$\theta = \tan^{-1}\frac{15}{43.5}$ = 19.02^0

Direction: N71^{0}W

(b) If the wind velocity and direction remain constant but the speed of the motorist is increase, find his speed when the wind appears to be blowing from the direction N45^{0}E.

$V_{WM} = V_W - V_M$

$\begin{pmatrix} -bsin45 \\ -bcos45 \end{pmatrix} = \begin{pmatrix} 46\sin71 \\ -46cos71 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}$

i components : -bsin45 = 46sin71

b = 61.5096

j components: -bcos45 = -46cos71 – a

a = 58.47kmh^{-1}

Example 12

To a man travelling due north at 10kmh^{-1}, a steady wind appears to blow from East. When he travels in the direction N60^{0}W at 8kmh^{-1}, it appears to come from south. Find the velocity of the wind.

$V_{WM} = V_W - V_M$

$\begin{pmatrix} -a \\ 0 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$V_W = \begin{pmatrix} -a \\ 10 \end{pmatrix}$ .....(i)

Also

$V_{WM} = V_W - V_M$

$\begin{pmatrix} 0 \\ b \end{pmatrix} = V_W - \begin{pmatrix} -8sin60 \\ b + 8cos60 \end{pmatrix}$ ..... (ii)

(i) and (ii)

$\begin{pmatrix} -a \\ 10 \end{pmatrix} = \begin{pmatrix} -8sin60 \\ b + 8cos60 \end{pmatrix}$

a = 8sin60 = $4\sqrt{3}$

10 = b + 8cos60

b = 6

$V_W = \begin{pmatrix} -a \\ 10 \end{pmatrix} = \begin{pmatrix} -4\sqrt{3} \\ 10 \end{pmatrix}$

$|V_W| = \sqrt{\left(-4\sqrt{3}\right)^2 + 10^2}$ = 12.17kmh^{-1}

$\theta = \tan^{-1}\left(\frac{10}{4\sqrt{3}}\right)$= 55.3^0

Direction: N(90 -53.3)^{0}W= N34.7^{0}W

Example 13

To a cyclist riding due north at 40kmh^{-1}, a steady wind appears to blow eastwards. On reducing his speed to 30kmh^{-}1 but moving in the same direction, the wind appears to come from southwest. Find the velocity of the wind

$V_{WC} = V_W - V_C$

$\begin{pmatrix} a \\ 0 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 40 \end{pmatrix}$

$V_W = \begin{pmatrix} a \\ 10 \end{pmatrix}$ .....(i)

Also

$V_{WC} = V_W - V_C$

$\begin{pmatrix} bsin45 \\ bcos45 \end{pmatrix} = V_W - \begin{pmatrix} 0 \\ 30 \end{pmatrix}$ ..... (ii)

(i) and (ii)

$\begin{pmatrix} a \\ 40 \end{pmatrix} = \begin{pmatrix} bsin45 \\ 30 + bcos45 \end{pmatrix}$

40 = 30 + bcos45

b = $10\sqrt{2}$

a = bsin45 = $10\sqrt{2}$sin45 = 10

$V_W = \begin{pmatrix} a \\ 40 \end{pmatrix} = \begin{pmatrix} 10 \\ 40 \end{pmatrix}$

$|V_W| = \sqrt{10^2 + 40^2}$ = 41.23kmh^{-1}

$\theta = \tan^{-1}\left(\frac{40}{10}\right)$= 75.96^0

Direction: N(90 -76)^{0}E= N14^{0}E

Exercise 2

1. Car A moving Eastward at 20ms^{-1} and car B is moving Northward at 10ms^{-1}. Find the
   (i) velocity of A relative to B [$10\sqrt{5}ms^{-1}$
   (ii) velocity of B relative to A [$10\sqrt{5}ms^{-1}$
2. A yatch and a trawler leave a harbour at 8am. The yatch travels due west at 10kmh-1 and trawler due east at 20kmh^{-1}
   (i) what is the velocity of the trawler relative to yatch [30kmh^{-1} east]
   (ii) how far apart are the boats at 9.30am [45km]
3. At 10.30am a car travelling at 25ms^{-1} due east overtakes a motor bike travelling at 10ms^{-1} due east. What is the velocity of the car relative to the motor bike and how far apart are the vehicle at 10.30am. [15ms^{-1} east, 900m]
4. Bird A has a velocity of (7i + 3j + 10k)ms^{-1} while bird B has a velocity (6i – 17k)ms^{-1}. Find the velocity of B relative to A [(-i -3j – 27k0ms^{-1}]
5. Joe rides his horse with a velocity $\begin{pmatrix} 5 \\ 24 \end{pmatrix}$kmh^{-1} while Jill is riding her horse with velocity $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$kmh^{-1}
   (i) Find Joe's velocity as seen by Jill $\left[\begin{pmatrix} 5 \\ 24 \end{pmatrix} \text{kmh}^{-1}\right]$
   (ii) What is Jill's velocity as seen by Joe. $\left[\begin{pmatrix} 0 \\ -124 \end{pmatrix} \text{kmh}^{-1}\right]$
6. In EPL football match, a ball is moving at 5ms^{-1}in the direction of N450E and the player is running due north at 8ms^{-1}. Find the velocity of the ball relative to the player. [5.69ms^{-1} at S38.38^{0}E]
7. An aircraft is flying at 250kmh^{-1} in direction N60^{0}E and a second aircraft is flying at 200kmh^{-1} in the direction N20^{0}W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft. [292kmh^{-1} at S77.9^{0}E]
8. A ship is sailing southeast at 20kmh^{-1} and a second ship is sailing due west at 25kmh^{-1}. Find the magnitude and direction of the velocity of the first ship relative to the second. [41.62kmh^{-1} at S70.13^{0}E]
9. What is the velocity of a cruiser moving at 20kmh^{-1} due to north as seen by an observer on a liner moving at 15kmh^{-1} in the direction N30^{0}W [10.3kmh^{-1} at N46.9^{0}E]
10. A car is being driven at 20ms-1 on a bearing of 0400. Wind is blowing from 3000 with speed of 10ms-1. Find the velocity of the wind as experienced by the driver of the car. [48.13ms^{-1} at S18.13^{0}W]
11. An aircraft is moving at 250kmh^{-1} in direction N60^{0}E. The second aircraft is moving at 200kmh^{-1} in a direction N20^{0}W. Find the velocity of the first aircraft as seen by the pilot of the second aircraft. [292kmh^{-1} at S77.9^{0}E]

12. To a pigeon flying with velocity of (-2i + 3j + k)ms^{-1}, a hawk appears to have a velocity of (i – 5j -10k)ms^{-1}.Find the true velocity of the hawk[(-i -2j – 9k)ms^{-1}]
13. To a cyclist riding at 3ms^{-1} due east, the wind appears to come from the south with the speed $3\sqrt{3}ms^{-1}$. Find the true speed and direction of the wind. [6ms^{-1} from S30^{0}W]
14. To the pilot of an aircraft A travelling at300kmh^{-1} due south, it appears that an aircraft B is travelling at 600kmh^{-1} in a direction N60^{0}W. Find the true speed and direction of the aircraft B. [520kmh^{-1} west]
15. Jane is riding her horse at 5kmh^{-1} due north and sees Suzan riding her horse apparently with velocity 4kmh^{-1}, N60^{0}E. Find Suzan's true velocity. [7.81kmh^{-1} N26.3^{0}E]
16. A eagle flying at 8ms^{-1} on a bearing of 240^0 sees a chick apparently running at 5ms^{-1} on bearing300^0. Find true velocity of the chick. [11.4ms^{-1} at 262.4^0]
17. A train is travelling at 80kmh^{-1} in direction N15^{0}E. A passenger on the train observes a plane apparently moving at 125kmh^{-1} in the direction N50^{0}E. Find the true velocity of the plane. [196kmh^{-1} N36.5^{0}E]
18. To an athlete jogging at 12kmh^{-1} on a bearing of N10^{0}E, the wind seems to come from a direction N20^{0}W at 15kmh^{-1}. Find the true velocity of the wind. [7.57kmh^{-1} N72.5^{0}W]
19. To a passenger on a boat which is travelling at 20kmh^{-1} on a bearing 230^0, the wind seems to be blowing from 250^0 as 12kmh^{-1}. Find the true velocity of the wind [9.64kmh^{-1} N24.8^{0}E]
20. On a particular day wind is blowing N30^{0}E at a velocity of 4ms^{-1} and a motorist is driving at 40ms^{-1} in the direction of S60^{0}E.
    (a) Find the velocity of the wind relative to the motorist. [40.2ms^{-1} at N54.28^{0}W]
    (b) If the motorist changes the direction maintaining his speed and the wind appears to blow due east. What is the new direction of the motorist [N85.03^{0}W]
21. A, B and C are three aircrafts. A has velocity (200i + 170j)ms^{-1}. To the pilot of A it appears that B has velocity (50i – 270j)ms^{-1}. To the pilot of B it appears that C has a velocity (50i + 170j)ms^{-1}. Find the velocities of B and C [(250i -100j)ms^{-1}, (300i + 70j)ms^{-1}]
22. To a bird flying due east at 10ms^{-1}, the wind seems to come from south. When the bird alters its direction of flight to N30^{0}E without altering its speed, the wind seems to come from the north-west. Find the true velocity of wind. [10.6ms^{-1} from S69.9^{0}W]
23. To an observer on a trawler moving at 12kmh^{-1} in the direction S30^{0}W, the wind appears to come from N600W. To an observer on a ferry moving at 15kmh^{-1} in a direction S80^{0}E, the wind appears to come from the north. Find the true velocity of the wind. [26.8kmh^{-1} N33.4^{0}W]

b. Relative position

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB respectively.

Position of A at time t is $R_{A(t=t)} = R_{A(t=0)} + t \times V_A$

Position of A at time t is $R_{Bt=t)} = R_{B(t=0)} + t \times V_B$

Position of A relative to B at time t is $R_{ABt=t)} = R_{A(t=0)} - R_{B(t=0)}$

$R_{ABt=t)} = (R_{A(t=0)} + t \times V_A) - (R_{B(t=0)} + t \times V_B)$

$R_{ABt=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

Example 14

The velocities of ships P and Q are (i +6j)kmh^{-1} and (-i + 3j)kmh^{-1}. At a certain instant, the displacement between the two ship is (7i + 4j)km. Find the

(a) Relative velocity of ship P to Q

$V_{PQ} = V_P - V_Q$

$V_{PQ} = \begin{pmatrix}1\\6\end{pmatrix} - \begin{pmatrix}-1\\3\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} kmh^{-1}$

(b) Magnitude of displacement between ships P and Q after 2hours.

$R_{PQt=t)} = (R_{P(t=0)} - R_{Q(t=0)}) + (V_{PQ})t$

$R_{PQt=t)} = \begin{pmatrix}7\\4\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}$

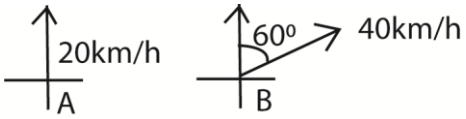
$R_{PQt=2)} = \begin{pmatrix}7\\4\end{pmatrix} + 2\,x\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}11\\10\end{pmatrix} km$

$|R_{PQ}| = \sqrt{11^2 + 10^2} = 14.87km$

Example 15

Two ship A and B move simultaneously with velocities 20kmh^{-1} and 40kmh^{-1}. Ship A moves in the northern direction while ship B moves in N60^{0}E. Initially ship B is 10km due west of A. Determine

(a) the relative velocity of A to B.

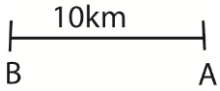


$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}0\\20\end{pmatrix} - \begin{pmatrix}40sin60\\40cos60\end{pmatrix} = \begin{pmatrix}-34.641\\0\end{pmatrix}$

$|V_{AB}| = \sqrt{(-34.641)^2 + 0^2}$ = 34.641

(b) the position of A relative to B at any time t



$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}10\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right] + t\begin{pmatrix}-34.641\\0\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}10 - 34.641t\\0\end{pmatrix} km$

Distance and time of closest approach

(Shortest distance and time of shortest distance)

When two particles are moving simultaneously with specific velocities, time will come when they are closest to each other **without** colliding.

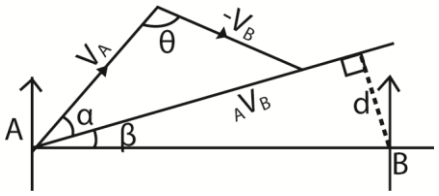
There are three methods used for the distance and time of closest approach, i.e. Geometrical, vector and differential method.

1. vector method
   Consider particle A and B moving with velocities VA and VB from point with position vector OA and OB respectively.
   For minimum distance to be attained then $V_{AB}.R_{AB(t=t)} = 0$. This gives the time.
   Then **shortest distance, d** = $|R_{AB(t=t)}|$
2. Differential method
   The minimum distance is reached when $\frac{d}{dt}|R_{AB(t=t)}|^2 = 0$. This gives time
   Then **shortest distance, d** = $|R_{AB(t=t)}|$

3. Geometrical method

If V_A and V_B are not given in vector form, then the velocity of B is reversed such that $V_{AB} = V_A + (-V_B)$ and the vector triangle is drawn as below.

The shortest distance, d will be perpendicular to V_{AB}



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos\theta$

$\frac{V_{AB}}{sin\theta} = \frac{V_B}{sin\alpha}$

Shortest distance, d =ABsinβ

Time to the shortest distance, t = $\frac{ABcos\beta}{V_{AB}}$

Example 16

A particle P starts from rest from a point with position vector (2j + 2k)m with a velocity (j + k)ms^{-1}. A second particle Q starts at the same time from a point whose position vector is (–11i – 2j – 7k)m with a velocity of (2i + j + 2k)ms^{-1}. Find

(i) the shortest distance between the particles

(ii)

(iii) how far each has travelled by this time.

$V_{PQ} = V_P - V_Q = \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\-1\end{pmatrix}$

$R_{PQ(t=t)} = \left(R_{P(t=0)} - R_{Q(t=0)}\right) + \left(V_{PQ}\right)t$

$R_{PQ(t=t)} = \left[\begin{pmatrix}0\\2\\2\end{pmatrix} - \begin{pmatrix}-11\\-2\\-7\end{pmatrix}\right] + \begin{pmatrix}-2\\0\\-1\end{pmatrix}t$

$R_{PQ(t=t)} = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}-2\\0\\-1\end{pmatrix}.\begin{pmatrix}11-2t\\4\\9-t\end{pmatrix} = 0$

-22 + 4t + 0 -9 + t = 0

t = $\frac{31}{5} = 6.2s$

(i) Then **shortest distance, d** = $\left|R_{PQ(t=t)}\right|$

$R_{PQ(t=6.2)} = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}$

$R_{PQ(t=6.2)} = \begin{pmatrix}11-2x\,6.2\\4\\9-6.2\end{pmatrix} = \begin{pmatrix}-1.4\\4\\2.8\end{pmatrix}$

$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2 + 4^2 + 2.8^2}$ = 5.08m

(ii) How far each has travelled

$R_{P(t=t)} = R_{P(t=0)} + (V_P)t$

$R_{P(t=6.2)} = \begin{pmatrix}0\\2\\2\end{pmatrix} + 6.2\,x\,\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\8.2\\8.2\end{pmatrix}$

$\left|R_{P(t=6.2)}\right| = \sqrt{0^2 + 8.2^2 + 8.2^2} = 11.6m$

$R_{Q(t=6.2)} = \begin{pmatrix}-11\\-2\\-7\end{pmatrix} + 6.2\,x\,\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1.4\\4.2\\5.4\end{pmatrix}$

$\left|R_{Q(t=6.2)}\right| = \sqrt{1.4^2 + 4.2^2 + 5.4^2} = 6.8m$

Method II

$\frac{d}{dt}\left|R_{AB(t=t)}\right|^2 = 0.$

$\left|R_{PQ(t=t)}\right|^2 = \begin{pmatrix}11-2t\\4\\9-t\end{pmatrix}^2$

$\left|R_{PQ(t=t)}\right|^2 = (11-2t)^2 + 4^2 + (9-t)^2$

$\left|R_{PQ(t=t)}\right|^2 = 218 - 62t + 5t^2$

$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = \frac{d}{dt}(218 - 62t + 5t^2)$

$\frac{d}{dt}\left|R_{PQ(t=t)}\right|^2 = -62 + 10t = 0$

t = 6.2s

$R_{PQ(t=6.2)} = \begin{pmatrix} 11 - 2x\ 6.2 \\ 4 \\ 9 - 6.2 \end{pmatrix} = \begin{pmatrix} -1.4 \\ 4 \\ 2.8 \end{pmatrix}$

$\left|R_{PQ(t=6.2)}\right| = \sqrt{(-1.4)^2 + 4^2 + 2.8^2}$ = 5.08m

Example 17

At 12 noon the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (-9i + 6j)km, v_A = (3i + 12j) kmh^{-1} and r_B = (16i + 6j), v_B = (-9i+ 3j)kmh^{-1} respectively

(i) Find how far apart the ships are at noon

$R_{AB(t=0)} = \left(R_{A(t=0)} - R_{B(t=0)}\right)$

$R_{AB(t=0)} = \begin{pmatrix} -9 \\ 6 \end{pmatrix} - \begin{pmatrix} 16 \\ 6 \end{pmatrix} = \begin{pmatrix} -25 \\ 0 \end{pmatrix}$

$\left|R_{AB(t=0)}\right| = \sqrt{(-25)^2 + 0^2} = 25km\ apart$

(ii) Assuming velocities do not change, find the least distance between the ships in the subsequent motion

$V_{AB} = V_A - V_B = \begin{pmatrix} 3 \\ 12 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \end{pmatrix}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \begin{pmatrix} -25 \\ 0 \end{pmatrix} + \begin{pmatrix} 12 \\ 9 \end{pmatrix} t = \begin{pmatrix} -25 + 12t \\ 9t \end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix} 12 \\ 9 \end{pmatrix}.\begin{pmatrix} -25 + 12t \\ 9t \end{pmatrix} = 0$

-300 + 144t + 81t = 0

t = $\frac{4}{3} hours$

Shortest distance, d = $\left|R_{PQ(t=\frac{4}{3})}\right|$

$R_{AB(t=\frac{4}{3})} = \begin{pmatrix} -25 + 12\ x\ \frac{4}{3} \\ 9x\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} -9 \\ 12 \end{pmatrix} km$

$\left|R_{AB(t=\frac{4}{3})}\right| = \sqrt{(-9)^2 + 12^2}$ = 15km

(iii) Find when their distance of closest approach occurs and the potion vectors of A and B

It occurs at 12.00 + $\frac{4}{3}\ x\ 60$ = 1.20pm

how far each travelled

$R_{A(t=t)} = R_{A(t=0)} + V_A t$

$R_{A(t=\frac{4}{3})} = \begin{pmatrix} -9 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 12 \end{pmatrix} x\ \frac{4}{3} = \begin{pmatrix} -5 \\ 22 \end{pmatrix} km$

$R_{B(t=t)} = R_{B(t=0)} + V_B t$

$R_{B(t=\frac{4}{3})} = \begin{pmatrix} 16 \\ 6 \end{pmatrix} + \begin{pmatrix} -9 \\ 3 \end{pmatrix} x\ \frac{4}{3} = \begin{pmatrix} 4 \\ 10 \end{pmatrix} km$

Example 18

At a certain time, the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (20j)km V_A = (9i -2j)kmh^{-1} at 14.00hrs

r_B = (i + 4j)km V_B = (4i + 8j)kmh^{-1} at 15.00hrs

Assuming velocities do not change, find

(a) the position vector of A at 15.00hrs

$$R_{A(t=t)} = R_{A(t=0)} + V_A t$$

At 16.00hrs: $R_{A(t=1)} = \begin{pmatrix} 0 \\ 20 \end{pmatrix} + \begin{pmatrix} 9 \\ -2 \end{pmatrix} x\ 1 = \begin{pmatrix} 9 \\ 18 \end{pmatrix} km$

(b) the least distance between A and B in the subsequent motion

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix} 9 \\ 18 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right] + \left[\begin{pmatrix} 9 \\ -2 \end{pmatrix} - \begin{pmatrix} 8 \\ 8 \end{pmatrix}\right]$t

$R_{AB(t=t)} = \begin{pmatrix} 8 + 5t \\ 14 - 10t \end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix} 5 \\ -10 \end{pmatrix}.\begin{pmatrix} 8 + 5t \\ 14 - 10t \end{pmatrix}$ = 0

t = 0.8hrs

Shortest distance, d = $\left|R_{PQ(t=0.8)}\right|$

$R_{AB(t=0.8)} = \begin{pmatrix} 8 + 5\ x\ 0.8 \\ 14 - 10\ x\ 0.8 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} km$

$R_{AB(t=0.8)} = \sqrt{12^2 + 6^2}$ = 13.42km

(c) time at which this least separation occurs.
15.00 + 0.8 x 60 = 15.48hrs

Example 19

At a certain time, the position vectors r and velocity vectors v of ship A and ship B are as follows

r_A = (-2i + 3j)km v_A =(12i -4j)kmh-1 at 11.45am

r_B = (8i + 7j)km v_B = (2i – 14j)kmh-1 at 12.00 noon

Assuming velocities do not change, find

(a) The least distance between A and B in the subsequent motion

OA = $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and v_A = $\begin{pmatrix} 12 \\ -4 \end{pmatrix} kmh^{-1}$

$R_{A(t=t)} = R_{A(t=0)} + V_A t$

12.00: $R_{A(t=\frac{1}{4})} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 12 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} km$

$V_{AB} = V_A - V_B = \begin{pmatrix} 12 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -14 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 8 \\ 7 \end{pmatrix}\right] + \begin{pmatrix} 10 \\ 10 \end{pmatrix} t$

$R_{AB(t=t)} = \begin{pmatrix} -7 + 10t \\ -5 + 10t \end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix} 10 \\ 10 \end{pmatrix}.\begin{pmatrix} -7 + 10t \\ -5 + 10t \end{pmatrix} = 0$

t = 0.6hrs

$R_{AB(t=0.6)} = \begin{pmatrix} -7 + 10\ x\ 0.6 \\ -5 + 10\ x\ 0.6 \end{pmatrix}$ =$\begin{pmatrix} -1 \\ 1 \end{pmatrix} km$

$\left|R_{AB(t=0.6)}\right| = \sqrt{(-1)^2 + 1^2}$ =1.4142km

(b) length of time for which A is within range, if ship B has guns within a range of up to 2km

$R_{AB(t=t)} = \begin{pmatrix} -7 + 10t \\ -5 + 10t \end{pmatrix}$

$\left|R_{AB(t=t)}\right| = 2km$

$\begin{pmatrix} -7 + 10t \\ -5 + 10t \end{pmatrix}^2 = 2^2$

$(-7 + 10t)^2 + (-5 + 10t)^2 = 4$

$100t^2 - 12t + 35 = 0$

t = 0.7hrs or t = 0.5hrs

Time for which they are in range

= 0.7 – 0.5 = 0.2h

Example 20

At 10am, ship A moves with a constant velocity (4i+ 20j) kmh^{-1} and ship B due north of A moves with a constant velocity (-3i – 4j) kmh^{-1}.

(a) Find the velocity of A relative to B

$$V_{AB} = V_A - V_B = \begin{pmatrix} 4 \\ 20 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \end{pmatrix} kmh^{-1}$$

(b) If the shortest distance between the two ships is 4.2km. Find the
   (i) time to the nearest minute when they are closest together
   (ii) original distance apart at 10am
   (iii) the bearing of B from A when they are closest together.

Solution

(ii) Let a km be the distance apart at 10am

$$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$$

$$R_{AB(t=t)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}\right] + \begin{pmatrix} 7 \\ 24 \end{pmatrix} t = \begin{pmatrix} 7t \\ -a + 24t \end{pmatrix} km$$

$$R_{AB(t=t)} = \begin{pmatrix} 7t \\ -a + 24t \end{pmatrix} km$$

$$\left|R_{AB(t=t)}\right| = 4.2km$$

$$\begin{pmatrix} 7t \\ -a + 24t \end{pmatrix}^2 = (4.2)^2$$

$$(7t)^2 + (-a + 24t)^2 = 17.64$$

$625t^2 - 48at + a^2 = 17.64$ .....(i)

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$$\begin{pmatrix} 7 \\ 24 \end{pmatrix}.\begin{pmatrix} 7t \\ -a + 24t \end{pmatrix} = 0$$

$49t - 24a + 576t = 0$

$a = \frac{625t}{24}$ .....(ii)

Substituting for a in eqn. (i)

$625t^2 - 48\left(\frac{625t}{24}\right)t + \left(\frac{625t}{24}\right)^2 = 17.64$

$53.1684t^2 = 17.67$

$t = \pm 0.57h = 0.576\ x\ 60 = 35minute$

(ii) $a = \frac{625t}{24} = \frac{625\ x\ 0.576}{24}$ = 15km

(iii) $R_{AB(t=0.576)} = \begin{pmatrix} 7x\ 0.576 \\ -15 + 24\ x\ 0.576 \end{pmatrix} km$

$$R_{AB(t=0.576)} = \begin{pmatrix} 4.032 \\ -1.176 \end{pmatrix}$$

$\theta = \tan^{-1}\left(\frac{1.176}{4.032}\right) = 16.3^0$

$Direction: E16.3^0S$

(c) length of time for which A is within range, if the visibility of ship B is within 12km

$$R_{AB(t=t)} = \begin{pmatrix} 7t \\ -a + 24t \end{pmatrix} = \begin{pmatrix} 7t \\ -15 + 24t \end{pmatrix} km$$

$$\left|R_{AB(t=t)}\right| = 12km$$

$$(7t)^2 + (-a + 24t)^2 = 144$$

$625t^2 - 720t + 81 = 0$

t = 1.026h or t = 0.126h

Time for which they are in range

= 1.026 – 0.126 = 0.9h

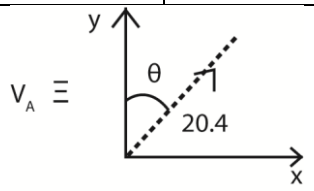
Example 21

At 12 noon a ship A is moving with constant velocity of 20.4kmh^{-1} in the direction Nθ^0E where tan θ = $\frac{1}{5}$. A second ship B is 15km due to north of A. Ship B is moving with constant velocity of 5kmh^{-1} in the

direction Sα^0W, where tan α= ¼. If the shortest distance between the ships is 4.2km, find the time to the nearest minute when the distance between the ships is shortest. (12mars)

Table of results

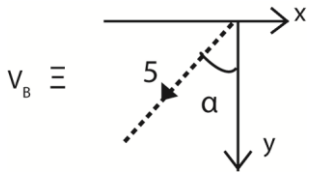
Vector	Magnitude	Direction
V_A	20.4kmh^{-1}	NθE
V_B	5kmh^{-1}	SαW



$V_A \equiv$

tan θ = $\frac{1}{5}$; θ = 11.3^0

$$V_A = \begin{pmatrix} 20.4\sin 11.3^0 \\ 20.4\cos\ 11.3^0 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$



$V_B \equiv$

tan α = $\frac{1}{5}$; α = 36.87^0

$$V_{-B} = \begin{pmatrix} -5\sin 36.87^0 \\ -5\cos\ 36.87^0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

B(0 , 15)

15km

A(0 , 0)

$r_A = \int V_A dt = \int \begin{pmatrix} 4 \\ 20 \end{pmatrix} dt = \begin{pmatrix} 4t \\ 20t \end{pmatrix} + c$

At t = 0, $r_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ => c = $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence $r_A = \begin{pmatrix} 4t \\ 20t \end{pmatrix}$

$R_B = \int V_B dt = \int \begin{pmatrix} -3 \\ -4 \end{pmatrix} dt = \begin{pmatrix} -3t \\ -4t \end{pmatrix} + c$

At t = 0, $r_B = \begin{pmatrix} 0 \\ 15 \end{pmatrix}$ => c = $\begin{pmatrix} 0 \\ 15 \end{pmatrix}$

Hence $r_B = \begin{pmatrix} -3t \\ 15 - 4t \end{pmatrix}$

$_Ar_B = r_A - r_B = \begin{pmatrix} 4t \\ 20t \end{pmatrix} - \begin{pmatrix} -3t \\ 15 - 4t \end{pmatrix} = \begin{pmatrix} 7t \\ 24t - 15 \end{pmatrix}$

$d_s = |\,_Ar_B| = \sqrt{(7t)^2 + (24t - 15)^5}$

but ds = 4.2

⇨ $\sqrt{(7t)^2 + (24t - 15)^5} = 4.2$

$\left(\sqrt{(7t)^2 + (24t - 15)^5}\right)^2 = 4.2^2$

$(7t)^2 + (24t - 15)^5 = 4.2^2$

$47t^2 + 576t^2 - 720t + 225 = 17.64$

$625t^2 - 720t + 207.36 = 0$

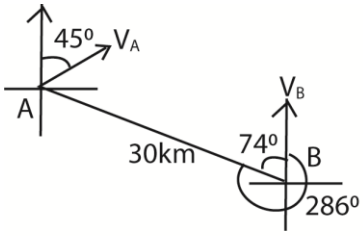
t = $\frac{720 \pm \sqrt{(-720)^2 - 4(625)(207.36)}}{2(625)}$ = 0.576hours

= 0.576 x 60 = 35minutes

Hence the time at which the distance is shortest is 12:35pm

Example 22

At noon a boat A is 30km from boat B and its direction from B is 2860. A is moving in the North-East direction at 16kmh-1 and B is moving in the north direction at 10kmh-1. Determine when they are closest to each other. What is the distance between them?



$R_{A(t=0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, R_{B(t=0)} = \begin{pmatrix} 30sin74 \\ -30cos74 \end{pmatrix} km$

$V_{AB} = \begin{pmatrix} 16sin45 \\ 16cos45 \end{pmatrix} - \begin{pmatrix} 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 11.314 \\ 1.314 \end{pmatrix} km/h$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}30sin74\\-30cos74\end{pmatrix}\right] + \begin{pmatrix}11.314\\1.314\end{pmatrix}t$

$R_{AB(t=t)} = \begin{pmatrix}-28.838 + 11.314t\\8.269 + 1.314t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0.$

$\begin{pmatrix}11.314\\1.314\end{pmatrix}.\begin{pmatrix}-28.838 + 11.314t\\8.269 + 1.314t\end{pmatrix} = 0$

t = 2.43h

$R_{AB(t=2.43)}$

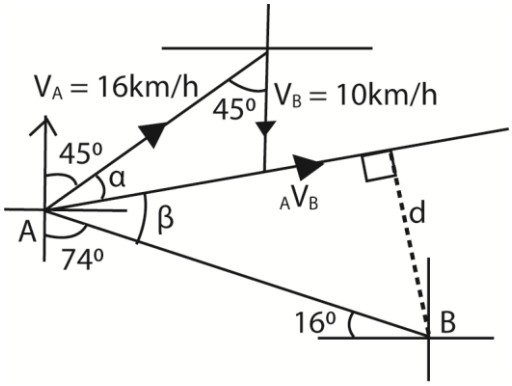
$= \begin{pmatrix}-28.838 + 11.314\ x\ 2.43\\8.269 + 1.314\ x\ 2.43\end{pmatrix} km$

$R_{AB(t=2.43)} = \begin{pmatrix}-1.345\\11.462\end{pmatrix} km$

$\left|R_{AB(t=2.43)}\right| = \sqrt{(-1.345)^2 + 11.462^2}$

= 11.54km

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos45$

$V_{AB} = \sqrt{16^2 + 10^2 - 2x16x10cos45}$

= 11.39km/h

$\frac{10}{sin\alpha} = \frac{11.39}{sin45}$

$\alpha = 38.38^0$

$45 + \alpha + \beta + 74 = 180^0$

$\beta = 22.62^0$

d = ABsinβ = 30sin22.62 = 11.54km

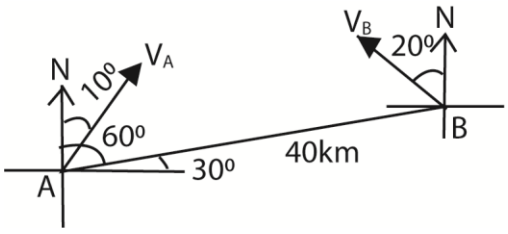
Time, t = $\frac{ABcos\beta}{V_{AB}} = \frac{30cos22.62}{11.39}$ = 2.43h

Time is 2.43h from noon or 2hour and 25.8minutes

It occurs 2.26pm at a distance 11.54km

Example 23

Two planes A and B are both flying above the Pacific Ocean. Plane A is flying on a course of 010^0at a speed 300kmh^{-1} and plane B is flying on a course of 340^0 at 200kmh^{-1}. At a certain instant, plane B is 40km from A. Plane A is then on a bearing of 060^0. After what time will they come closest together and what will be their minimum distance apart.



$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}300sin10\\300cos10\end{pmatrix} - \begin{pmatrix}-200sin20\\200cos20\end{pmatrix} = \begin{pmatrix}120.4985\\107.5038\end{pmatrix}$

$R_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix}40cos30\\40sin30\end{pmatrix} km$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}40cos30\\40sin30\end{pmatrix}\right] + t\begin{pmatrix}120.4985\\107.5038\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-34.641 + 120.4985t\\-20 + 107.5038t\end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}120.4985\\107.5038\end{pmatrix}\begin{pmatrix}-34.641 + 120.4985t\\-20 + 107.5038t\end{pmatrix} = 0$

-6324.2645 + 26076.9555t = 0

t = 0.2425h

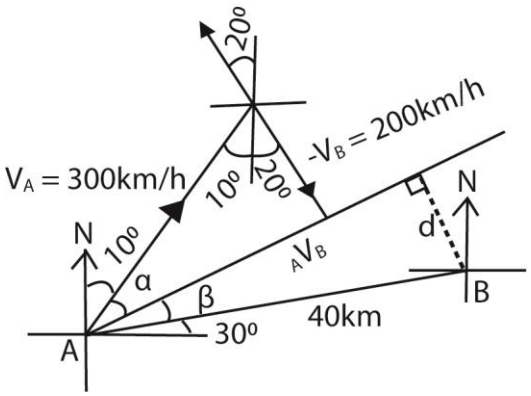
Least distance = $\left|R_{AB(t=0.2425)}\right|$

$R_{AB(t=\ 0.2425)} = \begin{pmatrix}-34.641 + 120.4985x0.2425\\-20 + 107.5038x0.2425\end{pmatrix}$

$R_{AB(t=\,0.2425)} = \begin{pmatrix} 5.4202 \\ 6.0692 \end{pmatrix} km$

$\left|R_{AB(t=0.2425)}\right| = \sqrt{(5.4202)^2 + (6.0692)^2} = 8.14m$

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos30$

$V_{AB} = \sqrt{300^2 + 200^2 - 2x300x200cos30}$

= 161.484km/h

$\frac{200}{sin\alpha} = \frac{161.484}{sin30}$

α = 38.26^0

α + β =50

β = 11.74^0

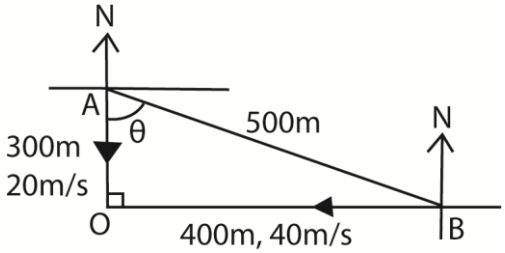
d = ABsinβ = 40sin11.74 = 8.14km

Time, t = $\frac{ABcos\beta}{V_{AB}} = \frac{40cos11.74}{161.484} = 0.2425$h

Example 24

At a given instant two cars are at a distance 300m and 400m from a point of intersection O of two roads crossing at right angles and are approaching O at uniform speeds of 20m/s and 40m/s respectively. Find

(i) Initial distance between the two cars
(ii) shortest distance between the cars
(iii) time taken to reach this point



$AB = \sqrt{300^2 + 400^2} = 500m$

$\theta = \tan^{-1}\left(\frac{400}{300}\right) = 53.1^0$

$R_{A(t=0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix} 50cos53.1 \\ 50sin53.1 \end{pmatrix} km$

$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix} 0 \\ -20 \end{pmatrix} - \begin{pmatrix} -40 \\ 0 \end{pmatrix} = \begin{pmatrix} 40 \\ -20 \end{pmatrix} ms^{-1}$

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 500cos53.1 \\ 500sin53.1 \end{pmatrix}\right] + t\begin{pmatrix} 40 \\ -20 \end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix} -399.842 + 40t \\ 300.21 - 20t \end{pmatrix}$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix} 40 \\ -20 \end{pmatrix}.\begin{pmatrix} -399.842 + 40t \\ 300.21 - 20t \end{pmatrix} = 0$

-21,997.88 +2000t = 0

t = 11s

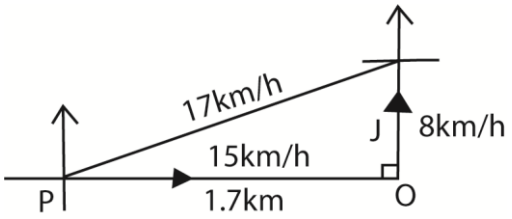
$R_{AB(t=11)} = \begin{pmatrix} -399.842 + 40x11 \\ 300.21 - 20\ x\ 11 \end{pmatrix} = \begin{pmatrix} 40.158 \\ 80.21 \end{pmatrix} m$

$R_{AB(t=11)} = \sqrt{(40.158)^2 + (80.21)^2}$ =89.701m

Example 25

A road running north-south crosses a road running east-west at a junction O. Initially Paul is on the east-west, 1.7km west of O and is cycling towards O at15mk/h. At the same time John is at O cycling due north at 8km/h. If Paul and John do not alter their velocities, find the

(i) relative velocity of Paul to John
(ii) shortest distance between Paul and John



$R_{P(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{J(t=0)} = \begin{pmatrix}1.7\\0\end{pmatrix} km$

$V_{PJ} = V_P - V_J$

$V_{PJ} = \begin{pmatrix}15\\0\end{pmatrix} - \begin{pmatrix}0\\8\end{pmatrix} = \begin{pmatrix}15\\-8\end{pmatrix} ms^{-1}$

$|V_{PJ}| = \sqrt{15^2 + (-8)^2}$ = 17km/h

$R_{PJ(t=t)} = (R_{P(t=0)} - R_{J(t=0)}) + (V_{PJ})t$

$R_{PJ(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}1.7\\0\end{pmatrix}\right] + \begin{pmatrix}15\\-8\end{pmatrix} t$

$R_{PJ(t=t)} = \begin{pmatrix}-1.7 + 15t\\-8t\end{pmatrix}$

For minimum distance: $V_{PJ}.R_{PJ(t=t)} = 0$

$\begin{pmatrix}15\\-8\end{pmatrix}.\begin{pmatrix}-1.7 + 15t\\-8t\end{pmatrix}$ =0

-25.5 +289t = 0

t = 0.088h

Least distance = $|R_{PJ(t=0.088)}|$

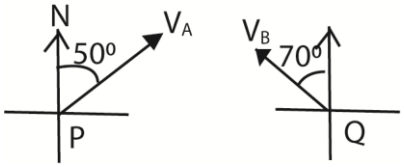
$R_{PJ(t=0.088)} = \begin{pmatrix}-1.7 + 15\ x\ 0.088\\-8x\ 0.88\end{pmatrix} = \begin{pmatrix}-0.38\\-0.704\end{pmatrix}$

$|R_{PJ}| = \sqrt{(-0.38)^2 + (-0.704)^2}$ = 0.8km

Example 26

Two airship P and Q are 100km apart, P being west of Q. Two Helicopters A and B fly simultaneously from P and Q respectively, at 11.00a.m. Helicopter A is flying with a constant speed of 400km/h in the direction N50^{0}E. Helicopter B flying at a constant speed of 500km in the direction N70^{0}W. Find the

(i) Time when the helicopters are closest together.
(ii) closest distance between the helicopters



$R_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} km, R_{B(t=0)} = \begin{pmatrix}100\\0\end{pmatrix} km$

$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix}400sin50\\400cos50\end{pmatrix} - \begin{pmatrix}-500sin70\\500cos70\end{pmatrix}$

$V_{AB} = \begin{pmatrix}776.264\\86.105\end{pmatrix} kms^{-1}$

$R_{AB(t=t)} = (R_{A(t=0)} - R_{B(t=0)}) + (V_{AB})t$

$R_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}100\\0\end{pmatrix}\right] + t\begin{pmatrix}776.264\\86.105\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-100 + 776.264t\\86.105t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}776.264\\86.105\end{pmatrix}\begin{pmatrix}-100 + 776.264t\\86.105t\end{pmatrix}.$

609999.869t =77624.4

t = 0.1273h = 0.1273 x 60= 8minutes

Time when they are closest

= 11.00 + 8= 11.08am

Least distance = $|R_{AB(t=0.1273)}|$

$R_{AB(t=0.1273)} \begin{pmatrix} -100 + 776.264\ x\ 0.1273 \\ 86.105\ x\ 0.1273 \end{pmatrix} km$

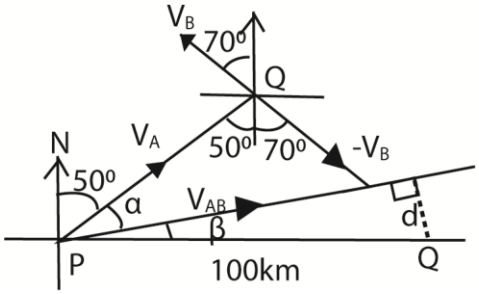
$R_{AB(t=0.1273)} = \begin{pmatrix} -1.182 \\ 10.961 \end{pmatrix} km$

$|R_{AB(t=0.1273)}| = \sqrt{(-1.182)^2 + 10.961^2}$

=11.0247km

Closest distance =11.025km

Alternatively



$V_{AB}{}^2 = V_A{}^2 + V_B{}^2 - 2V_AV_Bcos120$

$V_{AB} = \sqrt{400^2 + 500^2 - 2x400x500cos120}$

= 781.025km/h

$\frac{500}{sin\alpha} = \frac{781.025}{sin120}$

α = 33.67^0

α + β + 50 =90

β = 6.33^0

d = PQsinβ = 100sin6.33 = 11.025km

Time, t = $\frac{PQcos\beta}{V_{AB}} = \frac{100cos6.33}{781.025}$

t = 0.1273h = 0.1273 x 60= 8minutes

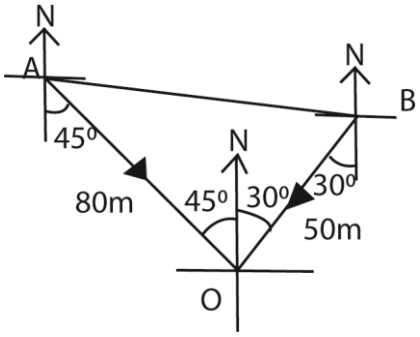
Time when they are closest

= 11.00 + 8= 11.08am

Example 26

Car A is 80m North-West of point O, Car B is 50m N300E of O. Car A is moving at 20m/s on a straight road towards O. Car B is moving at 10m/s on another straight road towards O. Determine the

(i) Initial distance between the cars
(ii) Velocity of A relative to B
(iii) shortest distance between the two cars as they approach O



$R_{A(t=0)} = \begin{pmatrix} -80cos45 \\ 80sin45 \end{pmatrix}, R_{B(t=0)} = \begin{pmatrix} 50sin30 \\ 50cos30 \end{pmatrix}$

$R_{AB(t=0)} = \begin{pmatrix} -80cos45 \\ 80sin45 \end{pmatrix} - \begin{pmatrix} 50sin30 \\ 50cos30 \end{pmatrix} = \begin{pmatrix} -81.5685 \\ 13.2673 \end{pmatrix}$

$[R_{AB(t=0)}] = \sqrt{(-81.5685)^2 + 13.2673^2}$ = 82 .6404m

(ii) $V_A = \begin{pmatrix} 20sin45 \\ -20cos45 \end{pmatrix}, V_B = \begin{pmatrix} -10sin30 \\ -10cos30 \end{pmatrix}$

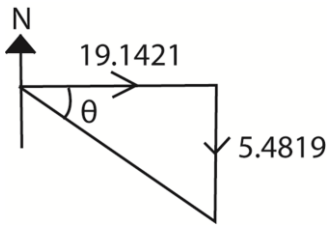
$V_{AB} = V_A - V_B$

$V_{AB} = \begin{pmatrix} 20sin45 \\ -20cos45 \end{pmatrix} - \begin{pmatrix} -10sin30 \\ -10cos30 \end{pmatrix}$

$V_{AB} = \begin{pmatrix} 19.1421 \\ -5.4819 \end{pmatrix}$

$|V_{AB}| = \sqrt{(19.1421)^2 + (-5.4819)^2}$

= 19.9116m/s



$\theta = \tan^{-1}\frac{5.4819}{19.1421} = 15.98^0$

Direction: E15.98^{0}S

(iii)

$R_{AB(t=t)} = \left(R_{A(t=0)} - R_{B(t=0)}\right) + (V_{AB})t$

$R_{AB(t=t)} = \begin{pmatrix}-81.5685\\13.2673\end{pmatrix} + t\begin{pmatrix}19.1421\\-5.4819\end{pmatrix}$

$R_{AB(t=t)} = \begin{pmatrix}-81.5685 + 19.1421t\\13.2673 - 5.4821t\end{pmatrix} km$

For minimum distance: $V_{AB}.R_{AB(t=t)} = 0$

$\begin{pmatrix}19.1421\\-5.4819\end{pmatrix}.\begin{pmatrix}-81.5685 + 19.1421t\\13.2673 - 5.4821t\end{pmatrix} = 0$

t = 4.1216s

Least distance = $\left|R_{AB(t=4.1216s)}\right|$

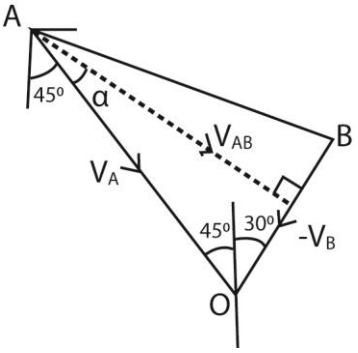
$R_{AB(t=t)} = \begin{pmatrix}-81.5685 + 19.1421\ x\ 4.1216\\13.2673 - 5.4821\ x\ 4.1216\end{pmatrix} m$

$R_{AB(t=t)} = \begin{pmatrix}-2.672\\-9.328\end{pmatrix} m$

$\left|R_{AB(t=t)}\right| = \sqrt{(-2.672)^2 + (-9.328)^2}$ =9.728m

Alternatively

Method II: Using geometric approach



Using cosine rule

$|V_{AB}|^2 = 20^2 + 10^2 - 2\ x\ 20\ x10\cos 75^0$

$|V_{AB}| = 19.912ms^{-1}$

$\frac{|V_{AB}|}{\sin 75^0} = \frac{10}{\sin\alpha}$

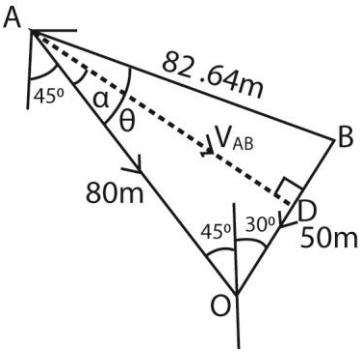
$\alpha = 29.02^0$

$45^0 + 29.02^0 = 74.02^0$

∴ The velocity of A relative to B is 19.912ms^{-1} due S74.02^{0}E

The shortest distance between the two cars as they approach O (04marks)

Using geometrical approach



$\frac{50}{\sin\theta} = \frac{82.64}{\sin 75^0}$; θ = 35.76^0

$< BAD = 35.76 - 29.02 = 6.74^0$

$\sin 6.74^0 = \frac{|r_{AB}|}{AB}$

$_Ar_B = 82.64\sin 6.74^0 = 9.699m$

∴ The shortest distance between the two cars they approach O is 9.699m

Revision exercise 3

1. 1.At 8am ship A and ship Bare 11km apart with B due west of A. a and B move with a constant velocities (-4i +3j)kh/h and (2i + 4j)km/h respectively. Find the the

(i) least distance between the two ship in the subsequent motion[1.81km]

(ii) time to the nearest minute at which this situation occurs[9.47am]

2. 2. At 7.30am, two ship A and B are 8km apart with B due north of A. A and B move with constant velocities (12j)km/h and (-5i)km/h respectively. Find the

(i) least distance between the two ship in the subsequent motion[3.08km]

(ii) time to the nearest minute at which this situation occurs[8.04am]

3. A and B are two tankers at 13.00hrs, tanker B bas position vector of (4i+8j)km relative to A. A and B move with constant velocities (6i + 9j)km/h and (-3i + 6j)km/h respectively. Find the
   (i) least distance between the two ship in the subsequent motion[6.32km]
   (ii) time to the nearest minute at which this situation occurs[13.40hrs]
4. At 12 noon the position vectors r and velocity vectors v of two ship A and B are as follows
   r_A = (5i + j)km, V_A = (7i + 3j)km/h and r_B = (8i + 7j)km, V_B = (2i –j)km/h
   (i) Assuming velocities do not change , find the least distance between the ships in subsequent motion [2.81]
   (ii) Find the time when their distance of closest approach occur [12.57pm]
5. At a certain time, the position vectors r and velocity vectors V of two ship A and B are as follows

   r_A = (3i + j)km, V_A = (2i + 3j)km/h at 11.00am

   r_B = (2i - j)km, V_B = (3i + 7j)km/h at 12.00noon

   Assuming velocities do not change; find the

   (i) The position vector of A at noon [5i + 4j]
   (ii) Distance between the ships at 12.00 noon [5.83km]
   (iii) The least distance between A and B in the subsequent motion[1.7km]
   (iv) Time at which the least separation occurs [1.21pm]

6. At 12 noon, the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   r_A = (13i + 5j)km, V_A = (3i - 10j)km/h

   r_B = (3i - 5j)km, V_B = (15i + 14j)km/h

   (i) Assuming the velocities do not change, find the least distance between the ships in subsequent motion [4.47km]
   (ii) The battle ship has guns with a range of up to 5km, find the length of time during which the cruiser is within range of the battle ships [10minutes]
7. At tine t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   $r_A = \begin{pmatrix}1\\2\\3\end{pmatrix}m, V_A = \begin{pmatrix}-6\\0\\1\end{pmatrix}m/s \text{ and } r_B = \begin{pmatrix}4\\-14\\1\end{pmatrix}m, V_A = \begin{pmatrix}-5\\1\\7\end{pmatrix}m/s$

   Assuming velocities do not change, find

   (i) The potion vectors of B relative to A at time t seconds $\left[\left(\begin{pmatrix}7\\2\\2\end{pmatrix}+\begin{pmatrix}9\\13\\-7\end{pmatrix}t\right)km\right]$
   (ii) The least distance between the ships in the subsequent motion [15.9m]
   (iii) The time taken to the closest distance $\left[\frac{25}{33}s\right]$

8. At time t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   r_A = (3i + j + 5k)m, V_A = (4i + j – 3k)m/s

   r_B = (i - 3j+ 2k) m, V_B = (i + 2j + 2k)m/s

   Assuming velocities do not change, find

   (i) The position vector of B relative to A at time t second $\left[\begin{pmatrix}2+3t\\4-t\\3-5t\end{pmatrix}m\right]$

   (ii) The value of t when A and B are closed $\left(\frac{13}{35}\right)$

   (iii) Least distance between A and B [4.917m]

9. At tine t = 0 the position vectors r and velocity vectors V of two battle ship A and battle B are as follows

   $r_A = (\beta)m, V_A = \begin{pmatrix}2\\1\\-5\end{pmatrix}m/s \text{ and } r_B = (2\beta)m, V_A = \begin{pmatrix}1\\-5\\1\end{pmatrix}m/s$

   Where β is a constant, assuming velocities do not change show that the least distance between the ships in the subsequent motion is $\frac{\beta}{73}$ and their distance of closest approach is $\frac{6\beta\sqrt{2}}{\sqrt{73}}$.

10. A lizard on a wall at point A, has a position vector $r_A = \begin{pmatrix}65\\40\\0\end{pmatrix}cm$. At time t= 0 seconds a fly has a position vector $r_F = \begin{pmatrix}37\\16\\22\end{pmatrix}cm$ and velocity vector $V_F = \begin{pmatrix}5\\2\\-1\end{pmatrix}cm/s$

    If the fly were to continue with this velocity, find the closest distance it would come to the lizard and the value of t when it occurs [$\sqrt{374}m$, 7s]

11. A particle P move with constant velocity (2i + 3j + 8k)m/s passes a point with position vector (6i – 11j + 4k)m. At the same instant particle Q passes through a point whose position vector is (i -2j + 5k)m moving at constant velocity of (3i + 4j- 7k)m/s. Find
    (a) Position of Q relative to P at that instant. [10.344m]
    (b) Shortest distance between the particles [10.32m]
    (c) Time that elapses before the particles are nearest to each other [0.0485s]

12. Two particles P and Q move with constant velocities (4i + j – 2k)m/s and (6i + 3k)m/s respectively. Initially P is at a point whose position vector is (i – 20j + 21k)m and Q is at a point whose position vector is (i + 3k)m. find
    (a) Time for which the distance between P and Q is least [2.2s]
    (b) Distance of P from the origin at the time when the distance between P and Q is least [28.8m]
    (c) Least distance between P and Q [24.14m]

Course of closest approach

If A is to pass as close as possible to B, then velocity of A must ne perpendicular to the relative velocity

$V_{AB}.V_A = 0$

Example 27

Two particles P and Q initially at positions (3i + 2j)m and (13i + 2j)m respectively begin moving. Particle P moves with a constant velocity (2i + 6j)m/s. A second particle Q moves with a constant velocity of (5j)m/s

(a) Find
  (i) Time when the particles are closest together.
  (ii) Bearing of particle O from Q when they are closest to each other.

(b) Given that half the time, the particle are moving closest to each other, particle P reduces its speed to half its original speed, in the direction to approach particle Q and the velocity of particle Q remains unchanged, find the direction of particle P.

Solution

$r_{P(t=0)} = \begin{pmatrix}3\\2\end{pmatrix} m, r_{Q(t=0)} = \begin{pmatrix}13\\2\end{pmatrix} m$

$V_{PQ} = V_P - V_Q = \begin{pmatrix}2\\6\end{pmatrix} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} m/s$

$r_{PQ(t=t)} = \left(r_{P(t=0)} - r_{Q(t=0)}\right) + \left(V_{PQ}\right)t$

$r_{PQ(t=t)} = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}13\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} t = \begin{pmatrix}-10+2t\\t\end{pmatrix} m$

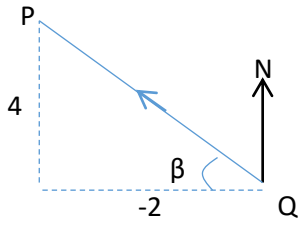
For minimum distance $V_{PQ}.r_{PQ(t=t)}$=0

$\begin{pmatrix}2\\1\end{pmatrix}.\begin{pmatrix}-10+2t\\t\end{pmatrix} = 0$

-20 +4t +t = 0

t = 4s

(ii) $r_{PQ(t=4)} = \begin{pmatrix}-10+2\,x\,4\\4\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix}$



$\beta = \tan^{-1}\frac{4}{2}$=63.43^0

The bearing of P from Q = (270 + 63.43)

=333.43^0

(b) At t = 2s, $V_P = \begin{pmatrix}1\\3\end{pmatrix} m/s$

Let P move at angle θ to x-axis

$V_{PQ} = \sqrt{10}\frac{cos\theta}{sin\theta} - \begin{pmatrix}0\\5\end{pmatrix} = \begin{pmatrix}\sqrt{10}co\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}$

If P is to approach Q; $8sin\theta$

$\begin{pmatrix}\sqrt{10}co\theta\\\sqrt{10}sin\theta - 5\end{pmatrix}.\left(\frac{\sqrt{10}cos\theta}{\sqrt{10}sin\theta}\right)$ = 0

10cos^{2}θ + 10sin^{2}θ -5$\sqrt{10}sin\theta$ =0

θ =$\sin^{-1}\frac{10}{5\sqrt{10}}$ = 39.2^0

Direction: N50.8^{0}E

Example 28

A motor boat B is travelling at a constant velocity of 10m/s due east and motor boat A is travelling at a constant speed of 8m/s. Initially A and B are 600m apart with A due south of B. Find

(a) course that A should set to get close as possible to B

Let A move at an angle θ to x-axis

$V_{AB} = \begin{pmatrix}8cos\theta\\8sin\theta\end{pmatrix} - \begin{pmatrix}10\\0\end{pmatrix} = \begin{pmatrix}8cos\theta - 10\\8sin\theta\end{pmatrix}$

If A is to approach B; $V_{AB}.V_A = 0$

$\begin{pmatrix}8cos\theta\\8sin\theta\end{pmatrix}.\begin{pmatrix}8cos\theta - 10\\8sin\theta\end{pmatrix}$ =0

θ = $\cos^{-1}\frac{64}{80}$ =36.9^0

Direction: N53.1^{0}E or E36.9^{0}N

(ii) Closest distance and time taken for the situation to occur

$r_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} m, r_{B(t=0)} = \begin{pmatrix}0\\600\end{pmatrix} m$

$r_{AB(t=t)} = (r_{A(t=0)} - r_{B(t=0)}) + (V_{AB})t$

$r_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0\\600\end{pmatrix}\right] + \begin{pmatrix}8cos36.9 - 10\\8sin36.9\end{pmatrix} t$

$r_{AB(t=t)} = \begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$

For minimum distance: $V_{AB}.r_{AB(t=t)} = 0$

$\begin{pmatrix}-3.603\\4.803\end{pmatrix}.\begin{pmatrix}-3.603t\\-600 + 4.803t\end{pmatrix}$ =0

t = 80s

$r_{AB(t=80)} = \begin{pmatrix}-3.603\ x\ 80\\-600 + 4.803x80\end{pmatrix}$

$= \begin{pmatrix}-288.24\\-215.76\end{pmatrix}$

Least distance,d = $|r_{AB(t=80)}|$

$|r_{AB(t=80)}| = \sqrt{(-288.24)^2 + (-215.76)^2}$

d = 360m

Example 29

A motor boat B is travelling at constant velocity of 14km/h due north and a motor boat A is travelling at constant speed 12km/h. Initially A and B are 5.2km apart with A due west of B. Find

(i) Couse that A should set in order to get as close as possible to B

Let A move at an angle θ to x-axis

$V_{AB} = \begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix} - \begin{pmatrix}0\\13\end{pmatrix} = \begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}$

If A is to approach B; $V_{AB}.V_A = 0$

$\begin{pmatrix}12cos\theta\\12sin\theta - 13\end{pmatrix}.\begin{pmatrix}12cos\theta\\12sin\theta\end{pmatrix}$=0

θ = $\sin^{-1}\frac{12}{13}$ =67.4^0

Direction: N22.6^0E or N67.4^0N

(ii) Closest distance and time taken for the situation to occur

$r_{A(t=0)} = \begin{pmatrix}0\\0\end{pmatrix} m, r_{B(t=0)} = \begin{pmatrix}5.2\\0\end{pmatrix} m$

$r_{AB(t=t)} = (r_{A(t=0)} - r_{B(t=0)}) + (V_{AB})t$

$r_{AB(t=t)} = \left[\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}5.2\\0\end{pmatrix}\right] + \begin{pmatrix}12cos67.4\\12sin67.4 - 13\end{pmatrix} t$

$r_{AB(t=t)} = \begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$

For minimum distance: $V_{AB}.r_{AB(t=t)} = 0$

$\begin{pmatrix}4.612\\-1.921\end{pmatrix}.\begin{pmatrix}-5.2 + 4.612t\\-1.921t\end{pmatrix}$ =0

t = 0.961h

$r_{AB(t=0.961)} = \begin{pmatrix}-5.2 + 4.612\ x\ 0.961\\-1.921\ x\ 0.961\end{pmatrix}$

$= \begin{pmatrix}-0.7725\\-1.8442\end{pmatrix}$

Least distance, d = $|r_{AB(t=0.961)}|$

$|r_{AB(t=0.961)}| = \sqrt{(-0.7725)^2 + (-1.8442)^2}$

d = 2km

Revision exercise 4

1. A ship A is moving with constant velocity of 18km/h in a direction N55^0E and is initially 6km from a second ship B, the bearing of A from B being N25^0W. If B moves with a constant speed of 15km/h. Find

(a) Course that B should set in order to get as close as possible to A [N21.4^{0}E]
(b) Closest distance and time taken for the situation to occur.[4.135km, t=0.437h]

2. Two aircraft A and B are flying at the same altitude with A initially 10km due north of B and flying at constant speed of 300m/s on a bearing of 060^0. If B flies at constant speed of 200m/s, find
   (a) Course that B should set in order to get as close as possible to A [E78.4^{0}N]
   (b) Closest distance and time taken for the situation to occur.[9.79km, t= 9.12s]
3. At 8am two boats A and B are 5.2km apart with A due west of B, and B travelling due north at a steady speed 13km/h. If A travels with a constant speed of 12km/h, show that for A to get as close as possible to B, A should set a course of Nθ0E where sinθ = $\frac{5}{13}$. Find the closest distance and time at which it occurs. [2km, 8.57 am]
4. Two aircraft A and B are flying at the same altitude with A initially 5km due north of B and B flying at constant speed of 300m/s on bearing of 060^0.If A flies at constant speed of 200m/s, find
   (a) Course that A should set in order to get as close as possible to B [108.2^0]
   (b) Time taken for the situation to occur.[5.4min]

5. A ship A moving with a constant speed of 24km/h in the direction N40^{0}E and is initially 10km from a second ship B, the bearing of A from B being N300W. If B moves with a constant speed of 22km/h; find
   (a) Course that B should set in order to get as close as possible to A [N16.4^{0}E]
   (b) Closest distance and time taken for the situation to occur.[6.89km, 45min]

Interception and collision

Consider two bodies A and B moving with V_A and V_B from points with position vectors OA and OB

Position of A after time t is

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

Position of B after time t is

$r_{B(t=t)} = r_{B(t=0)} + t\ x\ V_B$

For collision to occur $r_{A(t=t)} = r_{B(t=t)}$

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\left(r_{A(t=0)} - r_{B(t=0)}\right) + t(V_{AB})$=0

Hence $r_{AB(t=t)} = 0$

Example 30

The position vectors $r_A = (5i - 3j + 4k)m$ and $r_B = (7i + 5j - 2k)m$ are for two particles with velocities $V_A = (2i + 5j + 3k)m/s$ and $V_B = (-3i - 55j + 18k)m/s$ respectively. Show that if the velocities remain constant, a collision will occur

Solution

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$$\begin{pmatrix}5\\-3\\4\end{pmatrix} + t\begin{pmatrix}2\\5\\3\end{pmatrix} = \begin{pmatrix}7\\5\\-2\end{pmatrix} + t\begin{pmatrix}-3\\-15\\18\end{pmatrix}$$

$$\begin{pmatrix}-2\\-8\\6\end{pmatrix} = \begin{pmatrix}-5\\-20\\15\end{pmatrix}t$$

Along i direction: -2 = -5t; t = 0.4s

Along j direction: -8= -20t; t = 0.4s

Along k direction: 6 = 15t; t = 0.4s

Since t is the same in all directions, collision occurred

Example 31

At 12 noon the position vectors r and velocity vectors V of two ships A and B are as follows

$r_A = (i + 7j)m, V_A = (6i + 2j)m/s,\ r_B = (6i + 4j)m\ and\ V_B = (-4i + 8j)m/s$

Assuming velocities do not change

(i) Show that collision will occur
(ii) Find the time at which collision occurs
(iii) Find the position vector of the location during collision

Solution

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\begin{pmatrix}1\\7\end{pmatrix} + t\begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}6\\4\end{pmatrix} + t\begin{pmatrix}-4\\8\end{pmatrix}$

$\begin{pmatrix}-5\\3\end{pmatrix} = \begin{pmatrix}-10\\6\end{pmatrix}t$

Along the i direction: -5 = -10t; t = 0.5h

Along the j direction: 3 = 6t; t = 0.5h

Since t is the same in all directions

Collision occurred

(ii) time it occurred = 12:00 + 0.5 x 60

= 12:30pm

(iii) How far each had travelled

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

$r_{A(t=0.5)} = \begin{pmatrix}1\\7\end{pmatrix} + 0.5\begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}km$

Example 32

At 11:30am a battle ship is at a place with position vector (-6i + 12j)km and is moving with velocity vector (16i – 4j)km/h. At 12:00 noon a cruiser is at a place with position vector (12i – 15j) and is moving with velocity vector (8i + 16j)km/h. Assuming velocities do not change

(i) Show that collision will occur
(ii) Find the time at which collision occurs
(iii) Find the position vector of the location of collision

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

At 12:00: $r_{A(t=0.5)} = \begin{pmatrix}-6\\12\end{pmatrix} + 0.5\begin{pmatrix}16\\-4\end{pmatrix} = \begin{pmatrix}2\\10\end{pmatrix}$

For collision to occur

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\begin{pmatrix}2\\10\end{pmatrix} + t\begin{pmatrix}16\\-4\end{pmatrix} = \begin{pmatrix}12\\-15\end{pmatrix} + t\begin{pmatrix}8\\18\end{pmatrix}$

$\begin{pmatrix}-10\\25\end{pmatrix} = \begin{pmatrix}-8\\20\end{pmatrix}t$

Along the i direction: -10 = -8t; t = 1.25h

Along the j direction: 15 = 20t; t = 1.25h

Since t is the same in all directions collision occurred

(ii) time it occurred = 11:30 + 1.25 x 60

= 12:45pm

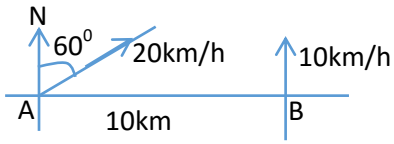
(iii) How far each had travelled

$r_{A(t=t)} = r_{A(t=0)} + t\ x\ V_A$

$r_{A(t=0.5)} = \begin{pmatrix}2\\10\end{pmatrix} + 1.25\begin{pmatrix}16\\-4\end{pmatrix} = \begin{pmatrix}22\\5\end{pmatrix}km$

Example 33

At 12:30 noon two ships A and B are 10km apart with B due east of A. A is travelling N600E at a speed of 12km/h and ship B is travelling due north at 10km/h. Show that, is the two ships do not change their velocities, they collide and fins to the nearest minute when collision occurs.



For collision to occur

$r_{A(t=0)} + t\ x\ V_A = r_{B(t=0)} + t\ x\ V_B$

$\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}20sin60\\20cos60\end{pmatrix} = \begin{pmatrix}10\\0\end{pmatrix} + t\begin{pmatrix}0\\10\end{pmatrix}$

$\begin{pmatrix}10\sqrt{3}\\10\end{pmatrix}t = \begin{pmatrix}10\\0\end{pmatrix} + t\begin{pmatrix}0\\10\end{pmatrix}$

Along the i direction: $10\sqrt{3}t$ = 10;

t = 0.5774h

Along the j direction: 10t = 10t

∴ t =0.5774 x 60 = 35minute

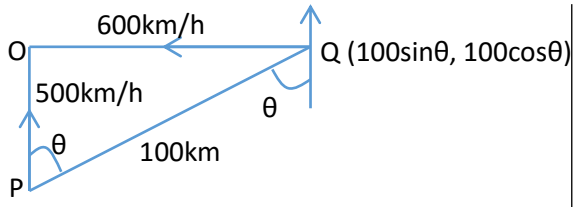
Collision occurred at 12:35minutes

Example 34

Two aircraft P and Q are flying at the same height. P is flying due north at 500km/h while Q is flying due west at 600km/h. When the aircrafts are 10km apart, the pilots realize that they are about to collide. The pilot of P changes Course to 345^0 and maintains the speed of 500km/h. The pilot Q maintains his course but increases speed. Determine the

(i) Distance each aircraft would have travelled if the pilots had not realized that they were about to collide

Solution



θ = $\tan^{-1}\frac{600}{500}$ = 50.2^0

For collision to occur

$r_{P(t=0)} + t\ x\ V_P = r_{Q(t=0)} + t\ x\ V_Q$

$\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}0\\500\end{pmatrix} = \begin{pmatrix}100sin\theta\\100cos\theta\end{pmatrix} + t\begin{pmatrix}-600\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}0\\500\end{pmatrix} = \begin{pmatrix}100sin50.2\\100cos50.2\end{pmatrix} + t\begin{pmatrix}-600\\0\end{pmatrix}$

Along i direction: 0 = 100sin50.2 – 600t

t = 0.128h

Distance moved by P

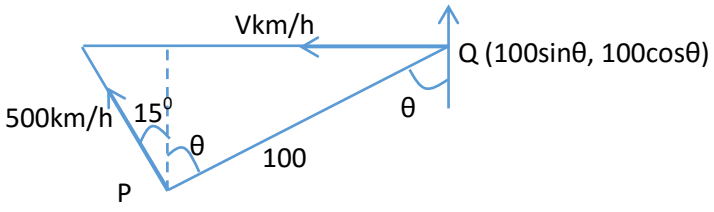
$\begin{pmatrix}0\\0\end{pmatrix} + 0.128\begin{pmatrix}0\\500\end{pmatrix} = \begin{pmatrix}0\\64\end{pmatrix} = 64km$

Distance moved by Q

$\begin{pmatrix}100sin\theta\\100cos\theta\end{pmatrix} + 0.128\begin{pmatrix}-600\\0\end{pmatrix} = \begin{pmatrix}0.0284\\64.011\end{pmatrix}$

=64.011km

(ii) New speed beyond which the aircraft Q must fly in order to avoid collision



For collision to occur

$r_{P(t=0)} + t\ x\ V_P = r_{Q(t=0)} + t\ x\ V_Q$

$\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}-500\sin 15\\500cos15\end{pmatrix} = \begin{pmatrix}100sin\theta\\100cos\theta\end{pmatrix} + t\begin{pmatrix}-V\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}-500\sin 15\\500cos15\end{pmatrix} = \begin{pmatrix}100sin50.2\\100cos50.2\end{pmatrix} + t\begin{pmatrix}-V\\0\end{pmatrix}$

Along j direction: 500cos15t = 64.011

t = 0.1325h

Along i direction:

-500sin15t = 76.8284 – Vt

V x 0.1325 = 76.8284 + 500sin15 x 0.1325

V =709.2837km/h

Course of interception

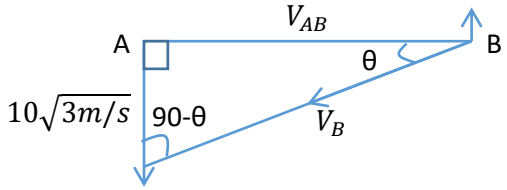
Suppose particle A with speed V_A is to intercept particle B with speed V_B, then

- Draw a sketch diagram showing the initial position and velocities of the two particles
- For interception to occur, the relative velocity must be in the direction of the initial displacement of the particles.

Example 35

At an instant a body A travelling south at $10\sqrt{3}$m/s is 150m west of B. Show that B will intercept A if B is travelling S30^{0}W at 20m/s and find the time that elapses before collision occurs.

Solution



$A = \begin{pmatrix}0\\0\end{pmatrix}m \qquad B = \begin{pmatrix}150\\0\end{pmatrix}m$

$V_A = \begin{pmatrix}0\\-10\sqrt{3}\end{pmatrix}m/s \qquad V_B = \begin{pmatrix}-20cos\theta\\-20sin\theta\end{pmatrix}m/s$

OA + t x v_A = OB + t x v_B

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\-10\sqrt{3}\end{pmatrix}t = \begin{pmatrix}150\\0\end{pmatrix} + \begin{pmatrix}-20cos\theta\\-20sin\theta\end{pmatrix}t$

j: 20sinθ =$10\sqrt{3}$

θ = $\sin^{-1}\frac{\sqrt{3}}{2}$ = 60^0

Bearing S30^{0}W

i: t = $\frac{150}{20cos\theta} = \frac{150}{20\cos 60} = 15s$

Alternatively

$\frac{sin90}{20} = \frac{sin\theta}{10\sqrt{3}}$

θ = 60^0: bearing S30^{0}W

Also, $\frac{sin90}{20} = \frac{\sin(90-\theta)}{V_{AB}}$

$V_{AB} = 10m/s$

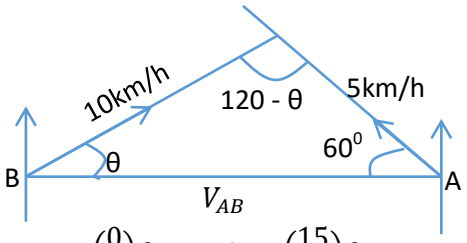
t = $\frac{AB}{VA_{AB}} = \frac{150}{10} = 15s$

Example 36

At 9:00am two ships A and B are 15km apart with B on a bearing of 270^0 from A. Ship A moves at 5km/h on a bearing of 330^0. If the maximum speed of B is 10km/h. Find the

(i) Direction B should set in order to intercept A as soon as possible
(ii) Time taken for the interception to occur.

Solution



$B = \begin{pmatrix}0\\0\end{pmatrix} km \quad A = \begin{pmatrix}15\\0\end{pmatrix} km$

$V_B = \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix} km/h \quad V_A = \begin{pmatrix}-5cos60\\5sin60\end{pmatrix} km/h$

OB + t x v_B = OA + t x v_A

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}10cos\theta\\10sin\theta\end{pmatrix} t = \begin{pmatrix}15\\0\end{pmatrix} + \begin{pmatrix}-5cos60\\5sin60\end{pmatrix} t$

j: 10sinθ =$5sin60$

θ = $\sin^{-1}\frac{5sin60}{10}$ = 25.7^0

Bearing E25.7^0N

i: t = $\frac{15}{10cos\theta+2.5} = \frac{15}{2.5+10\cos 25.7} = 1.303h$

t = 1.303 x 60 = 78mins

Alternatively

$\frac{sin60}{10} = \frac{sin\theta}{5}$

θ = 25.7^0: bearing E25.7^0N

Also, $\frac{sin90}{10} = \frac{\sin(120-\theta)}{V_{AB}}$

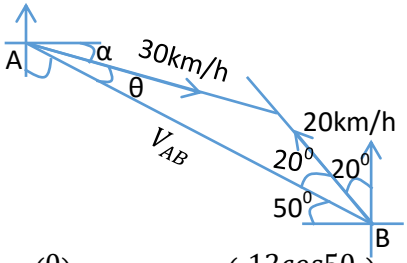
$V_{AB} = 11.515km/h$

t = $\frac{AB}{VA_{AB}} = \frac{15}{11.515}\ x\ 60 = 78minutes$

Example 37

At 12:00 noon two ship A and B are 12km apart with B on a bearing of 140^0 from A. Ship A moves at 30km/h to intercept B which is travelling at 20km/h on a bearing of 340^0. Find the

(i) Direction A should set in order to intercept B (ii) time taken for the interception to occur.



$A = \begin{pmatrix}0\\0\end{pmatrix} km \quad B = \begin{pmatrix}12cos50\\-12sin50\end{pmatrix} km$

$V_A = \begin{pmatrix}30cos\alpha\\30sin\alpha\end{pmatrix} km/h \quad V_B = \begin{pmatrix}-20cos20\\20sin20\end{pmatrix} km/h$

OA + t x v_A = OB + t x v_B

$\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}30cos\alpha\\30sin\alpha\end{pmatrix} t = \begin{pmatrix}12cos50\\-12sin50\end{pmatrix} + \begin{pmatrix}-20cos20\\20sin20\end{pmatrix} t$

i: $30tcos\alpha = 12cos50 + -20tcos20$

$t = \frac{7.713}{30cos\alpha+6.84}$ ..... (i)

J: $30tsin\alpha = -12sin50 + 20tsin20$

$t = \frac{9.193}{30sin\alpha+18.794}$ ..... (ii)

(i) = (ii): $\frac{7.713}{30cos\alpha+6.84} = \frac{9.193}{30sin\alpha+18.794}$

$231.39sin\alpha - 275.79cos\alpha = 82.078$

But sinα= $\frac{2T}{1+T^2}$ and cosα= $\frac{1-T^2}{1+T^2}$

$231.39\left(\frac{2T}{1+T^2}\right) - 275.79\left(c\frac{1-T^2}{1+T^2}\right) = 82.078$

$357.868T^2 + 462.78T - 193.712 = 0$

T = -1.626 or T = 0.333

∴ T = 0.333

sinα= $\frac{2T}{1+T^2}$

α =$\sin^{-1}\frac{2\ x\ 0.333}{1+0.333^0}$ =36.8^0

Bearing: E36.8^0S

$t = \frac{7.713}{30cos\alpha+6.84} = \frac{7.713}{30cos36.8+6.84}$

T = 0.25h=0.25x 60 =15minutes

Alternatively

$\frac{sin20}{30} = \frac{sin\theta}{20}$

θ = 13.2^0:

Bearing (50 – 13,2)^{0}S

E36.8^{0}S or S53.2^{0}E

Also, $\frac{sin20}{30} = \frac{\sin(180-(20+13.2))}{V_{AB}}$

$V_{AB} = 48.03km/h$

t = $\frac{AB}{VA_{AB}} = \frac{12}{48.03}\ x\ 60 = 15minutes$

Revision exercise 5

1. At 12:00 noon two ships A and B are 12km apart with B on a bearing of 250^0 from A. Ship A moves at 4km/h on a bearing of 320^0. If the maximum speed of B is 7km/h, find the
   (a) Direction B should set in order to intercept A [N37.6^{0}E]
   (b) Time taken for interception to occur. [99minutes]
2. Initially two particles A and B are 48m apart with B due north of A. A has a constant velocity of (5i + 4j)m/s and B a constant speed of 13m/s. Find the velocity of Y if it is to intercept A and find the time taken to do so [5i – 12j)m/s, 3s]
3. At 12 noon the position vectors, r and velocity vectors, V of two ships are
   $r_A = \begin{pmatrix}5\\2\end{pmatrix} km, V_A = \begin{pmatrix}15\\10\end{pmatrix} km/h\ and\ r_B = \begin{pmatrix}7\\7\end{pmatrix} km, V_A = \begin{pmatrix}9\\-5\end{pmatrix} km/h$
   Show that if the ships do not alter their velocities, a collision will occur and find the time at which it occurs and the position vector of its location [12.30pm, (10i + $\frac{16}{3}j$)km]
4. At 11.30 a jumbo jet has position vectors (-100i + 220j)km and it is moving with velocity vectors (300i + 400J)km/h. At 11:45am a cargo plane has a position vectors (-60i + 355j)km and is moving with velocity vectors (400i + 300j)km/h. Assuming velocities do not change
   (a) Show that the planes will crash
   (b) Find the time of the crash. [12.06pm]
   (c) Find the position vector of the crash [(80i + 460j)km]
5. At 2pm the position vectors, r and velocity vectors, V of three ships are as follows
   $r_A = (5i + j)km \quad V_A = (9i + 18j)km/h$
   $r_B = (12i + 5j)km \quad V_A = (-12i + 6j)km/h$
   $r_C = (13i - 3j)km \quad V_A = (9i + 12j)km/h$
   Assuming velocities do not change
   (a) Show that Ship A and B will collide and fins when and where collision occur. [2.20pm, (8i + 7j)km]
   (b) Find the position vector of C when A and B collide and find how far C is from the point of collision. [(16i + j)km, 10km]
   (c) When the collision occurs, C immediately changes its course but not its speed and streams directly to the scene. When does C arrive[3.00pm]
6. At 12 noon the position vectors, r and velocity vectors, V of three ships A, B and C are as follows
   $r_A = (10.5i + 6j)km \quad V_A = (9i + 18j)km/h$
   $r_B = (7i + 20j)km \quad V_A = (12i + 6j)km/h$
   $r_C = (10i + 15j)km \quad V_A = (6i + 12j)km/h$
   Assuming velocities do not change
   (a) Show that Ship A and B will collide and fins when and where collision occur. [1:10pm, (21i + 27j)km]
   (b) When the collision occurs, C immediately changes its course but not its speed and streams directly to the scene. When does C arrive[1:300pm]

7. In gulf water, a battleship streaming at 16km/h is5km southwest of a submarine. Find the course which the submarine should set in order to intercept the battle ship, if its speed is 12km/h. [N15^{0}W]
8. A boy hits a ball at 15m/s in a direction S80^{0}W. A girl 45m and S65^{0}W from the boy run at 6m/s to intercept the ball. Find in what direction the girl must rum to intercept the ball as quickly as possible and how long does it take her. [N24.7^{0}E, 2.35s]
9. A helicopter sets off from its bas and flies at 50m/s to intercept a ship which, when the helicopter sets off, is at a distance of 5km on a bearing 335^0 from the base. The ship is travelling at 10m/s on a bearing 095^0. Find the course that the helicopter pilot should set if he is to intercept the ship as quickly as possible and the time interval between the helicopter taking off and its reaching the ship. [N15^{0}W, 92.2s]
10. A life boat sets out a harbour at 9:10pm to go for assistance of a yatch which is, at the time, 5km due north of the harbour and drifting due west at 8km/h. If the life boat travels at 20km/h find:
    (a) Course the life boat should set so as to reach the yatch as quickly as possible [S23.6^{0}W]
    (b) Time when the boat arrives [9:27pm]
11. A coast guard vessel wishes to intercept a yatch suspected of smuggling. At 1am the yatch is 10km due east of the coast guard vessel and travelling due north at 15km/h. If the coast guard vessel travels at 20km/h,
    (a) In which direction should it steer in order to intercept the yatch? [N41.4^{0}E]
    (b) When would this interception occur. [1:45am]
12. The driver of a speed boat travelling at 75km/h wishes to intercept a yatch travelling at 20km/h in a direction N40^{0}E. Initially the speed boat is 10km from the yatch on a bearing S30^{0}E. Find
    (a) Course the speed boat should set so as to reach the yatch as quickly as possible. [N15.5^{0}W]
    (b) Time when the interception occurs[9minutes and 7 seconds]
13. A jet fighter travelling at 30km/h wishes to intercept a plane travelling at 20km/h in a course of 200^0. Initially the plane is 40km away on a bearing of 11^0 from the jet fighter. Find
    (a) Course the jet fighter should set so as to reach the plane as quickly as possible. [S5^{0}E]
    (b) Time taken for interception to occur. [48minutes and 24 seconds]
14. A batman hits a ball at 15m/s in a direction S80^{0}W. A fielder, 45m and S65^{0}W from the batsman, runs at 6m/s to intercept the ball. Assuming the velocities remain unchanged,
    (a) Find what direction the fielder must take to intercept the ball as quickly as possible. [N24.7^{0}E]
    (b) How long did it take him. [2.4s}

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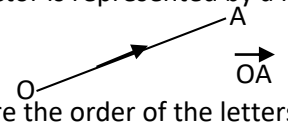
Vectors

A vector is a quantity which has both magnitude and direction. Examples include, force, displacement, acceleration, momentum and velocity.

Representation of a vector

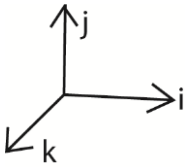
A vector is represented by a line with an arrow to indicate the direction of the vector.



where the order of the letters shows the direction

Vectors in dimensions

Vectors can be represented in three dimensions as I, j and k along the x, y and z- axes respectively



Resultant of vectors

When several vectors (V_1, V_2, V_3… V_n) are acting on a point object, the net vector R, is calculated as the vector sum

R = $V_1 + V_2 + V_3 + \ldots + Vn = \sum_{r=1}^{r=N} V_r$

Example 1

Find the resultant of the following vectors

(a) (2i + 3j + 3k) and (2i + 4j – 8k)

$$R = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -5 \end{pmatrix} or\ 4i + 7j - 5k$$

(b) (7i – 4j + 3k), (5i -2j + 8k), (i –k)

$$R = \begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 13 \\ -6 \\ 10 \end{pmatrix} or\ 13i - 6j + 10k$$

Example 2

The resultant of (5i -2j), (7i + 4j), (ai + bj) and (-3i + 2j) is (5i + 5j). Find the values of a and b.

$$R = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9 + a \\ 4 + b \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

9 + a = 5; a = -4 also 4 + b = 5; b = 1

Example 3

The resultant of the forces (3i + (a-c)j)N, ((2a + 3c)I + 5j)N and (4i, 6j)N acting on a particle is (10i + 12j)N. find

(i) Values of a and c

$$R = \begin{pmatrix} 3 \\ a - c \end{pmatrix} + \begin{pmatrix} 2a + 3c \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$$

2a + 3c + 7 = 10

2a + 3c = 3 ..... (i)

a –c +11 = 12

a – c = 1 ..... (ii)

(i) + 3(ii)

5a = 6; a = 1.2

from eqn. (ii)

c = 0.2

(ii) magnitude of (2a + 3c)I + 5j

R = (2a + 3c)I + 5j = (1.2 x 2 + 3 x 0.2)i + 5j = 3i + 5j

$|R| = \sqrt{3^2 + 5^2} = 5.831N$

Magnitude or modulus of a vector

This is the length of a vector

(i) Given R = xi + yj; $|R| = \sqrt{x^2 + y^2}$

(ii) Given R = xi + yj + zk; $|R| = \sqrt{x^2 + y^2 + z^2}$

Example 5

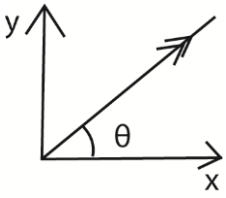
Find the magnitude of the following vectors

(a) 3i + 4j; $|R| = \sqrt{3^2 + 4^2}$ =5

(b) 3i + 2j – 6k; $|R| = \sqrt{3^2 + 2^2(-6)^2} = 7$

Direction of a vector

Consider R = xi + yj



$$\theta = \tan^{-1}\left(\frac{x}{y}\right)$$

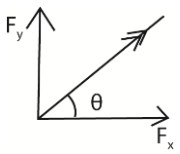
Example 6

Find the magnitude and direction of the resultant of each of the following

(a) (2i + 3j)N, (5t – 2j)N, (-3i, 3j)

$R = \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}5\\-2\end{pmatrix} + \begin{pmatrix}-3\\3\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$

$|R| = \sqrt{4^2 + 4^2}$ = 5.6569N

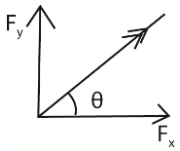


$\theta = \tan^{-1}\left(\frac{4}{4}\right)$= 45^0

(b) $\begin{pmatrix}2\\4\end{pmatrix}N, \begin{pmatrix}-6\\-5\end{pmatrix} Nand \begin{pmatrix}2\\1\end{pmatrix}N$

$R = \begin{pmatrix}2\\4\end{pmatrix}+, \begin{pmatrix}-6\\-5\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-2\\0\end{pmatrix}$

$|R| = \sqrt{(-2)^2 + 0^2}$ = 2N



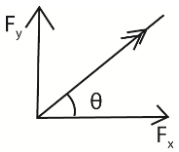
$\theta = \tan^{-1}\left(\frac{0}{-2}\right)$= 180^0

Example 7

Four forces of ai + (a-1)j, 3i + 2aj, 5i -6j, and –i -2j act on a particle. The resultant forces make an angle of 45^0 with horizontal. Find a. Hence determine the magnitude of the resultant force.

$R = \begin{pmatrix}a\\a-1\end{pmatrix} + \begin{pmatrix}3\\2a\end{pmatrix} + \begin{pmatrix}5\\-6\end{pmatrix} + \begin{pmatrix}-1\\-2\end{pmatrix}$

$= \begin{pmatrix}a+7\\3a-9\end{pmatrix}$



$\frac{a+7}{3a-9} = \tan^{-1}(45^0)$ = 1

a + 7 = 3a – 9, a = 8

$R = \begin{pmatrix}a+7\\3a-9\end{pmatrix} = \begin{pmatrix}8+7\\3\ x\ 8-9\end{pmatrix} = \begin{pmatrix}15\\15\end{pmatrix}$

$|R| = \sqrt{15^2 + 15^2}$ = 21.21N

Unit vector

This a vector whose magnitude is unit (1)

Unit vector of r denoted by r = $\frac{r}{|r|}$

Example 8

Find the unit vector of a = 6i -2j + 3k

Solution

a = $\frac{6\text{i} - 2\text{j} + 3\text{k}}{\sqrt{6^2 + (-2)^2 + 3^2}} = \frac{1}{7}(6\text{i} - 2\text{j} + 3\text{k})$

Parallel vectors

If vectors a and b are parallel, then one of them is a scalar multiple of the other.

If a vector r of magnitude $|r|$ moves in direction xi + yj + zk then, r =$|r|\left(\frac{\text{xi} + \text{yj} + \text{zk}}{\sqrt{x^2 + y^2 + z^2}}\right)$

Example 9

Find the vector, V which has a magnitude of 15 units and is parallel to 16i + 12j

V = 15 x $\frac{16\text{i} + 12\text{j}}{\sqrt{16^2 + 12^2}} = 15 \text{ x } \frac{16\text{i} + 12\text{j}}{20} = 12i + 9j$

Example 10

A body of velocity v and of magnitude 20m/s moves in the direction 6i + 8j. Find V.

V = 20 x $\frac{6i + 8j}{\sqrt{6^2+8^2}} = 20 \text{ x } \frac{6i + 8j}{10} = 12i + 16j$

Example 11

A force of magnitude 12N acts on a body in the direction 2i + j + 2k. Find the force

V = 12 x $\frac{2i + j + 2k}{\sqrt{2^2+1^2+2^2}} = 12x \frac{2i + j + 2k}{3} = 8i + 4j + 8k$

Example 12

The force A of magnitude 5N in the direction with unit vector $\frac{1}{5}(3i + 4j)$ and force B of magnitude 13N in the direction with unit vector$\frac{1}{13}(5i - 12j)$. Find the resultant forces of A and B.

FA = $\frac{1}{5}(3i + 4j)x5 = 3i + 4j$

FB = $\frac{1}{13}(5i - 12j)\ x\ 13 = 5i - 12j$

F = $\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}5\\-12\end{pmatrix} = \begin{pmatrix}8\\-8\end{pmatrix}$

$|F| = \sqrt{8^2 + (-8)^2}$ = 11.3137N

Example 13

A particle P moves through a displacement of 2m when acted o by two forces F_1 and F_2. Find the work done by the resultant force, if F_1 = i – j and F_2 = 10 N and acts in the direction 4i + 3j

Solution

F_1 = i – j

$F_2 = 10\ x\ \frac{4i + 3j}{\sqrt{4^2+3^2}}$ = 8i + 6j

F = $\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}8\\6\end{pmatrix} = \begin{pmatrix}9\\5\end{pmatrix}$

$|F| = \sqrt{9^2 + 5^2}$ = 10.2956N

W = $|F|\ x\ d$ = 10.2956 x 2 = 20.5912J

Revision exercise 1

1. Find the resultant of each of the following forces
   (a) (6i + 2j)N, (-5i + j)N, (3i – 3j)N. [(4i)N]
   (b) (2i + 4j)N, (3i -5j), (6i +2j)N, (-7i – 7j)N. [(4i -6j)N]
   (c) (2i + 3j -7k)N, (2i + 5k)N, (3j + 4k)N. [(4i + 6j + 2k)N]
2. The resultant of forces (5i + 7j), (ai + bj) and (bi –aj)N is a force (11i + 5j)N. Find a and b. [a = 4, b = 2]
3. Find the magnitude and direction of the resultant of each of the following;
   (a) (-2i + 5j)N, (I + 2j)N. [7.07N at 98.1^0]
   (b) (6i + 2j)N, (4i – 3j)N. [10.05N at 354.3^0]
   (c) (3i + 2j), (-5i + j)N. [3.61N at 124^0]
4. A force of magnitude 50N acts on a body in the direction 24i + 7j. Find the force. [(48i + 14j)]
5. Two forces F_1 and F_2 have magnitude αN and βN and act in the direction i -2j and 4i + 3j respectively. Given that the resultant of F_1 and F_2 is (48i + 14j). Find the magnitude of αN and βN. [α = $8\sqrt{5}$N and β = 50N]
6. If a = 3i + 4j, b = 4i + 20j and c = 5i – 19j; find the
   (i) resultant of a and b [(7i + 24j)]
   (ii) resultant of a and c [(8i – 15j)]
   (iii) vector is parallel to a and has magnitude of 15 unit [(9i + 12j)]

(iv) vector parallel to (a + b) and has a magnitude of 100 units [(28i + 96j]

7. If a = 2i + 5j, b = -7i + 7j and 14i. Find the;
   (i) resultant of a and b [(-5i + 12j)]
   (ii) resultant of a, b and c [(9i + 12j)]
   (iii) |b| [$7\sqrt{2}$]
   (iv) |a + b + c| [15units]
   (v) vector is parallel to a and has a magnitude of $5\sqrt{29}$ units. (10i + 25j)
   (vi) Vector is parallel to (a + b + c) and has magnitude 90 units. [(54i + 72j)]
8. If a = i -3j + 2k, b = 5i + 4j and c = 3i + j + 4k. Find the
   (i) resultant of a and b [(6i + j + 2k)]
   (ii) resultant of a, b and c.[(9i + 2j + 6k)
   (iii) |a| [$\sqrt{14}$]
   (iv) |a + b + c|[11units]
   (v) Vector parallel to (a + b + c) and has magnitude 5 units [$\frac{5}{11}(9i + 2j + 6k)$]
9. If a = 2i + 7j + 7k, b = 6i -3j + 2k and c = -4j -3k. find the
   (i) resultant a and b [8i +4jj + 9k]
   (ii) resultant a and c [2i + 3j + 4k
   (iii) |b| [7units]
   (iv) |a + b + c| [10 units]
   (v) vector is parallel to |a + b + c| and has magnitude of 50 units [40i + 30k]

Scalar products or dot products

The dot product of two vectors a and b inclined at angle θ is given by

$$\boldsymbol{a.b = |a||b|\, cos\theta}$$

Note

If two vectors are perpendicular then the angle between them is 90^0 and

$$\boldsymbol{a.b = |a||b|\, cos\, 90 = 0}$$

Example 14

If a = i – 2k and b = 3i – 3j + k, find

(i) a.b (ii) the angle between a and b

Solution

(i) $a.b = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = 3 + 0 + -2 = 1$

(ii) $\theta = \cos^{-1}\left(\frac{a.b}{|a||b|}\right) = \cos^{-1}\frac{1}{\sqrt{1^2+(2)^2}\sqrt{3^2+(-3)^2+1^2}} = 84.1^0$

Example 15

If p = 2i – j + 3k and q = i + 4j + 3k; find the angle between p and q.

Solution

p.q = $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ = 2 + -4 + 9 = 7

$\theta = \cos^{-1}\left(\frac{p.q}{|p||q|}\right) = \cos^{-1}\frac{7}{\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+4^2+3^2}}$ = 68^0

Example 16

If the angle between two vectors a = xi + 2j and b = 3i + j is 45^0.Find the two possible values of constant x.

Solution

$\begin{pmatrix} x \\ 2 \end{pmatrix}.\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \sqrt{x^2+2^2}.\sqrt{3^2+1^2}$

$3x + 2 = \sqrt{x^2+2^2}.\sqrt{10}.\frac{\sqrt{2}}{2}$

$(3x + 2)^2 = (x^2 + 4).\ 10 \times \frac{2}{4}$

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

$x = -4$ and $x = 1$

Example 17

If p = 2αi + 7j – k and q = 3αi + αj + 3k.Find the value of the scalar α if the vectors are perpendicular

Solution

$\begin{pmatrix} 2\alpha \\ 7 \\ -1 \end{pmatrix}.\begin{pmatrix} 3\alpha \\ \alpha \\ 3 \end{pmatrix} = 0$

$6\alpha^2 + 7\alpha - 3 = 0$

$\alpha = \frac{1}{3}$ and $\alpha = \frac{3}{2}$

Revision exercise 2

1. Find the scalar products for each of the following pairs of vectors.
   (i) a = 2i + j, b = i - 3j [-1]
   (ii) a = 3i, b = -2i + j [-6]
   (iii) a = 5i + j - 2k, b= 4i + 3j – 8k [39]
   (iv) 2i + 4j – 15k and -8i + 2j – k [7]
   (v) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ [4]
   (vi) $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ [6]
   (vii) $\begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ [8]
   (vi) $\begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ [2]

2. Find the angles between each of the following pairs of vectors
   (i) 3i + 4j and 5i – 12j [121^0]
   (ii) 3i and -2j [90^0]
   (iii) 2i + 3j – 6k and 2i + j + 2k [10^0]
   (iv) i + 2j – k and –I + 2j – k [48^0]
   (v) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ [82^0]
   (vi) $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ [92^0]

(v) (vii) $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} and \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ [120^0] (viii) $\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} and \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ [73^0]

3. If a = αi + 2j – k and b = 5i –αj + k. Find the value of the scalar α if the vectors are perpendicular [$\frac{1}{3}$]
4. If a = 2i + αj and b = -α – k. Find the value of the scalar α if the vectors are perpendicular [0]
5. If a = 4i + 5j and b = qi – 8j. Find the value of scalar q if the vectors are perpendicular. [10]
6. If a = 6i – j and b = 2i + pk. Find the value of scalar p if the vectors are perpendicular [12]
7. Given $\begin{pmatrix} q \\ 2+q \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 4-q \end{pmatrix}$ are perpendicular vectors. Find the value of q. [18]
8. If a = qi + 8j +(3q + 1)k and b = (q+1)I + (q-1)j – 2k. Find the value of the possible values of constant q if the vectors are perpendicular. [2 or -2]

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Vertical motion under gravity

When a body is projected **vertically downwards**, it is subjected to an acceleration of $9.8ms^{-2}$. i.e. $a = g = 9.8ms^{-2}$

Equations of motion become

$v = u + gt$; **$h = ut + \frac{1}{2}gt^2$;** **$v^2 = u^2 + 2gh$**

When a body is projected **vertically upwards**, it is subjected to a retardation of $9.8ms^{-2}$. i.e. $a = g = 9.8ms^{-2}$

Equations of motion become

$v = u - gt$; **$h = ut - \frac{1}{2}gt^2$;** **$v^2 = u^2 - 2gh$**

Maximum /greatest height

When a particle is projected vertically upwards, the final velocity is 0ms-1 at its maximum height

$v^2 = u^2 - 2gh$

$0 = u^2 - 2gh_{max}$ | $h_{max} = \frac{u^2}{2g}$

Time to reach maximum height

$v = u - gt$

$0 = u - gt$ | $t = \frac{u}{g}$

Time of flight

$T = \frac{2u}{g}$

Example 1

A stone is dropped from a point which is 40m above the ground. Find the time taen for the stone to reach the ground

$h = ut + \frac{1}{2}gt^2$

$40 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2$ | $t = \sqrt{\frac{40}{9.8}} = 2.857s$

Example 2

A ball is thrown vertically upwards with an initial speed of $30ms^{-1}$. Calculate

(i) Time taken to reach thrower

$T = \frac{2u}{g} = \frac{2\ x\ 30}{9.8} = 6.12s$

(ii) maximum height reached

$h_{max} = \frac{u^2}{2g} = \frac{30^2}{2\ x\ 9.8} = 45.92m$

Example 3

A particle is projected from the ground level vertically upwards with velocity of $19.6ms^{-1}$. Find

(i) greatest height reached

$h_{max} = \frac{u^2}{2g} = \frac{19.6^2}{2\ x\ 9.8} = 19.6m$

(ii) time taken by the particle to reach maximum height

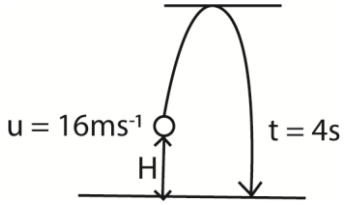
$t = \frac{u}{g} = \frac{19.6}{9.8} 2s$

(iii) Time of flight

T = 2t = 2 x 2 = 4s

Example 4

1. A stone is thrown vertically upwards with velocity $16ms^{-1}$ from a point H meters above the ground level. The stone hits the ground 4 seconds later.
   Calculate the
   (a) Value of H (03marks)



Using s = ut + ½ at^2; s = -H(below point of projection), u = $16ms^{-1}$, a = -g, t = 4s

$-H = 16 \times 4 - \frac{1}{2}\ x\ 9.8\ x\ 4^2$

H = 14.4m

   (b) Velocity of the stone as it hits the ground (02marks)

Using v = u + at; v = -v(below point of projection), a = -g, t = 4s

-v = 16 -9.8 x 4

v = $23.2ms^{-1}$

∴ the velocity of the stone as it hits the ground is $23.2ms^{-1}$

Example 6

A stone is thrown vertically upwards with a velocity of $21ms^{-1}$. Calculate the

(a) Maximum height attained by the stone (03marks)

$H = \frac{u^2}{2g} = \frac{21^2}{2\ x\ 9.8} = 225m$

(b) Time the stone takes to reach the maximum height. (02marks)

$t = \frac{u}{g} = \frac{21}{9.8} = 2.143s$

Example 7

A particle is projected vertically upwards with velocity ums-1. After t seconds another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collide after $\left(\frac{2}{2}+\frac{u}{g}\right)s$. Hence show that they meet at a height of $\frac{u^2-(gt)^2}{8g}$.

Solution

t_1 = time taken by 1st particle

t_2 = time taken by 2nd particle

$t_1 - t_2$ = t …….. (i)

t_1 and t_2 are roots of the equation

h = ut - $\frac{1}{2}gt^2$ or $gt^2 - 2ut + 2h = 0$

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$

$t_2 = \frac{2u-\sqrt{4u^2-8gh}}{2g}$

$\frac{2u+\sqrt{4u^2-8gh}}{2g} - \frac{2u-\sqrt{4u^2-8gh}}{2g}$ = t

$\sqrt{4u^2-8gh} = gt$ …… (ii)

From eqn (ii)

$h = \frac{4u^2-(gt)^2}{8g}$

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$ putting eqn. (ii)

$$t_1 = \frac{2u+gt}{2g} = \left(\frac{2}{2}+\frac{u}{g}\right)s$$

Example 8

A particle is projected upwards with velocity of 10ms-**1**. After 2s another particle is projected vertically upwards from the same point of projection with the same initial velocity. Find the height above the level of projection where the particle meet and time taken by the first particle before they meet.

t_1 = time taken by 1st particle

t_2 = time taken by 2nd particle

$t_1 - t_2$ = t …….. (i)

t_1 and t_2 are roots of the equation

h = ut - $\frac{1}{2}gt^2$ or $gt^2 - 2ut + 2h = 0$

$t_1 = \frac{20+\sqrt{400-8gh}}{2g}$

$t_2 = \frac{20-\sqrt{400-8gh}}{2g}$

$\frac{20+\sqrt{400-8gh}}{2g} - \frac{20-\sqrt{400-8gh}}{2g}$ = t

$\sqrt{400-8gh} = gt$ …… (ii)

From eqn (ii)

$h = \frac{400-(2\ x\ 9.8)^2}{8\ x\ 9.8} = 0.202m$

$t_1 = \frac{2u+\sqrt{4u^2-8gh}}{2g}$ putting eqn. (ii)

$t_1 = \frac{2u+gt}{2g} = \left(\frac{2}{2}+\frac{u}{g}\right) = \frac{2\ x10+9.8\ x\ 2}{2\ x\ 9.8} = 2.02\ s$

Revision exercise

1. A particle is projected vertically upwards with a velocity of 21ms^{-1}. How long it takes to reach a point 280m below the point of projection. [10s]
2. A particle is projected vertically upwards with a velocity of 17.5ms^{-1}. Find
   (i) how high the particle goes. [15.6m]

(ii) what time elapse before it's at a height of 10m [$\frac{5}{7}s$; $\frac{22}{7}s$

3. A particle is projected vertically upwards with velocity of 24.5ms^{-1}. Find
   (a) when its velocity is 4.9ms^{-1} [2s]
   (b) how long it takes to return to the point of projection.[5s]
   (c) at what time it will be 19.6m above the point of projection. [1s and 4s]
4. A particle is projected vertically upwards with a velocity of 35ms^{-1}. find
   (a) how long it takes to reach the greatest height.[3.57s]
   (b) distance it ascends during the 3rd second of motion. [10.5m]
5. Two objects are dropped from a cliff of height H. the second is dropped when the first has travelled a distanc e d. Prove that the instant when the first object reaches the bottom, the second is a distance $2\sqrt{DH} - D$ from the top of the cliff.
6. A particle is projected vertically upwards from point O with a speed of $\frac{4}{3}v\ ms^{-1}$. After it has travelled a distance of $\frac{2}{5}X\ m$ above O on its upward motion, another particle is projected vertically upwards from the same point with the same initial speed. Given that the particles collide at a height $\frac{2}{5}X\ m$ above O, prove that
   (i) the maximum height, H is given by $8v^2 = 9gH$
   (ii) when the particle collide 9X = 20H.
7. A particle is projected vertically upwards with velocity um/s. After t seconds another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Prove that the particles collide each other having a velocity of $\frac{1}{2}gt$.
8. A particle is projected vertically upwards with velocity 28m/s. After 2s another particle is projected vertically upwards from the same point of projection and with an initial velocity of 21m/s. Find when the two particles are at the same height and the velocity of each body at that instant. [4.9s after the first particle is projected, 20m/s, 7.4m/s]
9. A particle is projected vertically upwards with velocity 25m/s. After 4s another particle is projected vertically upwards from the same point of projection and with the same initial velocity. Find the time and height when the two particles meet.[4.55s after the first particle is projected, 12.288m]
10. A stone is dropped from the top of a tower. In the last second of its motion, it falls through a distance which is a fifth of the height of the tower. Find the height of the tower. [439.6m]

Thank You

Dr. Bbosa Science

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Work, energy and power

Work done by constant force

Work is said to be done when energy is transferred from one system to another

When a block of mass m (kg) rests on a smooth horizontal surface. When a constant force F (N) acts on the block and displace it by a distance s (m), then the work done by F is given by

$W = Fs$

Example 1

Find the work done against gravity when a body of mass 5kg is moved through a vertical distance of 2m.

Solution

$W = Fs = mgs$ | $W = 5 \times 9.8 \times 2 = 98J$

Example 2

A man building a wall lifts 50 bricks through a vertical distance of 3m. If each brick has a mass 4 kg. how much does the man do against gravity

Solution

$W = Fs = mgs$ | $W = 50 \times 4 \times 9.8 \times 3 = 5880J$

Example 3

A body of mass 2kg is moved vertically upwards at constant speed of $5ms^{-1}$. Find how much work is done against gravity in each second.

Solution

$W = Fs = mgs$ | $W = 2 \times 9.8 \times 5 \times 1 = 98J$

Example 4

A horizontal force pulls a body of mass 5kg a distance of 8m across a rough horizontal surface, coefficient of friction 0.25. The body moves with uniform velocity, find the work done against friction.

Solution

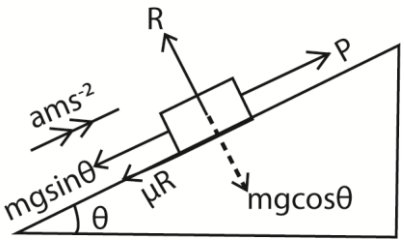
W = Fs = mgs | W = 0.25 x 5 x 9.8 x 8 = 98J

Example 5

A rough surface is inclined at $\tan^{-1}\left(\frac{7}{24}\right)$to the horizontal. A body of mass 5kg lies on the surface and is pulled at uniform speed of 75cms^{-1} up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is$\frac{5}{15}$. Find

(a) work done against gravity (b) work done against friction

Solution



$\theta = \tan^{-1}\left(\frac{7}{24}\right) = 16.3^0$

(a) work against gravity

W = mgsinθd

= 5 x 9.8sin16.3^0 x $\frac{75}{100}$ = 10.35J

(b) work against friction

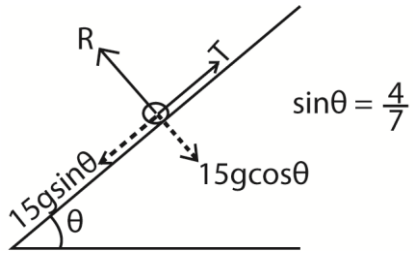
W = μRd but R = mgcosθ

=$\frac{5}{12}$ x 5 x 9.8cos16.3^0 x $\frac{75}{100}$ = 14.71J

Example 6

A particle of mass 15kg is pulled up a smooth slope by a light inextensible string parallel to the slope. The slope is 10.5m long and inclined at sin$^{-1}\left(\frac{4}{7}\right)$ to the horizontal. The acceleration of the particle is 0.98ms^{-2}. Determine the

(a) Tension in the string (03marks)



T – 15gsinθ = 15a

T – 15 x 9.8 x $\frac{4}{7}$ = 15 x 0.98

T- 84 = 14.7

T = 98.7N

(b) Work done against gravity when the particle reached the end of the slope. (02marks)

Work = force x distance

= 15gdsinθ

= 15 x 9.8 x 10.5 x $\frac{4}{7}$

= 882J

Revision exercise

1. Find the work done against gravity when a body of mass 1kg is raised through a vertical distance of 3m. [29.4J]
2. Find the work done against gravity when a person of mass 80kg climbs a vertical distance of 25m. [19600J]
3. A body of mass 200g is moved vertically upwards at a constant speed of $2ms^{-1}$. Find how much work is done against gravity in each second. [3.92J]
4. A body of mass 10kg is pulled a distance of 20m across a horizontal surface against resistance totalling 40N. If the body moves with a uniform velocity, find the work done against the resistance. [800J]
5. A horizontal force pulls a body of mass 3kg a distance of 20m across a rough horizontal surface, coefficient of friction$\frac{2}{7}$. The body moves with a uniform velocity and the only resistance is that due to friction. Find the work done.[168J]
6. A horizontal force drags a body of mass 4kg a distance of 10m across a rough horizontal floor at a constant speed. The work done against friction is 49J. Find the coefficient of friction between the body and the surface. [0.125]
7. A block of mass 15kg rests on a smooth plane inclined at an angle of 30^0 to the horizontal. The block is pulled at a uniform speed a distance of 10m up the line of greatest slope. Find the work done against gravitation force. [735J]
8. A surface inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body of mass 50kg lies on the surface and pulled at uniform speed a distance of 5m to the line of greatest slope against a resistance to totalling 50N. Find
   (a) work done against gravity [1470J]
   (b) work done against friction [250J]
9. A rough surface inclined at $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. A body of mass 130kg lies on the surface and is pulled at uniform speed a distance of 50m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface if $\frac{2}{7}$. Find
   (a) the frictional force acting [336N]
   (b) work done against friction[16800J]
   (c) work done against gravity [24500J]
10. A rough surface is inclined at 30^0 to the horizontal. A body of mass 100kg lies on the surface and is pulled at a uniform speed a distance of 20m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is 0.1. Find
    (a) work done against friction [1700J]
    (b) work done against gravity. [9800J]
11. A rough surface is inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body of mass 50kg lies on the surface and is pulled at a uniform speed a distance of 15m up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{1}{3}$. Find the total work done on the body. [6370J]
12. A rough surface is inclined at an angle θ to the horizontal. A body of mass mkg lie on the surface and is pulled at a uniform speed by a force acting along a lines of greatest slope. The coefficient of friction between the body and the surface is μ. show that the total work done on the body is $mgx(\sin\theta + \mu\cos\theta)$

Work – energy theorem

It state that the work done by the **net force** acting on a body is equal to the change in its energy. Consider a body of mass m accelerated from velocity, u by a constant force, F so that in a distance, s it gains velocity, v.

a = $\frac{v^2-u^2}{2s}$

resultant force = ma = $\frac{m(v^2-u^2)}{2s}$

But work done = F x s

W = $\frac{m(v^2-u^2)}{2s} x\ s = \frac{m(v^2-u^2)}{2}$

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$ work – energy theorem

Example 7

A constant force pushes a mass of 4kg in a straight line across a smooth horizontal surface. The body passes a point A with speed of 5ms^{-1} and then through a point B with a speed of 8ms^{-1}. B is 6m from A. Find the magnitude of force acting on the mass.

a = $\frac{v^2-u^2}{2s}$ = $\frac{8^2-5^2}{2\ x\ 6s}$ = 3.25ms^{-2}

F = ma = 4 x 3.25 = 13N

Alternatively

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

F x 6 = $\frac{1}{2}\ x\ 4\ (8^2 - 5^2)$

F = 13N

Example 8

A car of mass 1000kg moving at 50ms^{-1} skids to rest in 4s under a constant retardation. Calculate the magnitude of the work done by the force of friction

Using v = u + at

0 = 50 + 4a

a = -12.5ms^{-2}

Frictional force = ma

= 1000 x -12.5 = 1250N

s = ut + ½ at2

s = 50 x 4 + ½ x -12.5 x 4^2

= 100m

W = F x s = 1250 x 100

= 125 000J

Alternatively

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

W = $\frac{1}{2}\ x\ 1000\ (50^2 - 0^2)$

Work done =125 000J

Example 9

A body of mass 4kg is moving with an initial velocity of 4ms-1 on a plane. The kinetic energy of the body is reduced by 16J in a distance of 40m. Find the deceleration of the body

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

16 = $\frac{1}{2}\ x\ 4(5^2 -\ v^2)$

v^2 = 17

a = $\frac{v^2-u^2}{2s}$ = $\frac{17-5^2}{2\ x\ 40}$ = -0.1ms^{-2}

Example 10

A body of mass 5kg moves in a straight line across a horizontal surface against a constant resistance of magnitude 10N. The body passes through point A and then comes to rest at point B, 9m from A. Find the speed of the body when it is at A.

F = ma

-10 = 5a

a = $-2ms^{-2}$

$v^2 = u^2 + 2a$

$0^2 = u^2 + 2 \times -2 \times 9$

u = $6ms^{-2}$

Or

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

10 x 9 = $\frac{1}{2}\ x\ 5(u^2 - \ 0^2)$

u = $6ms^{-2}$

Example 11

A body of mass 5kg slides over a rough horizontal surface. in sliding 5m, the speed of the body decreases from 8ms-1 to 6ms-1, find

(a) Frictional force

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

F x 5 = $\frac{1}{2}\ x\ 5(8^2 - \ 6^2)$

F = 14N

F = μR

μ = $\frac{14}{5\ x\ 9.8}$ = 0.286

Alternatively; $v^2 = u^2 + 2as$

a = $\frac{v^2-u^2}{2s} = \frac{6^2-8^2}{2\ x\ 5}$

= $-2.8ms^{-2}$

F= ma = 5 x 2.8 = 14N

Example 12

A bullet of mass 15g is fired towards a fixed wooden block and enters the block when travelling horizontally at $400ms^{-1}$. It comes to rest after penetrating a distance of 25cm. Find the

(i) work done against resistance of the wood

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}\ x0.015(400^2 - \ 0^2)$ = 1200J

(ii) magnitude of resistance

W = F x s

1200 = F x 0.25

F = 4800N

Example 13

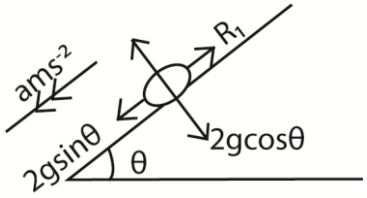
A particle of mass 5kg falls vertically against a constant resistance. The particle passes through two point A and B 2.5m apart with A above B. Its speed is $2ms^{-1}$ when passing through A and $6ms^{-1}$ when passing through B. Find the magnitude of the resistance.

W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$ | (5g – R) x 2.5 = $\frac{1}{2}\ x\ 5\ (6^2 - \ 2^2)$ | R = 17N

Inclined Planes

Example 14

A rough slope of length 5m is inclined at angle 30^0 to the horizontal. A body of mass 2kg is released from rest at the top of the slope and travels down the slope against a constant resistance. The body reaches the bottom of the slope with speed of $2ms^{-1}$, find the magnitude of the resistance.



W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

(2gsin30 – R_1) x 5 = $\frac{1}{2}\ x\ 2(2^2 - 0^2)$

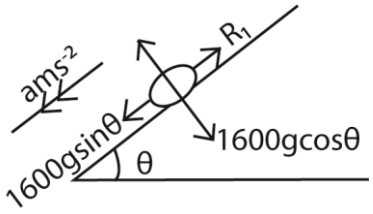
R_1 = 9N

Example 15

A car of mass 1600kg slides down a hill of slope 1 in 25. When the car descends 200m along the hill its speed increases from $3ms^{-1}$ to $10ms^{-1}$. Calculate

$\Delta K.E$ = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

= $\frac{1}{2}\ x\ 1600(10^2 - 3^2)$ = 72800J



$a = \frac{v^2-u^2}{2\ x\ s} = \frac{10^2-3^2}{2\ x\ 200}$ = $0.228ms^{-2}$

F = ma

1600gsinθ – R_1= 1600a

R1 = 1600 x 9.8 x $\frac{1}{25}$ – 1600 x 0.228

= 262.4N

OR; W = $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

(1600gsinθ – R_1) x 200 = $\frac{1}{2}\ x\ 1600(10^2 - 3^2)$

R = 262.4N

Revision exercise 2

1. A carton of mass 0.4kg is thrown across a table with velocity of 25ms-1. The resistance of the table to its motion is 50N. How far will it travel before coming to rest? What must the resistance be if it travels only 2m? [2.5m, 62.5N]
2. A and B are two points 4m apart on a smooth horizontal surface. A body of mass 2kg is initially at rest at A and is pushed by a force of constant magnitude acting in direction from a to B. The body reaches B with speed of $4ms^{-1}$. Find the magnitude of force [4N]
3. A and B are two points 3m apart on a smooth horizontal surface. A body of mass 6kg is initially at rest at A and is pushed towards B with a constant force of 9N. Find the speed of the body when it reaches B. [$3ms^{-1}$]
4. A constant force of magnitude 8N pushes a body of mass 4kg in a straight line across a smooth horizontal surface. The body passes through a point a with speed of $4ms^{-1}$ and then through appoint B 5m from A. Find the speed of the body at B. [$6ms^{-2}$]
5. A particle of mass 100g moves in a straight line across a horizontal surface against of constant magnitude. The particle passes through a point A with a speed of $7ms^{-1}$ and then through B with speed of $3ms^{-1}$, B being 2m from A. Find the magnitude of resistance. [1N]

6. A and B are two points 15m apart in the same vertical line, with A above B. A body of mass 5kg is released from rest at A and falls vertically against constant resistance of 25N. Find the speed of the body when it passes B. [12ms^{-1}]
7. A particle of mass 6kg is released from rest and falls freely under gravity. Find its speed when it has fallen a distance of 90m. [42ms-1]
8. A particle of mass 6kg is released from rest and falls freely under gravity. Find the distance it has fallen when its speed is 7ms^{-1}. [2.5m]
9. A body of mass 3kg is projected vertically upwards from a point A with speed 4ms^{-1}. The body passes through a point B 5m below A. Find the speed at B. [10.7ms^{-1}]
10. A particle of mass 2kg falls vertically against a constant resistance of 14N. The particle passes through two points A and B with a speed of 3ms^{-1} and 10ms^{-1} respectively. The distance AB. [16.25m]
11. A bullet of mass 8g is fired towards a fixed wooden block and enters the block when travelling horizontally at 300ms^{-1}. How far does the bullet penetrate if the wood provide a constant resistance of 1800N. [20cm]
12. A bullet of mass 50g travelling horizontally at 100ms^{-1} strikes a stationary block of wood and coming to rest through a distance of 5m. Calculate the average resistance of the block to the motion of the bullet. [50N]
13. A bullet of mass 50g travelling horizontally at 500ms^{-1} strikes a stationary block of wood and after travelling 10cm, it emerges from the block travelling at 100ms^{-1}. Calculate the average resistance of the block to the motion of the bullet. [60000N]
14. A bullet of mass 20g travelling horizontally at 210ms^{-1} strikes a stationary block of wood of thickness 0.1m and emerges from the block travelling at 50ms^{-1}. Calculate the average resistance of the block to the motion of the bullet.[4160N]
15. A smooth slope is inclined at $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A block of mass 4kg is released from rest at the top of the slope and travels down the slope, reaching the bottom of the slope with speed of 7ms^{-1}, find the length of the slope. [4.17m]
16. A point A is situated at the bottom of a smooth slope inclined at an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ to the horizontal. A body is projected from A with a speed of 14ms-1 along and up a line of greatest slope and the body comes to rest at a point B. Find distance AB. [26cm]
17. A rough slope of length 10m is inclined at an angle of $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A block of mass 50kgis released from rest at the top of the slope and travels down the slope, reaching the bottom of the slope with speed of 8ms^{-1}, find the
    (i) magnitude of the friction force. [134N]
    (ii) work done by friction force. [1340J]
    (iii) coefficient of friction [0.342]

18. Point A is situated at the bottom of a rough slope of length 10m is inclined at angle of $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal. A body is projected from A with a speed of 14ms-1 along and up a line of greatest slope. The coefficient of friction between the body and the plane is 0.25. The body first comes to rest at point B. Find the distance. [12.5m]

Power

Power is the rate of doing work

Power = $\frac{work\ done}{time}$

P = $\frac{F\ x\ d}{t}$

P = $F\ x\ \frac{d}{t}$

P = F x v

F = $\frac{P}{v}$

Example 16

What is the average rate at which work must be done in lifting a mass of 100kg a vertical distance of 5m in 7s.

P = $\frac{F\ x\ d}{t}$ = $\frac{100\ x\ 9.8\ x\ 5}{5}$ = 700W

Example 17

What is the rate at which work must be done lifting mass of 500kg at constant speed of 3ms^{-1}?

P = F x v = 500 x 9.8 x 3 = 14700W

Motion of cars

Consider a car being driven along a road, the forward or tractive force F_T moves the car is supplied by the engine working at constant rate of P watts.

Power = $\frac{work\ done}{time}$

P = $\frac{F_T\ x\ d}{t}$

P = $F_T\ x\ \frac{d}{t}$

P = F_T x v

F$_T$ = $\frac{P}{v}$

Example 18

A cyclist travels along a road at constant speed of 8ms^{-1}. If the resistance to motion is 50N, find the rate at which the cyclist is working.

Solution

F =ma at constant speed a = 0ms-2

F – R = m x 0

$\frac{P}{8} - 50 = 0$

P = 400W

Example 19

A car of mass 800kg is driven along a level road against a constant resistance to motion of 200N. With the engine working at a steady rate of 14kW, find

(i) acceleration of the car when its speed is 10ms-1

(ii) maximum speed at which the car can move

Solution

(i) F = ma

F_T – R = m x a

$\frac{14000}{10} - 200 = 800a$

a = 1.5ms^{-1}

(ii) F = ma at maximum speed a = 0

$F_T - R = m \times a$

$\frac{14000}{v} - 200 = 800 \times 0$

$v = 70ms^{-1}$

Example 20

A car of mass 500kghas an engine of maximum power 2.5kW

(a) Calculate the force resisting the motion of the car when it is travelling at its maximum speed of 72kmh^{-1} on a level road.

72kmh^{-1} = $\frac{72 \times 1000}{3600}$ = 20ms^{-1}

F = ma

$F_T - R = m \times a$

at maximum speed a = 0

$\frac{2500}{20} - R = 500 \times 0$

R = 125N

(b) If the resistance remains unaltered, find the acceleration of the car when travelling at 36kmh^{-1} on a level road with the engine working at the same rate.

$F_T - R = m \times a$

$\frac{2500}{10} - 125 = 500 \times a$

a = 0.25ms^{-2}

Example 21

A force on a particle of mass 15kg moves it along a straight line with velocity of 10ms-1. The rate at which work is done by the force is 50W. If the particle starts from rest, determine the time it takes to move a distance of 100m.

F = ma

$\frac{50}{10} = 15a$

a = 0.33ms^{-2}

s = ut + ½ at^2

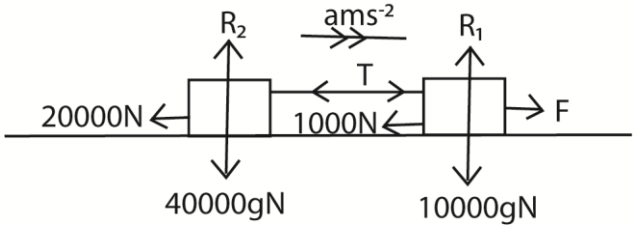
100 = 0 x t + $\frac{1}{2} \times 0.33 \times t^2$

t = 24.51s

Example 22

A car of mass 10 tonnes tows a trailer of mass 40 tonnes along a level road. The car exerts experience a resistance of 100N and the trailer a resistance of 2000N, if the car engine working at constant rate of 4,000kW. Find the acceleration produced and the tension in the two bar at the instant the speed is 72kmh^{-1}.

Solution



72kmh^{-1} =$\frac{72,000}{3600} = 20ms^{-1}$

F = $\frac{P}{v} = \frac{4,000,000}{20} = 20,000N$

10,000kg: 20000- (T+100) = 10000a

19,900 – T = 10000a …… (i)

40,000kg; T – 2000 = 40000a …… (ii)

(i) + (ii): a = 0.358ms^{-2}

Alternatively

50000kg: 20,000-200-100 = 5000a

17,900 = 50000a

a = 0.358ms^{-2}

T- 2000 = 40,000a

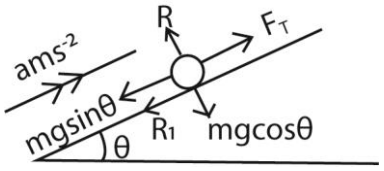
T =40000 x 0.358 + 2000

= 16,320N

Inclined planes

Example 23

A train of mass 20,000kg moves at constant speed of 72kmh^{-1} up a straight inclined plane against a frictional force of 128N. The incline is such that the train rise vertically 1m for every 100m travelled along the incline. Calculate the necessary power developed by the train



72kmh^{-1} = $\frac{72\ x\ 1000}{3600}$ =20ms^{-1}

sinθ = $\frac{1}{100}$

at constant speed, a = 0

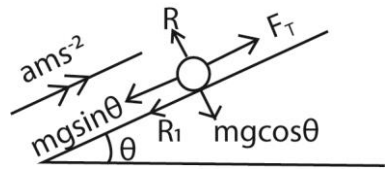
Force, $F_T = \frac{P}{v} - (mgsin\theta + R_1) =$ ma

$\frac{P}{20} - \left(20000x\ 9.8\ x\ \frac{1}{100} + 128\right)$ = 20000 x 0

P= 41760W

Example 24

A car of mass 1.5metric tonnes moves at constant speed of 6ms^{-1} up a slope inclined at $\sin^{-1}\frac{1}{7}$. Given that the engine of the car is working at constant rate of 18kW. Find the resistance to the motion.



sinθ = $\frac{1}{7}$

at constant speed, a = 0

Force, $F_T = \frac{P}{v} - (mgsin\theta + R_1) =$ ma

$\frac{P}{6} - \left(1500x\ 9.8\ x\ \frac{1}{7} + R_1\right)$ = 1500 x 0

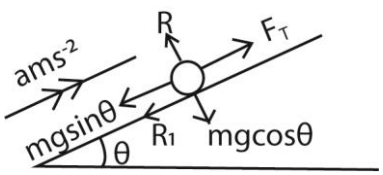
R_1 =900N

Example 25

A car of mass 800kg with engine working at constant rate of 15kW climbs a hill of inclination 1 in 98 against a constant resistance to motion of 420N. Find the

(i) the acceleration when the speed is 10ms^{-1}

(ii) maximum speed of the car up the hill



(ii) at constant speed, a = 0

Force, $F_T = \frac{P}{v} - (mgsin\theta + R_1) =$ ma

$\frac{P}{10} - \left(800x\ 9.8\ x\ \frac{1}{98} + 420\right)$ = 800 x a

a =1.25ms^{-2}

$\frac{15000}{v} - \left(8000x\ 9.8\ x\ \frac{1}{98} + 420\right)$ = 800 x 0 v = 30ms^{-1}

Example 26

A car of mass 1000kg has a maximum speed of 40ms^{-1} on a level road and the engine s working at 32kW against constant resistance

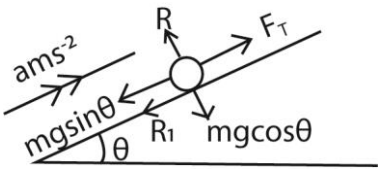
(i) find the resistance to motion

$F_T - \mathrm{R}_1 =$ ma; at constant speed a = 0

$\frac{32000}{40} - R_1 = 1000\ x\ 0$

$R_1 = 800N$

(ii) Given that the resistance in both cases varies as the speed, find the rate at which the engine must work for the car to ascend a slope of 1 in 98 at constant speed of 20ms^{-1}.



R_1 = kv

k = $\frac{800}{40} = 20$

Force, $F_T = \frac{P}{v} - (mgsin\theta + R_1) =$ ma

at constant speed a =0

$\frac{P}{20} - \left(1000x\ 9.8\ x\ \frac{1}{98} + 20\ x\ 20\right)$ = 1000 x 0

P = 10,000W

Example 27

A car of mass 1000kg has maximum speed of 150kmh^{-1} on a level road and working at 60kW

(i) find the coefficient of friction between the car and the road if all resistance is due to friction

150kmh-1 = $\frac{150000}{3600}$ = 41.67ms^{-1}

F = ma

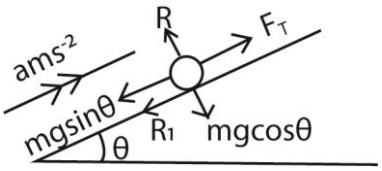
F_T – μR = ma

At constant speed, a = 0

$\frac{60000}{41.67} - \mu(1000\ x\ 9.8) = 1000\ x\ 0$

μ = 0.147

(ii) Given that the tractive force remains unaltered and the non-gravitation resistance in both cases varies as square of the speed, find the greatest slope on which a speed of 120kmh^{-1} could be maintained.



μR = kv^2

k = $\frac{0.147\ x\ 1000\ x\ 9.8}{(41.67)^2}$ = 0.8297

120kmh^{-1} = $\frac{120\ x1000}{3600}$ = 33.33ms^{-1}

Force, $F_T = \frac{P}{v} - (mgsin\theta + R_1) =$ ma

at constant speed a =0

$\frac{60000}{41.67} - (1000x9.8sin\theta + 0.8297(33.33)^2$

$= 1000\ x0$

θ = 3.04^0

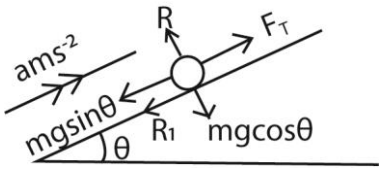
Example 28

A car of mass 900kg with the engine working at a constant rate of 7.35kW climbs a hill of inclination 1 in 63 against constant resistance to motion. Find the

(i) resistance to motion when the car is travelling with a constant speed of $15ms^{-1}$

(ii) maximum speed of the car when travelling down the same slope with the engine working at the same rate as before and the resistance to motion unchanged.

Solution



$F_T = \frac{P}{v} - (mgsin\theta + R_1) = ma$

$\frac{7350}{15} - \left(900\ x\ 9.8\ x\ \frac{1}{63} + R_1\right) = 900\ x\ 0$

$R_1 = 350N$

(ii) $F_T = \frac{P}{v} + (mgsin\theta - R_1) = ma$

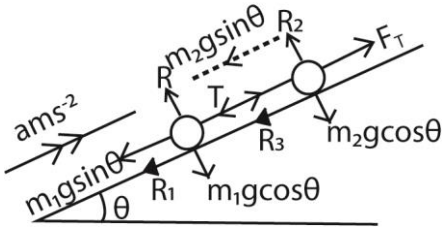
$\frac{7350}{v} - \left(900\ x\ 9.8\ x\ \frac{1}{63} - 350\right) = 900\ x\ 0$

$v = 35ms^{-1}$

Example 28

A car of mass 1200kg is pulling a trailer of mass 300kg up a slope 1 in 10, the resistance to motion for car and trailer is $0.2Nkg^{-1}$. Given that the car moved at a constant speed of $1.5s^{-1}$ for 5 minutes, find the

(i) Tension in coupling between the car and the trailer
(ii) work done by the engine of the car during this time
(iii) total resistance, if the engine develops a power of 15kW at maximum speed of $120kmh^{-1}$ on level road.



At constant speed, a = 0; $sin\theta = \frac{1}{10}$

300kg: $T - (m_1gsin\theta + 0.2m_1) = m_1 \times 0$

$T = 300 \times 9.8 \times \frac{1}{10} + 0.2 \times 300 = 354N$

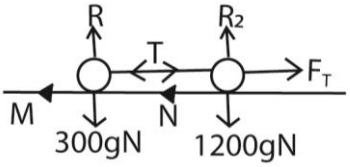
1200kg: $F_T - (T + m_2gsin\theta + 0.2m_2) = 1200 \times 0$

$FT = (354 + 1200 \times 9.8 \times \frac{1}{10} + 0.2 \times 1200)$

$= 1770N$

$W = FT \times d = 1770 \times 1.5 \times 5 \times 60 = 796{,}500J$

(iii)



$120kmh^{-1} = \frac{120\ x\ 1000}{3600} = 33.33ms^{-1}$

$F_T = \frac{P}{v} = \frac{15000}{33.33} = 450N$

1500kg: $450 - (M + N) = 1500 \times 0$

$(M + N) = 450$

Example 30

A car of mass mkg has an engine which works at a constant rate of 2H kW. The car has constant speed of V ms^{-1} a long a horizontal road.

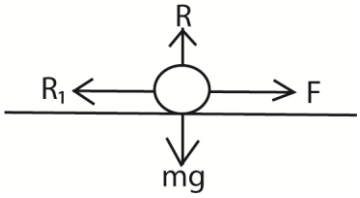
(a) Find in terms of m, H, V and θ the acceleration of the car when travelling
   (i) up a road of inclination θ with a speed of $\frac{3}{4}V\ ms^{-1}$.
   (ii) down the same road with a speed of $\frac{3}{5}V\ ms^{-1}$, the resistance to the motion of the car apart from gravitational force, being constant

(b) If an acceleration in (a)(ii) above is 3 times that of (a)(i) above, find the angle of inclination θ of the road.

(c) If the car continues directly up the road, in case (a)(i) above, show that its maximum speed is $\frac{12}{13}V\ ms^{-1}$

Solution

(a) When the car moves along a horizontal road



With constant speed, F = R_1

But power, P = Fv = R_1v

⇨ 2000H = Fv = R_1v

R_1v= 2000H

$R_1 = \frac{2000H}{v}$

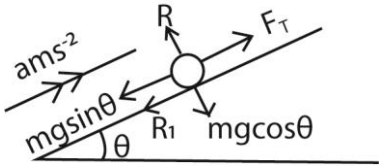
(i) When ascending with velocity $\frac{3v}{4}ms^{-1}$

Power = Fv

2000H = $\frac{3v}{4}F$

F = $\frac{8000H}{3v}$

Let a = acceleration



Net accelerating force = F – (mgsinθ + R_1)

ma = $\frac{8000H}{3v}$ - (mgsinθ + $\frac{2000H}{v}$)

ma = $\frac{2000H}{3v}$ − mgsinθ

a = $\frac{2000H-3mvgsin\theta}{3mv}$

(ii) When descending with velocity $\frac{3v}{5}ms^{-1}$

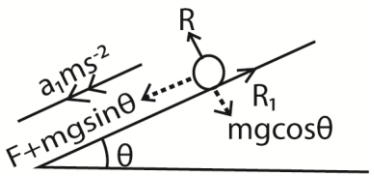
Power = Fv

2000H = $\frac{3v}{5}F$

F = $\frac{10000H}{3v}$

Let a_1 = acceleration

Net accelerating force = F +mgsinθ - R_1) = ma_1



$ma_1 = \frac{10000H}{3v} + mgsin\theta - \frac{2000H}{v}$

$a_1 = \frac{4000H+3mvgsin\theta}{3mv}$

(b) Given $a_1 = 3a$

$\frac{4000H+3mvgsin\theta}{3mv} = 3\left(\frac{2000H-3mvgsin\theta}{3mv}\right)$

$\frac{4000H+3mvgsin\theta}{3mv} = \frac{2000H-3mvgsin\theta}{mv}$

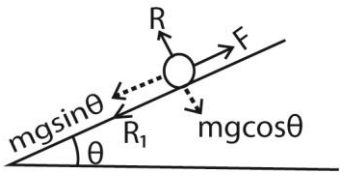
$4000H + 3mvgsin\theta = 6000H - 9mvgsin\theta$

$2000H = 12mvgsin\theta$

$sin\theta = \frac{2000H}{12mvg}$

$\theta = \sin^{-1}\left(\frac{2000H}{12mvg}\right)$

(c) With maximum speed, a = 0



Net accelerating force = F – (mgsinθ + R_1)

m x 0 = F – (mgsinθ + R_1)

F = R_1 + mgsinθ

Let v1 be maximum velocity

Power, P = Fv_1

2000H = Fv_1

2000H = (R_1 + mgsinθ)v_1

Substituting for R and sinθ

$2000H = \left(\frac{2000H}{v} + mg\left[\frac{2000H}{12mvg}\right]\right)v_1$

$2000H = \left(\frac{2000H}{v} + \frac{2000H}{12v}\right)v_1$

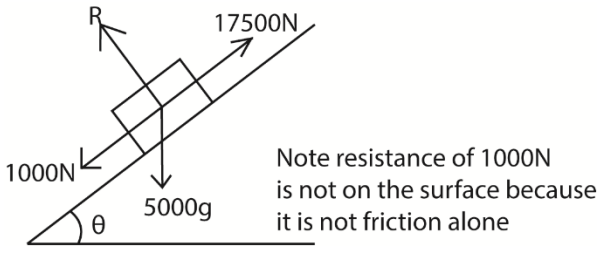
$1 = \left(\frac{1}{v} + \frac{1}{12v}\right)v_1$

$1 = \left(\frac{13}{12v}\right)v_1$

$v_1 = \frac{12v}{13}$

Example 31

The engine of a lorry of mass 5,000kg is working at a steady rate of 350 kW against a constant resistance force of 1,000N. The lorry ascends a slope of inclination θ to the horizontal. If the maximum speed of the lorry is 20ms^{-1}, find the value of θ (05marks)



Resultant force = 17500 – (1000 + 5000sinθ)

5000a = 17500 – (1000 + 5000sinθ)

At maximum speed a = 0

⇨ 0 = 17500 – (1000 + 5000sinθ)
16500 = 5000sinθ
θ = 19.7^0

Example 32

A car is working at 5kW and is travelling at constant speed of 72kmh^{-1}. Find the resistance to motion

Power = 5kW = 5000w

V = 72kmh^{-1} = $\frac{72\ x\ 1000}{3600} = 20ms^{-1}$

At constant speed, F = R

power = Fv= Rv

R = $\frac{5000}{20} = 250N$

Revision exercise 3

1. A car is drive along a level road against a constant resistance to motion of 400N. Find the maximum speed at which the car can move when its engine works at steady rate of 8.8kW. [22ms^{-1}]
2. What is the maximum speed which a car can travel along road when its engine is developing 24kW and there is a resistance to motion of 800N. [30ms^{-1}]
3. A car is driven along a level road against a constant resistance to motion of 400N. Find the maximum speed at which the car can move when its engine works at a steady rate of 4kW. [10ms^{-1}]
4. A car is working at 14kW and is travelling at constant speed of 75kms^{-1} along a level road. Find the resistance to motion. [187N]
5. A car of mass 1000kg is driven along a level road against constant resistance to motion of 200N. With the engine of car when working at steady rate of 8kW, find
   (i) acceleration of the car when its speed is 5ms^{-1} [1.4ms^{-2}]
   (ii) the maximum speed of the car. [40ms^{-1}]
6. A car of mass 900kg is driven along a level road against a constant resistance to motion of 300N. With the engine working at steady rate of 12kW, find
   (i) acceleration of the car when its speed is 10ms^{-1} [1ms^{-2}]
   (ii) the maximum speed of the car. [40ms^{-1}]
7. A car of mass 1600kg is freewheeling down a hill of slope 1 in 25. When the car descends 200m along a hill, its speed increases from 3ms^{-1} to 10ms^{-1}, find the average resistance to motion [263.2N]
8. A train of mass 100 tonnes has an engine of maximum power 60kW.
   (i) Calculate the force resisting the motion of the car when it is travelling at its maximum speed of 108kmh^{-1} on a level road. [2000N]
   (ii) If the resistance remains unaltered, find the acceleration of the car when travelling at 54kmh^{-1} on a level road with the engine working at the same rate.[0.02ms^{-2}}
9. A cyclist of mass 75kg moves on a level road and working at a rate of 210W against a constant resistance of 21N
   (i) find the maximum speed that a cyclist can attain. [10ms^{-1}]
   (ii) with the resistance and the rate of working unchanged, find the maximum speed can ascend a slope of inclination 1 in 15. [3ms^{-1}]
10. A car of mass 900kg moves on a level road at maximum speed of 48ms^{-1} against a constant resistance of 350N.
    (i) find the rate at which the engine is working. [16.8kw]
    (ii) with the resistance and rate of working unchanged, find the maximum speed the car ascends a slope of inclination 1 in 18 [20ms-1]
11. A car of mass 900kg working at a rate of 19.2kW moves on a level road at maximum speed of 20ms^{-1} against a constant resistance of (160 + av^2)N, where a is a constant and v is the speed of the car in ms^{-1}.
    (i) find the value of a [a = 2]

(ii) with the resistance and the rate of working unchanged, find the maximum speed of the car can ascend a slope of inclination of 1 in 30. [1.40667ms^{-1}]

12. A car of mass 100 metric tonnes moves with a constant speed of 654kmh^{-1} up a slope of inclination$\sin^{-1}\left(\frac{1}{59}\right)$. Given that the engine of the car is working at constant rate of 369kW. Find the resistance to motion. [5000N]
13. A man of mass 70kg rides a bicycle of mass 15kg at steady speed of 4.0ms-1 up a road which rises 1.0m for every 20m of its length. What is his power if resistance to motion is 20N. [250N]
14. A lorry of mass 2000kg moving at 10ms^{-1} on horizontal surface is brought to rest in a distance of 12.5m by the brakes being applied.
    (i) calculate the average retarding force. [8000N]
    (ii) what power must the engine produce if the constant speed of 10ms-1, frictional resistance being 200N. [22kW]
15. A lorry of mass 2.4 metric tonnes carrying goods of 9.8 metric tonnes ascends a hill of 1 in 8 with an acceleration of 0.45ms^{-2}.
    (i) calculate the tractive force of the engine. [20,435N]
    (ii) If at a speed of 36kmh^{-1}, the engine develops a power of 235kW, calculate the resistance of the lorry. 3,065N]
16. A car of mass 2 tonnes moves from rest down a road of inclination $\sin^{-1}\left(\frac{1}{20}\right)$to the horizontal. Given that the engine develops a power of 64.8kW when travelling at a speed of 54kmh^{-1} and the resistance to motion is 500N, find the acceleration. [2.4ms^{-2}]
17. The force opposing the motion of a car is (a + bv^2)N, where a and b are constants and v is the speed of the car in ms^{-1}. the power required to maintain a steady speed of 20ms^{-1} and at 30ms^{-1} is 15.3kW.
    (a) Find the values of a and b. [a = 150, b = 0.4]
    (b) power developed for a steady speed of 40m^{-1} [31.6kW]
18. A car is driven at uniform speed of 48kmh^{-1} up a smooth incline of 1 in 8. If the total mass of the car is 800kg and the resistance are neglected, calculate the power at which the car is working. [13076W]
19. A truck of mass 2000kg moving at constant speed of 10ms^{-1} up a hill which is inclined at an angle α, where sinα = 0.25. There is a constant force of 400N opposing the motion.
    (i) calculate the power used[53kW]
    (ii) If the power is instantaneously increased to 75kW, calculate the rate at which the truck begins to accelerate. [1.1ms^{-2}]
20. With its engine working at a constant rate of 9.8kW, a car of mass 800kg can descend a slope of 1 in 56 at twice the steady speed that it can ascend the same slope, the resistance to motion remaining the same throughout. Find the magnitude of the resistance and the speed of ascent. [420N, 17.5ms^{-1}]
21. The frictional resistance to motion is (kv)N where k is a constant and v is the speed of a car in ms-1. The car ascends a hill of inclination 1 in 10 at a steady speed of 8ms^{-1}. The power exerted by the engine is 9.76kW.
    (i) find the value of k [k = 30]
    (ii) Find the steady speed at which the car ascends the hill if the power exerted by the engine is 12.8kW. When the car is travelling at this speed, the power exerted by the engine is increased by 2kW, find the immediate acceleration of the car. [10ms^{-1}, 0.2ms^{-1}]

Pump raising and ejecting water

Consider a pump which is used to raise water from the source and then eject it at a given speed. The work done per second gives the rate (power) at which the pump is working.

Power of the pump = P.E given to water per second + K.E given to water per second

Example 32

A pump raises water through a height of 3.0m at a rate of 300kg per minute and delivers it with a velocity of 8.0ms^{-1}. Calculate the power output of the pump.

solution

Power of the pump = P.E given to water per second + K.E given to water per second

work done per second = (mass per second x g x h) + $(\frac{1}{2}\ x\ mass\ per\ second\ x\ v^2)$

$$= \frac{300}{60}\ x\ 9.8\ x\ 3 + \frac{1}{2}\ x\ \frac{300}{60}\ x\ 8^2 = 310W$$

Example 33

A pump draws 6m3 of water of density 1000kgm^{-3} from a well 9m below the ground in every minute and issues it at a speed of 12ms^{-1}. Find the rate at which the pump is working.

Solution

Power of the pump = P.E given to water per second + K.E given to water per second

work done per second = (mass per second x g x h) + $(\frac{1}{2}\ x\ mass\ per\ second\ x\ v^2)$

$$= \frac{6\ x\ 1000}{60}\ x\ 9.8\ x\ 9 + \frac{1}{2}\ x\ \frac{6\ x\ 1000}{60}\ x\ 12^2 = 16020W$$

Example 34

A pump raises water through a vertical distance of 10m in one and half minutes, and discharges it at a speed of 25ms-1. Show that the power developed is approximately 2.25kW.

Solution

Power of the pump = P.E given to water per second + K.E given to water per second

work done per second = (mass per second x g x h) + $(\frac{1}{2}\ x\ mass\ per\ second\ x\ v^2)$

$$= \frac{2\ x\ 1000}{90}\ x\ 9.8\ x\ 10 + \frac{1}{2}\ x\ \frac{2\ x\ 1000}{90}\ x\ 25^2 = 2247.22W \approx 2.25kW$$

Example 35

A pump draws 3.6m^3 of water of density 1000kgm^{-3} from a well 5m below the ground in every minute and issues it at ground level through a pipe of cross-section area 40cm^2. Find

(i) the speed with which water leaves the pipe.
(ii) the rate at which the pump is working
(iii) if the pump is only 80% efficient, find the rate at which it must work
(iv) find the power wasted.

Solution

(i) volume per second = area x velocity

$\frac{3.6}{60} = \frac{40}{10000}$v

$v = 15ms^{-1}$

(ii) Power of the pump = P.E given to water per second + K.E given to water per second

work done per second = (mass per second x g x h) + ($\frac{1}{2}\ x\ mass\ per\ second\ x\ v^2$)

$= \frac{3.6\ x\ 1000}{60}\ x\ 9.81\ x\ 5 + \frac{1}{2}\ x\ \frac{3.6\ x\ 1000}{60}\ x\ 15^2 = 9693W$

(iii) efficiency = $\frac{power\ output}{power\ input}\ x\ 100\%$

80% = $\frac{9693}{power\ input}\ x\ 100\%$

power input = 12116.25W

(iv) Power wasted = power input – power output

= 12116.25 – 9693 = 2423.25W

Revision exercise 4

1. A pump with power output of 600W raises water from a lake a height of 3.0m and delivers it with a velocity of 6.0ms^{-1}. What is the mass of water removed from the lake in one minute? [7500kg]
2. In every minute a machine pumps 300kg of water along a horizontal hose from rest at one end to eject at a speed of 4ms^{-1} at the other. Find the average rate at which the machine is working? [40W]
3. In every minute a pump draws 6m^3 of water from a well and issues it at a speed of 5ms^{-1} from a nozzle situated 4m above the level from which the water is drawn. Find the average rate at which the pump is working. [5.17W]
4. In every minute a pump draws 5m^3 of water from a well and issues it at a speed of 6ms^{-1} from a nozzle situated 6m above the level from which water was drawn. Find the average rate at which the pump is working. [6.4kW]
5. A pump draws water from a tank and issues it at a speed of 8ms^{-1} from the end of a pipe of cross-section area of 0.01m^2 situated at 10m above the level from which the water is drawn. find
   (i) the mass of water issued from the pipe in each second. [80kg]
   (ii) the rate at which the pump is working. [10.4kW]
6. A pump draws water from a tank and issues it at a speed of 10ms^{-1} from the end of the pipe of cross-sectional area 5cm^2 situated at 4m above the level from which the water is drawn. Find the rate at which the pump is working. [446W]
7. In every minute a pump draws 2.4m^3 of water from a well 5m below the ground and issues it at ground level through a pipe of cross-sectional area 50cm^2. Find the
   (i) speed with which the water leaves the pipe. [8ms^{-1}]
   (ii) rate at which the pump is working if its only 75% efficient. [3.24kW]
8. In every minute a pump working at 3.48kW raises 15m3 of water from underground tank and issues it from the end of a pipe situated at ground level. The water leaves the pipe with a speed of 10ms-1 and the pump is 50% efficient. Find
   (i) the cross-sectional area of the pipe. [25cm^2]

(ii) the depth below the ground level from which the water is drawn. [2m]

9. In each minute a pump working at 825W draws $0.3m^3$ of water from a well and issues from the nozzle situated 5m above the level from which the water was drawn. If the pump is 60% efficient, find
   (i) the velocity with which water is ejected. [10ms-1]
   (ii) the area of a cross-section of the nozzle. [$5cm^2$]
10. A pump raises 75kg of water a vertical distance of 20m in 14s. Find the average rate at which the pump is working. [1.05kW]

Thank you

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